Question 412: the number ρ

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abstract

This note presents some formulas related with the real root of the equation:

$x^5+x^4+x^3+x^2+x-1=0$.

1. Introduction: The number ρ .

- Definition:

$$\rho=\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}+...\right)^6\right)^6=0.5086603916... \tag{1}$$

2. Equations for ρ .

$$\rho^5+\rho^4+\rho^3+\rho^2+\rho-1=0 \tag{2}$$

$$\rho^6-2\rho+1=0 \tag{3}$$

- Remark 1: the equation $f(x)=x^5+x^4+x^3+x^2+x-1$ is not solvable by radicals.
- Remark 2: $x^6-2x+1=(x-1)\left(x^5+x^4+x^3+x^2+x-1\right)$.

3. Linear recurrences for ρ .

$$u_{n+6}=2u_{n+5}-u_n \, , u_1=1, u_2=2, u_3=4, u_4=8, u_5=16, u_6=32 \tag{4}$$

$$u_n=\{1,2,4,8,16,32,63,124,244,480,944,1856,...\} \tag{5}$$

$$\frac{u_n}{u_{n+1}}\to\rho \tag{6}$$

$$u_{n+5}=u_{n+4}+u_{n+3}+u_{n+2}+u_{n+1}+u_n \, , u_1=1, u_2=1, u_3=2, u_4=4, u_5=8 \tag{7}$$

$$u_n=\{1,1,2,4,8,16,31,61,120,236,464,912,...\} \tag{8}$$

$$\frac{u_n}{u_{n+1}}\to\rho \tag{9}$$

4. Recurrences for ρ .

$$u_{n+1} = \frac{1}{2} + \frac{1}{2} u_n^6 \quad , u_1 = 0 \Rightarrow u_n \to \rho \tag{10}$$

$$u_{n+1} = \frac{29 - u_n}{57 - 29 u_n^5} \quad , u_1 = 0 \Rightarrow u_n \to \rho \tag{11}$$

$$u_{n+1} = \frac{59 - 2u_n}{116 - 59 u_n^5} \quad , u_1 = 0 \Rightarrow u_n \to \rho \tag{12}$$

$$u_{n+1} = \frac{88 - 3u_n}{173 - 88 u_n^5} \quad , u_1 = 0 \Rightarrow u_n \to \rho \tag{13}$$

$$\rho = \frac{1}{2} + y \quad , y_{n+1} = \frac{1}{2}\left(\frac{1}{2} + y_n\right)^6 , y_1 = 0 , y_n \to y \tag{14}$$

$$u_{n+1} = \frac{1}{2 - u_n^5} \quad , u_1 = 0 \Rightarrow u_n \to \rho \tag{15}$$

$$u_{n+1} = \frac{1}{2} + \frac{1}{2\left(2 - u_n^5\right)^6} \quad , u_1 = 0 \Rightarrow u_n \to \rho \tag{16}$$

$$u_{n+1} = \frac{\left(2 - u_n^5\right)^5}{2\left(2 - u_n^5\right)^5 - 1} \quad , u_1 = 0 \Rightarrow u_n \to \rho \tag{17}$$

$$u_{n+1} = \frac{32}{64 - \left(1 + u_n^6\right)^5} \quad , u_1 = 0 \Rightarrow u_n \to \rho \tag{18}$$

5. Identities

$$\left(1 + \rho + \rho^2\right)\left(1 + \rho^3\right) = 2 \tag{19}$$

$$\left(1 + \rho\right)\left(1 - \rho + \rho^2\right)\left(1 + \rho + \rho^2\right) = 2 \tag{20}$$

$$\left(1 - \rho^3\right)\left(1 + \rho^3\right) = 2\left(1 - \rho\right) \tag{21}$$

$$\rho\left(1 + \rho\right)\left(1 + \rho^2\right) = 1 - \rho^5 \tag{22}$$

6. Series

$$\rho = \sum_{n=0}^{\infty} \binom{6n}{n} \frac{2^{-6n-1}}{5n+1} \tag{23}$$

$$\rho = \frac{1}{2} F\left(\left\{ \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6} \right\}, \left\{ \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{6}{5} \right\}, \frac{729}{3125} \right) \tag{24}$$

❖ Remark 3: F is the hypergeometric function.

7. Pi Formulas

$$\pi = 3\sum_{n=0}^{\infty}\left(\frac{\rho}{8}\right)^{n}\sum_{k=0}^{[n/6]}\binom{2n-10k}{n-5k}\binom{n-5k}{k}\frac{(-1)^{k}\,2^{14k}}{2n-10k+1} \qquad (25)$$

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^{n}\rho^{2n+1}}{2n+1}+4\sum_{n=0}^{\infty}\frac{c_n}{5n+5}\left(1-\rho^{5n+5}\right) \qquad (26)$$

$$c_n = 5^{-n}\left\{\left(\frac{1}{2}+i\right)(2-i)^{n}+\left(\frac{1}{2}-i\right)(2+i)^{n}\right\} \quad ,n=0,1,2,3,... \qquad (27)$$

$$c_n = 5^{-n}a_n \quad , a_n = 4a_{n-1}-5a_{n-2} \, , a_0 = 1, a_1 = 4 \qquad (28)$$

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}(-\rho)^{n}\sum_{k=0}^{[n/6]}\binom{n-5k}{k}\frac{(2/3)^{n-5k}\,2^{-k}}{2n-10k+1} \qquad (29)$$

$$\pi = 4\sum_{n=0}^{\infty}\frac{c_n}{n+1}\rho^{n+1} \qquad (30)$$

$$c_0=1, c_1=2, c_2=-1, c_3=4, c_4=1, c_5=-2, c_6=-1, c_7=-8, c_8=1, c_9=-8 \qquad (31)$$

$$c_n = -c_{n-2}-c_{n-4}-3c_{n-6}-3c_{n-8}-c_{n-10} \quad ,n\ge 10 \qquad (32)$$

$$\pi = 4\sum_{n=0}^{\infty}\rho^{2n+1}\left(\frac{(-1)^{n}}{2n+1}+\frac{2\rho\, c_n}{2n+2}\right) \qquad (33)$$

$$c_n = -c_{n-2}-2c_{n-3}-c_{n-4} \, ,n\ge 4 \quad ,c_0=1, c_1=2, c_2=-1, c_3=-4 \qquad (34)$$

8. The sequence n_k .

$$n_k = \{128,1312,12996,127291,1246181,12199025,...\} \qquad (35)$$

$$n_{k+1} = floor\left(\left(\frac{1}{2}\left(\frac{1}{2}+\sum_{m=1}^{k}\frac{1}{n_m}\right)^{6}-\sum_{m=1}^{k}\frac{1}{n_m}\right)^{-1}\right) \quad ,n_1=128 \, ,k=1,2,3,... \qquad (36)$$

$$\rho = \frac{1}{2}+\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}+... \qquad (37)$$

9. Complex roots of the equation: $x^5+x^4+x^3+x^2+x-1=0$.

$$z1 = z = \sqrt[5]{2}\left(e^{2\pi i/5}-\frac{1}{10\sqrt[5]{2}}\sum_{n=0}^{\infty}\frac{\left(1+(n/5)\right)_n}{(n+1)!}\left(\frac{e^{-2\pi i/5}}{2\sqrt[5]{2}}\right)^{n}\right) \qquad (38)$$

$$z2=\overline{z}=\sqrt[5]{2}\left(e^{-2\pi i/5}-\frac{1}{10\sqrt[5]{2}}\sum_{n=0}^{\infty}\frac{\left(1+\left(n/5\right)\right)_n}{\left(n+1\right)!}\left(\frac{e^{2\pi i/5}}{2\sqrt[5]{2}}\right)^n\right) \tag{39}$$

$$w1=w=\sqrt[5]{2}\left(e^{4\pi i/5}-\frac{1}{10\sqrt[5]{2}}\sum_{n=0}^{\infty}\frac{\left(1+\left(n/5\right)\right)_n}{\left(n+1\right)!}\left(\frac{e^{-4\pi i/5}}{2\sqrt[5]{2}}\right)^n\right) \tag{40}$$

$$w2=\overline{w}=\sqrt[5]{2}\left(e^{-4\pi i/5}-\frac{1}{10\sqrt[5]{2}}\sum_{n=0}^{\infty}\frac{\left(1+\left(n/5\right)\right)_n}{\left(n+1\right)!}\left(\frac{e^{4\pi i/5}}{2\sqrt[5]{2}}\right)^n\right) \tag{41}$$

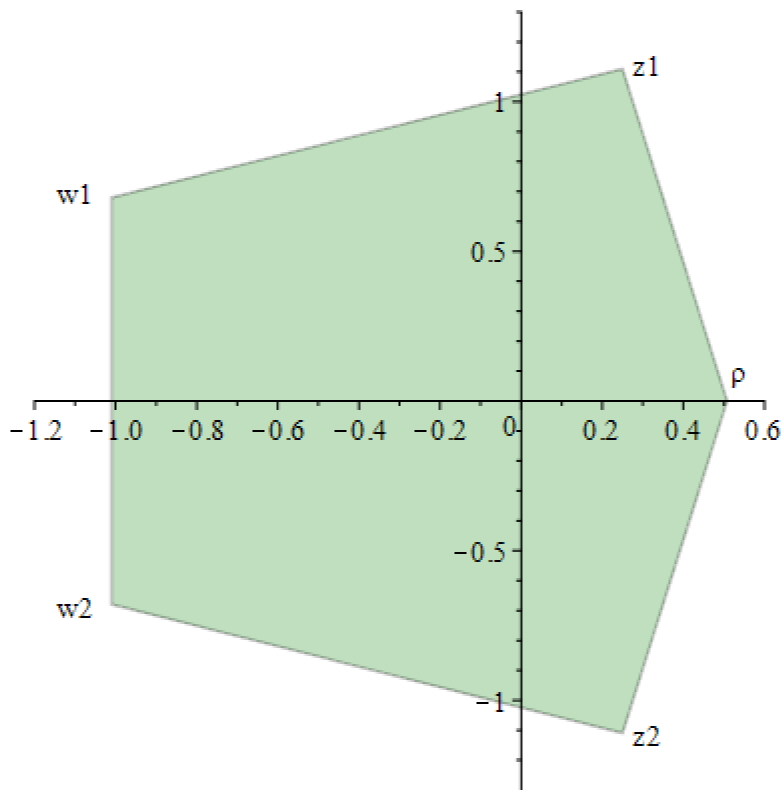

Fig. 1. Roots of $x^5+x^4+x^3+x^2+x-1=0$.

10. Integrals

$$\rho=\frac{2304}{\pi}\int_0^{2\pi}\frac{e^{xi}\left(5e^{xi}-4\right)}{4096-6144e^{xi}+729e^{6xi}}\,dx \tag{42}$$

$$\pi = 3\int_0^1 \sin^{-1}\left(\frac{1+\rho^{12}-2\rho^{12x}}{1-\rho^{12}}\right)dx \qquad (43)$$

References

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