PROOF OF THE COLLATZ CONJECTURE

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ABSTRACT

This article contains a proof of the Collatz conjecture. It is shown that it is more efficient to start calculating the Collatz function $C(n)$ from odd numbers of the form $6m \pm 1$. Then, it is proved that if we calculate using the formula $((6n \pm 1) \cdot 2^q - 1)/3$ on the basis of the sequence of numbers $6n \pm 1$, increasing the exponent of two by 1 at each iteration, then to each number of the form $6n \pm 1$, there will correspond a set whose elements are numbers of the forms $3t, 6m - 1$, and $6m + 1$. Moreover, all sets do not intersect. It is further shown that if we construct micrographs of numbers by combining the numbers $6n \pm 1$ with their elements of the set $3t, 6m - 1$, and $6m + 1$ and then combine the micrographs by combining equal numbers $6n \pm 1$ and $6m \pm 1$, we can then create a tree-like fractal graph of numbers. A tree-like fractal graph of numbers, each vertex of which corresponds to numbers of the form $6m \pm 1$, is a proof of the Collatz conjecture, as any of its vertices is connected with a finite vertex that is directly connected with unity.

KEYWORDS: Collatz conjecture, *3 x + 1* problem, Syracuse problem, Ulam's problem, Kakutani's problem, proof

MSC CLASSIFICATION CODE: 11D04

1. INTRODUCTION

The Collatz conjecture, also known as the $3x + 1$ problem, the Syracuse problem, Ulam's problem, and Kakutani's problem, is one of the unresolved problems in mathematics. The following papers devoted to the $3x + 1$ problem [1-5] can be noted.

The Collatz function $C(n)$ is defined on natural numbers as follows:

$$C(n) = \begin{cases} n/2, \ if \quad n - even, \\ 3n+1, \ if \quad n - odd. \end{cases} \tag{1}$$

To explain the Collatz conjecture, take any natural number n. If the number is even, then divide it by 2, and if the number is odd, then multiply by 3 and add 1 (we obtain $3n + 1$). On the number obtained, we perform the same actions, and so on. The Collatz conjecture is that regardless of the initial number n is taken, sooner or later, we arrive at unity.

2. THE STARTING NUMBER

According to the Collatz conjecture, the calculation can begin with any natural number greater than 1. Nevertheless, it is obvious that it is more efficient to start from an odd number, as any even number divided by 2 (one or several times) turns into an odd number.

Since odd numbers are divided into multiple and non-multiples of the number 3, the following question arises:

Question 1. From which odd numbers is it more efficient to start the calculation?

As a result of calculating the Collatz function from any odd number, a number of the form $3t + 1$ is formed, which does not have a factor of 3.

Thus, it can be stated that all natural numbers that are multiples of 3 through one operation of $3m + 1$ and division by certain powers of two, in the case of an even number, turn into odd numbers having the forms $6n - 1$ and $6n + 1$. Exceptions are numbers that after the operation $3m+1$ are equal to powers of two. Let us formalize this fact in the form of a theorem.

Theorem 1. If any natural number that is a multiple of 3 is multiplied by 3 and then added to 1 and the resulting even number is divided by a certain power of 2 until an integer is obtained, then this number will have the form $6m-1$ *or* $6n+1$.

It follows from Theorem 1 that it is more efficient to begin the calculation of the Collatz function with odd numbers having the form $6m-1$or $6m+1$.

Note. Different letters m and n are used in the notation of numbers of the same type 6m∓1 and 6n∓1 to emphasize the numbers input into and output from the calculations.

3. REVERSE CALCULATION

From the logic, it follows that if we carry out a reverse calculation, then numbers must be obtained from which the direct calculation yields a starting number. To confirm this assumption, we will perform calculations using the following formula:

$$N = ((6n \mp 1)\cdot 2^q - 1)/3, \text{ где } q = 0,1,2,3 \qquad (2)$$

Calculations performed according to formula (2) based on the sequence of numbers $6n \mp 1$ show that to each term of the sequence, there will correspond infinitely many alternating integers of the forms $3t$, $6m+1$, and $6m-1$.

It follows from formula (2) that integers are obtained according to formula (2) only if multiples of 3 are formed in the bracket. Taking this into account, we will write the following equation:

$$(6n \mp 1)\cdot 2^q - 1 = 3t. \qquad (3)$$

Hence, we obtain:

$t = ((6n \mp 1) \cdot 2^q - 1)/3.$ (4)

It is known that if we calculate according to the formula $t = (2^q - 1)/3$, increasing the exponent of two by one at each iteration, then integers are formed for even exponents of the degree. In addition, even multiples of 3, starting with $q = 6$, generate multiples of 3, and for other even exponents of the degree, numbers of the form $6m - 1$ and $6m + 1$ are alternately formed.

We note the following point: because in formula (4), the power of two is multiplied by the number $6n \mp 1$, according to formula (4), integers can be formed for even and odd exponents. In this case, the alternation of numbers of the forms $3t$, $6m - 1$, and $6m + 1$ will be preserved for any number $6n \mp 1$. Since in the calculations using formula (4), the number $6n \mp 1$will be fixed, only the exponent of the power of two will change. This is an important result, so we will also provide it in the form of a theorem.

Theorem 2. If, on the basis of each term of a continuous sequence of numbers having the form $6n \mp 1$*, a calculation is performed using the formula* $((6n \mp 1) \cdot 2^q - 1)/3$*, increasing the exponent of two by 1 at each iteration, then to each number of the form* $6n \pm 1$ *will correspond an ordered set of numbers whose elements are alternating numbers of the form* $3t$, $6m - 1$*, and* $6m + 1$*, where* $t, m, n = 1, 2, 3$.

Thus, if we calculate according to formula (2), then each term of a sequence of numbers of the form $6n \mp 1$ will form a set of numbers K_n consisting of elements corresponding to numbers of the form $3t$, $6m + 1$, and $6m - 1$:

$K_n = \{k |\ k = 6m \mp 1, 3t;\ \ m, t \in N\}.$ (5)

In other words, in the reverse calculation, numbers of the form $6n \mp 1$ split into three numbers of the forms $3t$, $6m + 1$, and $6m - 1$. With an increase in the degree of two, the number of such sets, consisting of three integers, will also increase; there are an infinite number of them.

The scheme of splitting the number of the form $6n \mp 1$ into three numbers of the forms $3t$, $6m + 1$, and $6m - 1$, called a micrograph, is shown in Fig. 1.

Note that in Fig. 1, only two sets of three integers of the form $3t$, $6m + 1$, and $6m - 1$ formed when splitting a number of the form $6n \mp 1$ are shown. While the solid lines show the vertices and edges corresponding to the numbers of the first set, the dotted lines show the vertices and edges of the second set. The number of sets of three numbers of the form $3t$, $6m + 1$, and $6m - 1$ depends on the degree of the deuce; therefore, the larger the boundary value of the exponent of the deuce, the more sets of three numbers there will be.

It is not difficult to understand that sets whose elements are numbers obtained by formula (2), corresponding to each number of the form $6n \mp 1$, are disjoint sets, i.e., the elements of the set of one number of the form $6n \mp 1$will not be repeated in the set of any other number of the same kind. Nevertheless, below, we mathematically show the impossibility of repeating numbers in different sets.

Suppose that two numbers of the form $6m \mp 1$ formed from two different numbers of the form $6n \mp 1$, as a result of the calculation using formula (2), are equal:

$$((6n_1 \mp 1) \cdot\ 2^{q_1} - 1)/3 = ((6n_2 \mp 1) \cdot\ 2^{q_2} - 1)/3.$$

From this equation, after reduction, we obtain $(6n_1 \mp 1) \cdot\ 2^{q_1} = (6n_2 \mp 1) \cdot\ 2^{q_2}$; hence, we have the following relation:

$$\frac{(6n_1 \mp 1)}{(6n_2 \mp 1)} = 2^{q_2 - q_1}. \qquad (6)$$

Obviously, the above relation does not have solutions of natural numbers, as the left side of the equation will certainly be an odd or fractional number, and the right side of the equation will always be an even number.

The disjoint sets of the form (5) corresponding to each number of the form $6n\mp1$ exclude the formation of cyclic operations in calculating the Collatz function.

4. PROOF OF CONJECTURE

Since two vertices of a micrograph of any number of the form $6n \mp 1$ are numbers of the form $6m \mp 1$ and the third vertex corresponds to a multiple of 3, it is possible to combine micrographs by combining vertices with equal numbers of the forms $6m \mp 1$ and $6n \mp 1$.

Fig. 2 shows an example of combining two micrographs, each of which consists of only one set of three numbers. For convenience, multiples of 3 are not shown on the graphs. For example, micrographs of numbers 5 and 13 are used. Since one of the vertices of the micrograph of the number 5 corresponds to the number 13, the main vertex of the micrograph of the number 13 and the vertex 13 of the number 5 are combined. The result is the graph shown in Fig. 2(c).

If we combine micrographs of numbers of the form $6n \mp 1$ into a large graph, taking into account the numbers on the vertices of the micrographs and showing the direction of the formation of numbers by the condition of the Collatz function, we obtain a tree-like oriented graph similar to the graph shown in Fig. 3.

The tree-like oriented graph (Fig. 3), which consists of all possible combinations of numbers formed in calculating the Collatz function, is a classic example of a fractal graph.

Note that in Fig. 3, each micrograph is formed from only one set of three integers of the forms $3t,\ 6m+1$, and $6m-1$. If we show other sets of three numbers, the graph is multidimensional.

A tree-like oriented graph whose vertices correspond to numbers of the form $6m \mp 1$ is a proof of the validity of the Collatz conjecture, as any of its vertices is connected with a finite vertex that has a direct connection with unity.

The tree-like fractal graph shown in Fig. 3 indicates that each vertex has its own multiple of 3, which corresponds to Theorem 2. At the same time, the multiples of 3, as shown in Fig. 3, do not influence the formation of the graph structure. If we begin the calculation with multiples of 3, then the path is joined to the vertex corresponding to a number of the form $6m \mp 1$. Then, the calculation path will be continued along the structure of the graph.

It should be emphasized that the numbers corresponding to the vertices of one graph are not repeated in other graphs, i.e., each graph is unique, although the graph forms are the same.

As seen from Fig. 3, the final vertex of a graph having a direct connection to unity has a special significance, as it is the basis of the graph. For this connection, the following question arises:

Question 2. Which numbers of the form $6m \mp 1$ *can be a finite vertex of a graph, and how many such numbers exist?*

Numbers of the form $6m \mp 1$ that are the final vertex of the graph, i.e., odd numbers of the form $6m \mp 1$ that form an even number equal to a power of two when multiplied by 3 and then added to 1, are infinitely many.

Such numbers correspond to even exponents of the power of two (starting with $q = 4$), except for even exponents of multiples of 3. Such numbers can be calculated from the following formula:

$$6m \mp 1 = (2^q - 1)/3, \tag{7}$$

where $q \geq 4$ is an even number not divisible by 3.

For example, if the numbers 5, 85 and 341, which correspond to numbers of the form $6m \mp 1$, are multiplied by 3 and then added to 1, then the even numbers 16, 256 and 1024 are formed, which are powers of two.

Since there are infinitely many numbers of the form $6m \mp 1$ corresponding to the finite vertex of a graph, the number of tree graphs is also infinitely large; the tree-like graphs form a forest of graphs.

Thus, the proof of the Collatz hypothesis is based on the following facts and patterns:

1. Any even number in the calculation of the Collatz function turns into an odd number, and any odd multiple of 3 in the calculation of the Collatz function becomes a number of the form $6n - 1$ or $6n + 1$, where $n \in \mathbb{N}$.
2. From point 1, it follows that it is more efficient to start the calculation of a function from numbers of the form $6m \mp 1$, where $m \in \mathbb{N}$. This means that to prove the Collatz conjecture, it is important to investigate a sequence of numbers of the form $6m \mp 1$.
3. In the inverse calculation performed according to formula (2), to each number of the form $6n \mp 1$, there will correspond a set whose elements are alternating numbers of the forms $3t$, $6m + 1$ and $6m - 1$, with all sets being disjoint.

4. If we combine the elements of the sets corresponding to each number $6n \mp 1$ in the form of a graph, each vertex of which corresponds to one number of the form $6m \mp 1$, then we obtain a tree-like oriented graph.

5. Numbers of the form $6m \mp 1$ that are the final vertex of the graph, i.e, odd numbers of the form $6m \mp 1$ that form an even number equal to a power of two when multiplied by 3 and then added to 1, are infinitely many. Such numbers, which are directly related to unity, are calculated using formula (7).

6. Tree-like oriented graphs, the number of which is infinite, are proof of the validity of the Collatz conjecture, as any vertex of each graph is connected with a finite vertex that has a direct connection with unity.

Since all of the above facts and patterns are supported by indisputable evidence, it can be argued that the Collatz hypothesis is correct and that it is proved.

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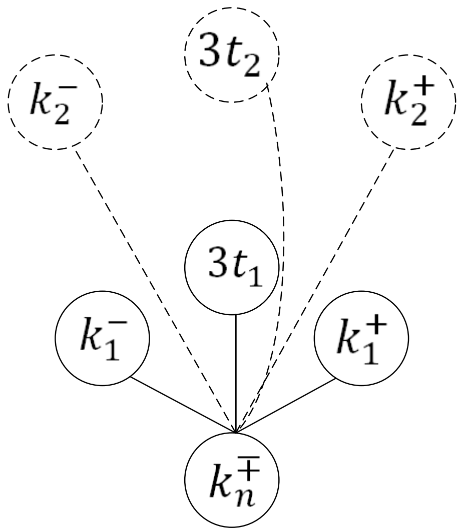
FIGURE CAPTIONS

Fig. 1 Micrograph of numbers of the form $k_n^{\mp} = 6n \mp 1$

Fig. 2 Example of combining two micrographs

Fig. 3 Tree-like oriented graph

FIGURES



Note: $k_1^- = 6m_1 - 1$, $k_1^+ = 6m_1 + 1$, $k_2^- = 6m_2 - 1$, and $k_2^+ = 6m_2 + 1$.

Fig. 1

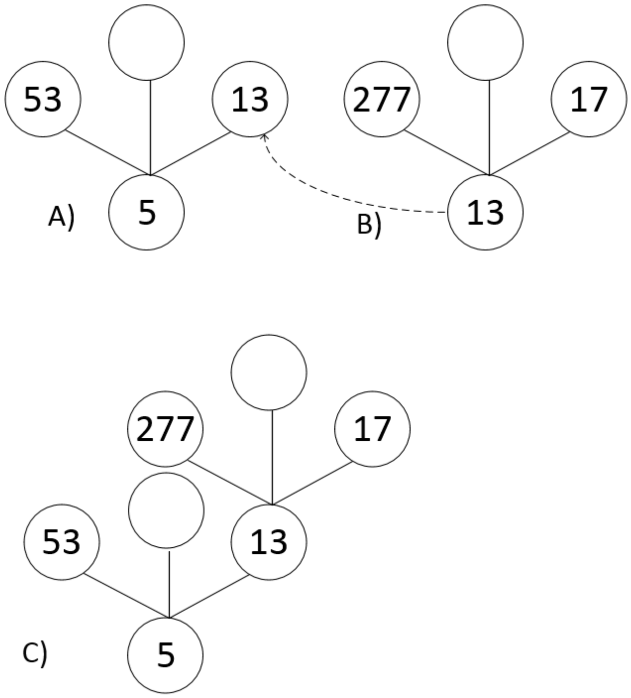
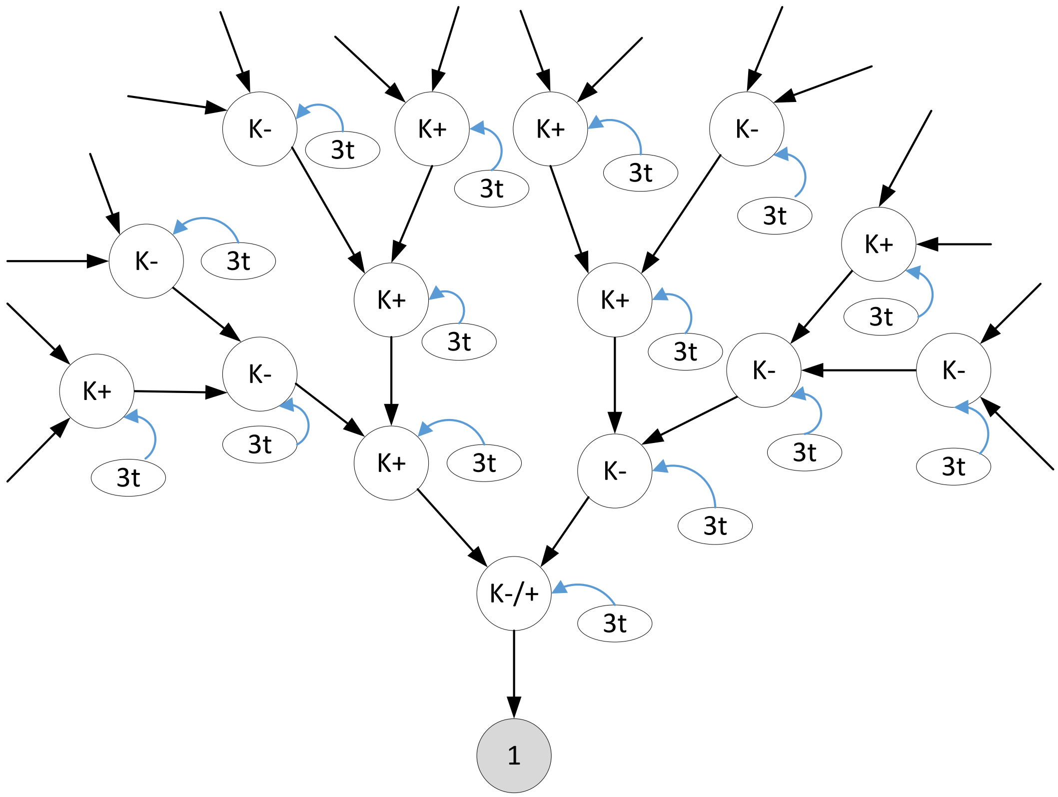


Fig. 2



Note: $k- = 6m - 1$, $k+ = 6m + 1$, and $k\mp = 6n \mp 1$.

Fig. 3