The Irrationality of Trigonometric and Hyperbolic Functions

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Introduction

Niven shows that for rational, non-zero r, $\cos r$ and $\cosh r$ are irrational [6]. His method is similar to that of his famous irrationality of π proof: functions are defined, integrals involving integration by parts are used, and a contradiction is arrived at [5]. Parks makes a similar argument, arguably simpler, for the cosine case [7]. Zhou recently proved the cosine and hyperbolic cosine results using recursive integrals [8]. In this article, our pattern does not involve integrals; just multiplication and derivatives of polynomials are needed.

The pattern is to start with an exponential equation where a sum of exponential values equals a rational number. So, for cosh this equation is $e^r + e^{-r} = a/b$. Then a polynomial, f, is defined. It has a zero root of multiplicity $p-1$, p a prime, and the exponents in the sum, r, $-r$, for cosh, as additional roots of multiplicity p. Using $e^x F(0) = F(x) + \epsilon$, proven below, and simple multiplication, this gives $0 = F(0)(e^r + e^{-r} - a/b) = -a/bF(0) + F(r) + F(-r) + \epsilon$. As the ϵ value grows power wise in the degree of f and multiplicity in f translates into factorial values in F, division by $(p-1)!$ gives a contradiction for large enough p. Details follow.

As the transform $e^x F(0) = F(x) + \epsilon$ is good for complex variables, as well as real, the identity $2\cos r = e^{ri} + e^{-ri}$ is similar to the pattern for cosh and its identity $2\cosh r = e^r + e^{-r}$ just given. Using other identities the corresponding irrationality of other trigonometric and hyperbolic functions are obtained. Corresponding results for inverses of these functions are easily proven as well.

The methods used in this article are also used in [3, 4], giving the transcendence of e, easiest, and π, hardest, respectively.

Lemmas

Definition 1. Given a polynomial $f(z)$, lowercase, the sum of all its derivatives is designated with $F(z)$, uppercase.

Definition 2. For non-negative integers n, let $\epsilon_n(z)$ denote the infinite series

$$\frac{z}{n+1} + \frac{z^2}{(n+1)(n+2)} + \cdots + \frac{z^j}{(n+1)(n+2)\dots(n+j)} + \dots.$$

Lemma 1. *If $f(z) = cz^n$, then*

$$F(0)e^z = F(z) + \epsilon, \tag{1}$$

where ϵ has polynomial growth in n.

Proof. As $F(x) = c(x^n + nx^{n-1} + \cdots + n!)$, $F(0) = cn!$. Thus,

$$\begin{aligned} F(0)e^x &= cn!(1 + x/1 + x^2/2! + \cdots + x^n/n! + \dots) \\ &= cx^n + cnx^{(n-1)} + \cdots + cn! + cx^{n+1}/(n+1)! + \dots \\ &= F(x) + cx^n(x/(n+1) + x^2/(n+1)(n+2) + \dots) \\ &= F(x) + f(x)\epsilon_n(x). \end{aligned}$$

Now $f(x)$ has polynomial growth in n and $\epsilon_n(x) \leq e^x$, so the product has polynomial growth in n. $\square$

Lemma 2. *If F is the sum of the derivatives of the polynomial $f(z) = c_0 + c_1 z + \cdots + c_n z^n$ of degree n, then*

$$e^z F(0) = F(z) + \epsilon, \tag{2}$$

where ϵ has polynomial growth in the degree of f.

Proof. Let $f_j(x) = c_j x^j$, for $0 \leq j \leq n$. Using the derivative of the sum is the sum of the derivatives,

$$F = \sum_{k=0}^{n} (f_0 + f_1 + \cdots + f_n)^{(k)} = F_0 + F_1 + \cdots + F_n,$$

where F_j is the sum of the derivatives of f_j. Using Lemma 1,

$$e^x F_j(0) = F_j(x) + \epsilon \tag{3}$$

and summing (3) from $k = 0$ to n, gives

$$e^x F(0) = F(x) + n\epsilon.$$

As the finite sum of functions with polynomial growth in n also has polynomial growth in n, we arrive at (2). $\square$

Lemma 3. *If polynomial $f(x)$ has root r of multiplicity p, then $f^{(k)}(r) = 0$, for $0 \le k \le p-1$ with each term of $f^{(k)}$, $p-1 < k \le n$ is a multiple of $p!$, where n is the degree of $f(x)$.*

Proof. Suppose $r = 0$ then $f(x) = x^p(b_n x^n + \cdots + b_0)$ has $b_0 x^p$ as its term with minimal exponent. Using the derivative operator, $D(x^n) = nx^{n-1}$, repeatedly, we see the 0 through $p-1$ derivatives of $f(x)$ will have a positive exponent of x in each term. This implies that $r = 0$ is a root, as needed for this case. Using the produce of p consecutive natural numbers is divisible by $p!$, terms of subsequent derivatives will be multiples of $p!$, .

If $r \neq 0$, then $f(x) = (x-r)^p Q(x)$, for some polynomial $Q(x)$. Let $g(x) = f(x+r) = x^p Q(x+r)$. As $g^{(k)} = f^{(k)}$ for all k, $g^{(k)}(0) = f^{(k)}(r)$, and the $r = 0$ case applies. $\square$

Lemma 4. *If a and b are Gaussian integers and p is a prime, $p > |a|$, then $|a(p-1)! + bp!|$ is a non-zero integer divisible by $(p-1)!$, but not by p.*

Proof. As $a(p-1)! + bp!$ is of the form $A - B + (C - D)i$ with $A - B \neq 0$ or $C - D \neq 0$, both results follow. $\square$

Lemma 5. *Let polynomial $f(z)$ have root $r = 0$ of multiplicity $p-1$ then, for large enough p, $p \nmid |F(0)|$.*

Proof. We can write $f(x) = x^{p-1}(b_j x^j + \cdots + b_0)$. The $p-1$ derivative is $(p-1)!b_0$ and, using Lemma 3, all subsequent terms of derivatives are multiples of $p!$. Now if $p > |b_0|$, then $p \nmid |F(0)|$, using Lemma 4. $\square$

Applications

Theorem 1. *For non-zero rational r, $\cosh r$ is irrational.*

Proof. Suppose not. Suppose $2\cosh r = a/b$ where a/b is a rational number. As 0 is not in the range of cosh, we can assume $a/b \neq 0$. Using the exponents of $2\cosh = e^r + e^{-r}$, define

$$f(z) = d^{2p-1}[(z+r)(z-r)]^p = (dz)^{p-1}(dz-c^2)^p$$

where $r = c/d$. Then $f(z)$ is an integer polynomial. Next

$$0 = F(0)\left(e^r + e^{-r} - \frac{a}{b}\right).$$

Using Lemma 2

$$0 = bF(r) + bF(-r) - aF(0) + b\epsilon$$

and, using Lemmas 3, 4, and 5, this gives a contradiction. $\square$

As Lemma 3 applies to complex polynomials as well as real, the proof for $\cos r$ is similar.

Theorem 2. *For non-zero rational r, $\cos r$ is irrational.*

Proof. Suppose not. Suppose $2\cos r = a/b$ where a/b is a rational number. We exclude $a/b = 0$ as $\cos k\pi/2 = 0$ has cos with an irrational argument. Using the exponents of $2\cos = e^{ri} + e^{-ri}$, define

$$f(z) = d^{2p+p-1}z^{p-1}[(z+ri)(z-ri)]^p = (dz)^{p-1}((dz)^2+c^2)^p$$

where $r = c/d$. Then $f(z)$ is an integer polynomial. Next

$$0 = F(0)\left(e^{ri} + e^{-ri} - \frac{a}{b}\right).$$

Using Lemma 2 and multiplying by b gives

$$0 = bF(ri) + bF(-ri) - aF(0) + b\epsilon$$

and, using Lemmas 3, 4, and 5, this gives a contradiction. $\square$

Conclusion

Once $\cos r$ and $\cosh r$ are proven irrational, sec r and sech r are easy consequences, being reciprocals of these functions [6]. As $\cos 2r = \cos^2 r - \sin^2 r = 1 - 2\sin^2 r$, the rationality of $\sin r$ would imply that of cos, a contradiction. Assume $\tan r$ is rational. Then using

$$\cos 2r = \frac{1-\tan^2 r}{1+\tan^2},$$

$\cos r$ would be rational too, a contradiction. As csc r and cot r are the reciprocals of $\sin r$ and $\tan r$, the former two are proven irrational. Similarly, the hyperbolic functions all follow the same program. Inverse functions have an easy proof: if $f^{-1}(r) = a/b$ then $f(f^{-1}(r)) = r = f(a/b)$, a contradiction of $f(r)$ is irrational.

One might wonder if Theorem 2 could be modified to allow for the $a/b = 0$ case. This would allow for a proof of the irrationality of π, a nice result. We need $F(0)$, but not necessarily that 0 is a root of f for Lemma 2 to work. We need to ensure that

$$0 = F(0)(e^{ri} + e^{-ri}) = F(ri) + F(-ri) + \epsilon.$$

Using $f(z) = d^{2p-1}(z-ri)^{p-1}(z+ri)^p$ will do the trick. Note a benefit of Lemma 2 over the use of the mean value theorem and real and complex integrals to establish its validity, as in [1, 2], is just this: one of the roots can assume the role of the odd power out, a multiplicity of $p-1$.

References

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