SUPPLEMENTARY

1. Evaluation of computer assisted proofs on Gödel incompleteness theorems

Introduction

The Gödel incompleteness theorem states in effect that sequences of logic symbols can be assigned to strings of natural numbers, but because it is assumed there are more natural number than sequences of logic symbols, the symbols are incomplete as a self-referencing mechanism to describe repeatedly themselves as yet more numbers. (The Gödel completeness theorem states in effect that the sequences of logic symbols may be consistent to form a logic system that is sufficiently complete enough to prove theorems as tautology.)

The arguments ultimately turn on the mapping of *sequences* of symbols into *strings* of natural numbers.

The arguments also assume a function to map numbers as a domain into symbols as an image in a co-domain, also collectively named a range. The incompleteness theorem is that such a function exists and operates where all of its domain is larger than the smaller image existing within all of the co-domain. The completeness theorem is that all of the image, existing solely for its purposes, is self-sufficient unto the existence of itself.

We are interested in mapping the image in the co-domain as sequences of logic symbols back into the originating domain which is named a preimage. Attempting such an inverse function is not allowed by the one-way definition of a function to an image as dictated by the incompleteness theorem. We show this invertive approach or reverse process is not tautologous by evaluating the misuse of the application of quantified operators in the one-way functional mapping in the first place.

Our experimental results in mapping are the definitive evidence that intuistionistic logic and subsequently constructivistic logics are not complete (deny the completeness theorem) and hence deny the incompleteness theorem. The most important evidence is that those logics can exist only by ignoring the law of excluded middle (LEM) that "p or not p is a tautology", (p+~p)=(p=p). We argue that any system denying the LEM is not tautologous and hence unworkable.

Experimental questions

To better evaluate computer assisted proofs of Gödel incompleteness theorem(s), we ask two questions.

Q1. Are there refutations of Gödel incompleteness theorems using proof assistants?; and

Q2. In the PowerEpsilon proof language (Zhu, 2013), what is the first mistake found, and does it affect the results?

Experimental answers

A1.1 First order logic (FOL) can be represented mistakenly in numeric symbols (Meyer, 2017), and the Godel numbering function is flawed (Meyer, 2009). Used as examples are objections to Gödel theorems at:

jamesrmeyer.com/pdfs/ff_harrison.pdf (A1.1.1)
jamesrmeyer.com/pdfs/ff_oconnor.pdf (A1.1.2)
jamesrmeyer.com/pdfs/ff_shankar.pdf (A1.1.3)

Each assessment is negative for the computer assisted tool in evaluation of the theorems, with the effective admonition against the proof assistants as garbage-in, garbage-out.

A1.2 The website does not evaluate PowerEpsilon (PE), so we supplied a copy of (Zhu, 2013) with the question of: "What is mistaken in this monograph". The response was that a sequence of symbols is assumed to be expressed in PE if the sequence indexes into the variable type of natural numbers. This is theoretical mistake for practicing PE, but not a logical mistake. This led us to ask Q2.

A2. The first equation we evaluated in (Zhu, 2013) was for induction that we validated as tautologous using Meth8-VŁ4 (James III, 2017c). We reiterate it here due to its brevity:

From 2.2.4, page 7, we evaluate an equation for "[m]athematical induction as an inference rule formalized as a second-order axiom".

We assume the Meth8 apparatus.

$$\forall P \,.\, P(0) \wedge \forall k \,.\, P(k) \Rightarrow P(k+1) \Rightarrow \forall n \,.\, P(n) \tag{1.1}$$

LET: p P; q k; r n;

& And; + Or; > Imply; = Equivalent to; @ Not Equivalent to;

universal quantifier, modal necessity; (r@r) 0 [Zero]; (r=r) 1 [One]

T tautology; F contradiction

Result fragment is the repeating row from the truth table of 16-values.

$$\forall P \,.\, P(0) \wedge \forall k \,.\, P(k) \Rightarrow P(k+1) \Rightarrow \forall n \,.\, P(n) \tag{1.1}$$

(((#p&p)&(r@r))&((#q&p)&q)) > ((p&(q+(r=r)))>((#r&p)&r)) ; TTTT ; (1.2)

From the script rendition in Eq 1.2, Meth8 validates Eq 1.1 as tautologous.

We remark that in Meth8 we can coerce two numeric values of 0 or 1 to be represented by logical T tautology or F contradiction, due to the bivalent definition of VŁ4 based on a 2-tuple.

We also remark that a free variable in Meth8 is simply another variable.

The next expression in the text is the FOL axiom of the law of excluded middle (LEM):

$$\forall P \,.\, P \vee \neg P \tag{2.1}$$

We assume the Meth8 apparatus, where the designated proof value is T tautology. Truth tables are in 16-values as presented row major and horizontally.

LET: # ∀; p P; + Or; ~ Not; = Equivalent to; (p=p) Tautology.

$$\forall P \,.\, P \vee \neg P \tag{2.1}$$

(#p&(p+~p)) = (p=p) ; `FNFN FNFN FNFN FNFN` (2.1.1)

Eq. 2.1.1 is not tautologous.

We distribute the universal quantifier directly to the variable in the antecedent at Eq. 2.1.1:

(#p+#~p) = (p=p) ; `FNFN FNFN FNFN FNFN` (2.1.2)

The result of Eq. 2.1.2 and 2.1.1 is the same, as not tautologous.

We are reminded that the LEM without the universal quantifier is:

(p+~p) = (p=p) ; `TTTT TTTT TTTT TTTT` (2.1.3)

Eq. 2.1.3 is tautologous.

We attempt to coerce the universal quantifier onto the LEM to make it tautologous as follows:

(#(p+~p)=#(p=p))) = (p=p) ; `TTTT TTTT TTTT TTTT` (2.1.4)

Eqs. 2.1.4 is tautologous.

We determine that Eq. 2.1 is mistaken.

For the second part of Q2, how does Eq. 2.1, now as not tautologous, affect the resulting conclusion in PE to prove the theorem of incompleteness? We answer that Eq. 2.1 has no affect. Our reasoning is that the Gödel formulas are mistaken, due to non-uniform representation of quantification, but mapped with fidelity by the rules of PE. In other words, the source of error reverts fully to Gödel.

Conclusions

We conclude that from a logical mistake in PE for rending the LEM, the correct implementation of LEM for any logic system is necessary to comply with the bivalence of Meth8-VŁ4.

What follows is that our previous evaluationsof intuisionistic and constructivist logics in fact verify those logics as unworkable (James III, 2017b; 2017a).

2. Mistaken equations (2) from Gödel's text

The Gödel incompleteness theorem (Meyer, 2014) contains examples of two contradictions in first order logic (FOL), as an axiom in system P and a proposition for generic schema.

We assume the Meth8 apparatus implementing variant system VŁ4 for:

T tautology (designated proof value); F contradiction; C contingency (falsity); N non-contingency (truth). Truth table results in 16-values are row-major and horizontally.

2.1. "This axiom represents the axiom of reducibility (the axiom of comprehension of set theory)" in formal system P, Section 2, Proposition IV.1:

(∃ u)(v ∀(u(v) ≡ a)) (2.4.1.0)

LET: p r s a u v ; % existential quantifier, # universal quantifier, as undistributed

%r&(s&#((r&s)=p)) ; `FFFF FFFF FFFF FNFN` (2.4.1.1)

LET: p r s a u v ; % existential quantifier, # universal quantifier, as distributed

(%r&s)&(%r&((#r&#s)=#p)) ; `FFFF FFFF CCCC CTCT` (2.4.1.2)

Eqs. 2.4.1.1 and 2.4.1.2 are not tautologous.

2.2. "Relation (class) is called arithmetical" in Section 3, Proposition 6:

x > y ≡ ~(∃z)[y = x+z] ; (3.6.0)

LET: p q r x y z ; % existential quantifier, as undistributed

(p>q)=(~%r&(q=(p+r))) ; `NTFN FTFF NTFN FTFF` (3.6.1)

LET: p q r x y z ; % existential quantifier, as distributed

(p>q)=((~%r&q)=(~%r&(p+r))) ; `TNCT TFTT TNCT TFTT` (3.6.2)

Eqs. 3.6.1 and 3.6.2 are not tautologous.

Using Meth8-VŁ4 we can not find tautology in these examples. We conclude that the use of quantified operators by Gödel was mistaken as inconsistent, or not bivalent, or both.

3. Cut elimination equations (11) from constructivist rendition of Gödel

This section relies on the first order logic (FOL) expressions as a perfect implementation of Gödel's axioms, rules, and theorems in the programming language of PowerEpsilon (Zhu, 2013), a proof-development system based on constructive type theory. With that exposition, Meth8 was capable to validate as tautologous the theorems of Gödel.

LET: p x; q y; s s; # for all; % for some (one) ; & And; \ Not And (/);

> Imply; < Not Imply

The designated proof value is T tautology. Other values are: F contradiction; C contingency (a falsity value); and N non-contingency (a truth value).

Truth tables are presented as the 16-values in row major, horizontally.

When rendering quantified operators from the text to the script of Meth8, we explicitly distribute quantified operators for clarity and portability. For example $\forall p . (p \vee \neg p)$ is equivalent to $\forall p . (p) \vee \forall p . (\neg p)$.

We examine FOL expressions to replicate results in the text:

[Section 4.4. FOL axioms replicated and confirmed tautologous. Section 4.5. FOL inference rules; we stopped at 4.5.2.13 with functions which are programming language dependent, then commenced again at 4.5.2.14.1.]

At 4.5.2.15 for universal quantifier:

$\underline{\forall Y . \Gamma \vdash X[Y/v]}$

$\Gamma \vdash \forall v . X$ (4.5.2.15.1)

LET: p X; q Y; r v; s upper_case_Gamma; # for all; % for some

((#q&s)>(p&(q\r)))>(s>(#r&p)) ; TTTT TTTT TTCT TTTT (4.5.2.15.1.1)

$\underline{\Gamma \vdash \forall v . X}$

$\forall Y . \Gamma \vdash X[Y/v]$ (4.5.2.15.2)

((s>(#r&p))>(#q&s))>((#q&s)>(p&(q\r))) ; TTTT TTTT TTCT TTCC (4.5.2.15.2.1)

$\underline{\Gamma \vdash Y \ \Gamma \vdash \forall v . X}$

$\Gamma \vdash X[Y/v]$ (4.5.2.15.3)

((s>q)&(s>(#r&p)))>(s>(p&(q\r))) ; TTTT TTTT TTTT TTTC (4.5.2.15.3.1)

At 4.5.2.16 for existential quantifier:

$\underline{\exists Y . \Gamma \vdash X[Y/v]}$

$\Gamma \vdash \exists v . X$ (4.5.2.16.1)

((%q&s)>(%q&(p&(q\r))))>(s>(%r&p)) ; TTTT TTTT CCTC CTTT (4.5.2.16.1.1)

$\underline{\Gamma \vdash \exists v . X}$

$\exists Y . \Gamma \vdash X[Y/v]$ (4.5.2.16.2)

(s>(%r&p))>((%q&s)>(%q&(p&(q\r)))) ; TTTT TTTT TTTT TTTC (4.5.2.16.2.1)

$\underline{\Gamma \vdash Y \ \Gamma \vdash X[Y \backslash v]}$

$\Gamma \vdash \exists v . X$ (4.5.2.16.3)

((s>q)&(s>(p&(q\r))))>(s>(%r&p)) ; TTTT TTTT TTTC TTTT (4.5.2.16.3.1)

At 4.5.2.21 for universal and existential quantifiers:

$\underline{\Gamma \vdash \neg \forall v . X}$

$\Gamma \vdash \exists v . \neg X$ (4.5.2.21.3) (s>(~#r&p))>(s>(%r&~p)) ;   TTTT TTTT TFTF TNTN   (4.5.2.21.3.1)

Γ ⊢ ¬∃v . X

Γ ⊢ ∀v . ¬X   (4.5.2.21.4)

(s>(~%r&p))>(s>(#r&~p)) ;   TTTT TTTT TCTC TTTT   (4.5.2.21.4.1)

Meth8 does not replicate those quantified expressions in Sections 4.5.2.15, 4.5.2.16, or 4.5.2.21. Some of the truth tables come close to tautology by pattern.

At 8.2.4 for completeness and incompleteness theorems:

Completeness of logic system:

∀p . (∃Γ . Γ ⊢ p ∨ ∃Γ . Γ ⊢ ¬p)   (8.2.3.1)

(#p&(%s&(s>p)))+(#p&(%s&(s>~p))) ;   FFFF FFFF FNFN FNFN   (8.2.3.1.1)

Incompleteness of logic system:

∃p . (¬∃Γ . Γ ⊢ p ∧ ¬∃Γ . Γ ⊢ ¬p)   (8.2.3.2)

(#p&(~%s&(s>p)))&(#p&(~%s&(s>~p))) ;   FNFN FNFN FFFF FFFF   (8.2.3.2.1)

Completeness of formula set:

∀p . (Γ ⊢ p ∨ Γ ⊢ ¬p)   (8.2.4.1)

(#p&(s>p))+(#p&(s>~p)) ;   FNFN FNFN FNFN FNFN   (8.2.4.1.1)

Incompleteness of formula set:

∃p . (Γ /⊢ p ∧ Γ /⊢ ¬p)   (8.2.4.2)

(%p&(s<p))&(%p&(s<~p)) ;   FFFF FFFF FFFF FFFF   (8.2.4.2.1)

Meth8 does not replicate those quantified theorems in Sections 8.2.3 or 8.2.4. Eq. 8.2.4.2.1 is validated as contradictory.

References

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