Rule of necessitation: true, but not tautologous

1. The axiom or rule of necessitation **N** states that if p is a theorem, then necessarily p is a theorem:

   If $\vdash p$ then $\vdash \Box p$.

We show this is non-contingent (a truth), but not tautologous (a proof). We evaluate axioms (in bold) of **N**, **K**, **T**, **4**, **B**, **D**, **5** to derive systems (in italics) of *K*, *M*, *T*, *S4*, *S5*, *D*.

We assume the Meth8 apparatus implementing system variant VŁ4, where:

necessity, universal quantifier; % possibility, existential quantifier;
\> Imply; = Equivalent to; (p=p) Tautology

Definition	Axiom	Symbol	Name	Meaning	2-tuple	Ordinal
1	p=p	T	Tautology	proof	11	3
2	p@p	F	Contradiction	absurdum	00	0
3	%p>#p	N	Non-contingency	truth	01	1
4	%p<#p	C	Contingency	falsity	10	2

The designated proof value is T tautology. Note the meaning of (%p>#p): a possibility of p implies the necessity of p; and some p implies all p. In other words, if a possibility of p then the necessity of p; and if some p then all p. This shows equivalence and interchangeability of respective modal operators and quantified operators, as proved in Appendix.
(That correspondence is proved by VŁ4 corrections to the vertices of the Square of Opposition and subsequent corrections to the syllogisms of Modus Cesare and Modus Camestros.)

Results are the 16-value truth table as row-major and horizontal; tautology is all "`TTTT`".

N: If $\vdash p$ then $\vdash \Box p$. (**N**.1.1)
p>#p ; `TNTN TNTN TNTN TNTN` (**N**.1.2)

Eq. 1.2 is minimally true at a level of non-contingency (`NNNN NNNN NNNN NNNN`), but not a proof at a level of tautology (`TTTT TTTT TTTT TTTT`).

The definitions of the other axioms are as follows (Steward, Stoupa, 2004):

K: $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$; no conditions (**K**.1.1)
#(p>q)>(#p>#q) ; `TTTT TTTT TTTT TTTT` (**K**.1.2)

T: $\Box p \rightarrow p$; reflexive (**T**.1.1)
#p>p ; `TTTT TTTT TTTT TTTT` (**T**.1.2)

4: $\Box p \rightarrow \Box \Box p$ (**4**.1.1)
#p>##p ; `TTTT TTTT TTTT TTTT` (**4**.1.2)

B: p → □ ◊ p ; reflexive and symmetric (**B**.1.1)
p>#%p ; TTTT TTTT TTTT TTTT (**B**.1.2)

D: □ p → ◊ p ; serial (**D**.1.1)
#p>%p ; TTTT TTTT TTTT TTTT (**D**.1.2)

5: ◊ p → □ ◊ p (**5**.1.1)
%p>#%p ; TTTT TTTT TTTT TTTT (**5**.1.2)

The definitions of systems are as follows:

K:= **K** (no conditions) (*K*.1.1)
#(p>q)>(#p>#q) ; TTTT TTTT TTTT TTTT (*K*.1.2)

alternatively, **K** & **N** is used (viz, en.wikipedia.org/wiki/Modal_logic) (*K*.2.1)
(#(p>q)>(#p>#q))&(p>#p) ; TNTN TNTN TNTN TNTN (*K*.2.2)

Eq. *K*.2.2 subsequently taints all results as having some value of truth (`TNTN`), but *not* tautology (`TTTT`).

D:= *K* & **D** (serial) (*D*.1.1)
(#(p>q)>(#p>#q))&(#p>%p) ; TTTT TTTT TTTT TTTT (*D*.1.2)

M:= *K* & **T** (*T*.1.1)
(#(p>q)>(#p>#q))&(#p>p) ; TCTT TCTT TCTT TCTT (*T*.1.2)

S4:= *M* & **4** ; reflexive and transitive (*S4*.1.1)
((#(p>q)>(#p>#q))&(#p>p))&(#p>##p) ; TTTT TTTT TTTT TTTT (*S4*.1.2)

B:= *M* & **B** (*B*.1.1)
((#(p>q)>(#p>#q))&(#p>p))&(p>#%p) ; TTTT TTTT TTTT TTTT (*B*.1.2)

S5:= *M* & **5** ; reflexive and Euclidean (*S5*.1.1)
((#(p>q)>(#p>#q))&(#p>p))&(%p>#%p) ; TTTT TTTT TTTT TTTT (*S5*.1.2)

alternatively, *M* & **B** & **4** (*S5*.2.1)
(((#(p>q)>(#p>#q))&(#p>p))&(p>#%p))&(#p>##p) ;

2. We also evaluated (Steward, Stoupa, 2004) to derive by replication some systems of interest.

K: [](p ⊃q) ⊃([]p ⊃[]q) (3.1.1)
#(p>q)>(#p>#q) ; TTTT TTTT TTTT TTTT (3.1.2)

Axiom **T**: []p ⊃p (3.2.1)
#p>q ; TTTT TTTT TTTT TTTT (3.2.2)

M, obtained by extending system **K** with rule **T** [not Gödel's system T] (3.3.1)
(#(p>q)>(#p>#q))>(#p>q) ; TCTT TCTT TCTT TCTT (3.3.2)

"The strongest system from these modal logics that is perfectly straightforward to formulate in a sequent system and to prove cut-free is system **G-M** (for Gentzen system **M**)".

We remark that the subsequent derivations of *S4*, *B*, and *S5* are tautologous, as are **K** and **T** as demonstrated in section 1.

2. We found other mistakes in (Steward, Stouppa, 2004).

2.1. "The following lemma is a straightforward exercise in theoremhood over **K**:

LEMMA 6 If A ⊃B is a theorem of **M**, then so are: (L.6.0.1)
1. A ∧C ⊃B ∧C; (L.6.1.1)
2. A ∨C ⊃B ∨C; (L.6.2.1)
3. []A ⊃[]B; (L.6.3.1)
4. <>A ⊃<>B." (L.6.4.1)

To map Eq. L.6.0.1 we use Eq. 3.3.2.

((#(p>q)>(#p>#q))>(#p>q)) > (p>q) ; TNTT TNTT TNTT TNTT (L.6.0.2)

We then reuse Eq. L.6.0.2 to map L.6.1.2 - 6.4.2.

(((#(p>q)>(#p>#q))>(#p>q))>(p>q)) > ((p&r)>(q&r)) ; TTTT TCTT TTTT TCTT (L.6.1)
(((#(p>q)>(#p>#q))>(#p>q))>(p>q)) > ((p+r)>(q+r)) ; TCTT TTTT TCTT TTTT (L.6.2)
(((#(p>q)>(#p>#q))>(#p>q))>(p>q)) > (#p>#q) ; TCTT TCTT TCTT TCTT (L.6.3)
(((#(p>q)>(#p>#q))>(#p>q))>(p>q)) > (%p>%q) ; TCTT TCTT TCTT TCTT (L.6.4)

2.2. These inference rules were flagged by Meth8, with page number for equation.

LET: p uc_Gamma; q uc_Delta; r A; s B

(p&r)>(%p&#r) ; 1.#1 ; TTTT TNTN TTTT TNTN (315, []1)
(%p&r)>(%p&#r) ; TTTT NNNN TTTT NNNN (323, []2)
((%p&q)&r)>((%p&#q)&#r) ; TTTT TTNN TTTT TTNN (324, []5)

"we recommend the reader works ... example (A ⊃B ⊃C) ⊃(A ⊃C) ⊃B ⊃C" (321.1)
((((p>q)>r)>(p>r))>q)>r ; TFFF TTTT TFFF TTTT (321.2)

We conclude that **N** the axiom or rule of necessitation is *not* tautologous Because system *M* as derived and rendered is not tautologous, system *G-M* also *not* tautologous.

What follows is that systems derived from using *M* are tainted, regardless of the tautological status of the result so masking the defect, such as systems *S4*, *B*, and *S5*.

We also find that Gentzen-sequent proof is suspicious, perhaps due to its non bi-valent lattice basis in a vector space.

References

Steward, Charles; Stouppa, Phiniki. (2004). A systematic proof theory for several modal logics; also at textproof.com/supervision/phiniki04sbm.pdf