JOURNEY TO THE END OF GÖDEL THEOREMS

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The two theorems for incompleteness and completeness developed by Gödel in the 1930's provoked a conceptual revolution in the history of mankind. The claim that not all truths can be demonstrated had a pervasive and enduring influence, leading to deep relativistic standpoints in physics, biology, philosophy and social sciences. Here, based on recently-developed logic tools, we demonstrate that the two Gödel theorems are not tautologous, and hence are unworkable.

The first incompleteness theorem demonstrates that, if an observer is limited to syntactic approaches, he is not capable to obtain truths (Gödel, 1931). As an example, Gödel states that the phrase: "this proposition is not demonstrable" could be either a) false, and in this case is absurd, because it is false and demonstrable, or b) true, and in this case is not demonstrable. Given a group of arithmetical axioms, there is always a true arithmetic proposition that cannot be demonstrated starting from the axioms, if we admit just the syntactic methods, as dictated by the Hilbert model (Mancosu, 1998). Hence, syntactic methods are inadequate to understand the complete properties of the same model that is easily understandable through semantics. Therefore, while the human brain is able to grasp the semantic notion of truth, non-physicalist truth cannot be verified by computers. The second incompleteness theorem states that is not possible to verify the truth of a group of arithmetic axioms (Gödel, 1931). Yet such a state of affairs may be overcome because every true arithmetic proposition can be demonstrated starting from Peano axioms (Segre 1994), which use a semantic model. However, the certainty of the semantic model as correct, complete and truthful cannot be known. The existence of axioms is based on the questionable intuition that such axioms are deemed true. Therefore, if unquestioned and consistent syntactic methods are used, as proposed by Hilbert, then all truths cannot be proved. By alternative example, if uncertain semantic models are used, then all arithmetic truths possibly could be known, but the correctness of such models is uncertain. Such conclusions become, in the years to come, the tenets for all scientific developments, including quantum dynamics, theoretical physics, biology, neuroscience and sociology. After Gödel, the truths developed by the human mind and by scientific procedures cannot anymore describe the whole realm of reality. This achievement, together with the Schrödinger theorem, became the standpoint that prevents the observer to fully understand his own world.

JOURNEY TO THE END OF THE FIRST-ORDER LOGIC

Here we introduce a recently-developed logic approach, termed Meth8, in order to demonstrate that the two Gödel theorems are logically untenable. Logic is a way to do abstract arithmetic on formulas. The goal is to disprove or prove equations with logic values, and the result is a truth table of outcome values, where simple true (or false) builds into proofs (or contradictions). When the designated truth value is specified such as Tautology, then that is the cue for what value the practitioner seeks to constitute as a proof. This paper is about a logic named VŁ4 and its proof assistant named Meth8 (James III, 2016a, 2016b).

Meth8 is an automated tool to check the logic on input and produce truth tables on output at each step. Propositions in words (or formulas in symbols) are mapped into logic scripts of one-character symbols. VŁ4 is based on the 2-tuple, a bivalent object in bits: {11,01,10,00}. The 2-tuple is a ring, module and group, but it is not a vector space. The logical values are assigned as based on the left- or right-side of the pair of bits. The atomic values are: Zero 0 as False, One 1 as True. The bits are in pairs, so we chose a Left- or Sinister-side and Right- or Dexter-side for each of the four logical values. We also assign the Sinister-side as False, and the Dexter-side as True (**Figure A**). This binary logic system is B4. VŁ4 rehabilitates the modal logic system Ł4, by specifying five models to describe the combinations of logical results. VŁ4 is also based on a simplified truth and proof definition: tautology is proof; contradiction is not proof; truth is non-contingency; and falsity is contingency. When VŁ4 was applied to the Square of Opposition, the assumed basis of all logic, it realized that updates were required for the Square to be modernized (Westerstahl, 2012). Indeed, the vertices of the Square were slightly redefined: this caused its edges to have definitions as logical equations. Those, in turn, were applied to the syllogisms of the Square, so that a correction to two of them was found. What followed from correcting the assumptions in the Square proved that the modal operators are logically equivalent to the quantified operators. This simplified the theory of proofs, to the extent that VŁ4 can evaluate equations in any of the major high order logic systems. In other words, VŁ4 can compare and contrast results from other systems (such as Belnap, 1977; Dugundji, 1940; Kleene, 1938; Lewis and Langford, 1959) .

In greater detail, the basis of mathematical logic is the Square of Opposition, with four sides and four vertices. VŁ4 updated the formulas of vertices to be exact. Consequently, the four sides obtained unique descriptive equations. The Square also produces combinations for 256-syllogisms. From those, 24-syllogisms were adopted historically as valid modes of argument. VŁ4 found and corrected two of the syllogisms, named Modus Camestros and Modus Cesare. Exact formulas for the vertices of the Square enabled correct definitions for the modal operators of necessity [] and possibility <> and for the quantified operators of the universal quantifier ∀ and existential quantifier ∃. VŁ4 then proved the respective operators were interchangeable and equivalent: necessity means “for all”; and possibility means “for one” to overcome. Because of this unique quality, VŁ4 is capable to describe other logic systems, and to validate them at tautologous. We resuscitate Ł4 by looking at the combinations of modal operators on the four truth values and for five models. VŁ4 has two sets of logical values: FNCT and UIPE. The letters mean: contradiction, non-contingency, contingency, tautology; and unevaluated, improper, proper, evaluated. The designated proof values are tautology (T) and evaluated (E). The truth tables for Model 1 and for the Model 2 group are described in **Figures B** and **C**, with ~ not, # necessity, and % possibility.

A

Left-false	Right-true	Meaning for each side	Reduction	Value names
1	1	True it is false, *or* True it is true.	False, *or* True	tautology (proof)
1	0	True it is false, *and* False it is true.	False, *and* False	contingent (falsity)
0	1	False it is false, *and* True it is true.	True, *and* True	non-contingent (truth)
0	0	False it is false, *and* False it is true.	True, *and* False	absurdum (contradiction)

B

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  VŁ4      ~VŁ4              1          21    22    231   232
M1  M2   ~M1  ~M2           # %         # %   # %   # %   # %
 F   U     T    E       F.  F C     U.  U U   U E   U P   U I
 C   I     N    P       C.  F C     I.  I I   U E   I E   U I
 N   P     C    I       N.  N T     P.  P P   U E   U P   P E
 T   E     F    U       T.  N T     E.  E E   U E   I E   P E
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C

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1 & . F,F,F,F . F,C,F,C . F,F,N,N . F,C,N,T
1 \ . T,T,T,T . T,N,T,N . T,T,C,C . T,N,C,F
1 + . F,C,N,T . C,C,T,T . N,T,N,T . T,T,T,T
1 - . T,N,C,F . N,N,F,F . C,F,C,F . F,F,F,F
1 < . F,F,F,F . C,F,C,F . N,N,F,F . T,N,C,F
1 = . T,N,C,F . N,T,F,C . C,F,T,N . F,C,N,T
1 > . T,T,T,T . N,T,N,T . C,C,T,T . F,C,N,T
1 @ . F,C,N,T . C,F,T,N . N,T,F,C . T,N,C,F

2 & . U,U,U,U . U,I,U,I . U,U,P,P . U,I,P,E
2 \ . E,E,E,E . E,P,E,P . E,E,I,I . E,P,I,U
2 + . U,I,P,E . I,I,E,E . P,E,P,E . E,E,E,E
2 - . E,P,I,U . P,P,U,U . I,U,I,U . U,U,U,U
2 < . U,U,U,U . I,U,I,U . P,P,U,U . E,P,I,U
2 = . E,P,I,U . P,E,U,I . I,U,E,P . U,I,P,E
2 > . E,E,E,E . P,E,P,E . I,I,E,E . U,I,P,E
2 @ . U,I,P,E . I,U,E,P . P,E,U,I . E,P,I,U
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Figure. Details of VŁ4. **A**: The binary logic system used in VŁ4. **B**: Truth tables for Model 1 and for the Model 2 groups, with ~ not, # necessity, and % possibility. **C**: The connectives by Model types 1 and 2 are: & And, \ Not and, + Or, - Not or, < Not imply, = Equivalent, > Imply, @ Not equivalent. The look up truth tables for connectives by Model types 1 and 2 are presented horizontally as row major.

WAS GÖDEL WRONG? THE PROOF

This demonstration relies on the first order logic (FOL) expressions as a perfect implementation of Gödel's axioms, rules, and theorems in the programming language of PowerEpsilon (Zhu, 2013), a proof-development programming system based on constructive type theory. With that exposition, Meth8 is capable to validate as tautologous the theorems of Gödel.

LET: p x; q y; s s; # for all; % for some; & And; \ Not And /; > Imply; < Not Imply

The designated proof value is T tautology. Other values are: F contradiction; C contingency (a falsity value); and N non-contingency (a truth value).
Truth tables are presented as the 16-values in row major, horizontally.
When rendering quantified operators from the text to the script of Meth8, we explicitly distribute quantified operators for clarity and portability. For example ∀p . (p ∨ ¬p) is equivalent to ∀p . (p) ∨ ∀p . (¬p).
We examine FOL expressions to replicate results in the text:
[Section 4.4. FOL axioms replicated and confirmed tautologous.
Section 4.5. FOL inference rules; we stopped at 4.5.2.13 with functions which are programming language dependent, then commenced again at 4.5.2.14.1.]

At 4.5.2.15 for universal quantifier:

LET: p X; q Y; r v; s upper_case_Gamma; # for all; % for some

∀Y . Γ ⊢ X[Y/v]
Γ ⊢ ∀v . X (4.5.2.15.1)

((#q&s)>(p&(q\r)))>(s>(#r&p)) ; TTTT TTTT TTCT TTTT (4.5.2.15.1.1)

Γ ⊢ ∀v . X
∀Y . Γ ⊢ X[Y/v] (4.5.2.15.2)

((s>(#r&p))>(#q&s))>((#q&s)>(p&(q\r))) ; TTTT TTTT TTCT TTCC(4.5.2.15.2.1)

Γ ⊢ Y Γ ⊢ ∀v . X
Γ ⊢ X[Y/v] (4.5.2.15.3)

((s>q)&(s>(#r&p)))>(s>(p&(q\r))) ; TTTT TTTT TTTT TTTC (4.5.2.15.3.1)

At 4.5.2.16 for existential quantifier:

∃Y . Γ ⊢ X[Y/v]
Γ ⊢ ∃v . X (4.5.2.16.1)

((%q&s)>(%q&(p&(q\r))))>(s>(%r&p)) ; TTTT TTTT CCTC CTTT (4.5.2.16.1.1)

Γ ⊢ ∃v . X
∃Y . Γ ⊢ X[Y/v] (4.5.2.16.2)

(s>(%r&p))>((%q&s)>(%q&(p&(q\r)))) ; TTTT TTTT TTTT TTTC (4.5.2.16.2.1)

Γ ⊢ Y Γ ⊢ X[Y\v]
Γ ⊢ ∃v . X (4.5.2.16.3)

((s>q)&(s>(p&(q\r))))>(s>(%r&p)) ; TTTT TTTT TTTC TTTT (4.5.2.16.3.1)

At 4.5.2.21 for universal and existential quantifiers:

Γ ⊢ ¬∀v . X
Γ ⊢ ∃v . ¬X (4.5.2.21.3)

(s>(~#r&p))>(s>(%r&~p)) ; TTTT TTTT TFTF TNTN (4.5.2.21.3.1)

$\Gamma \vdash \neg\exists v . X$
$\Gamma \vdash \forall v . \neg X$ (4.5.2.21.4)

(s>(~%r&p))>(s>(#r&~p)) ; TTTT TTTT TCTC TTTT (4.5.2.21.4.1)

Meth8 does not replicate those quantified expressions in Sections 4.5.2.15, 4.5.2.16, or 4.5.2.21. Some of the truth tables come close to tautology by pattern.

At 8.2.4 for completeness and incompleteness theorems:

Completeness of logic system:

$\forall p . (\exists\Gamma . \Gamma \vdash p \vee \exists\Gamma . \Gamma \vdash \neg p)$ (8.2.3.1)

(#p&(%s&(s>p)))+(#p&(%s&(s>~p))) ; FFFF FFFF FNFN FNFN (8.2.3.1.1)

Incompleteness of logic system:

$\exists p . (\neg\exists\Gamma . \Gamma \vdash p \wedge \neg\exists\Gamma . \Gamma \vdash \neg p)$ (8.2.3.2)

(#p&(~%s&(s>p)))&(#p&(~%s&(s>~p))) ; FNFN FNFN FFFF FFFF (8.2.3.2.1)

Completeness of formula set:

$\forall p . (\Gamma \vdash p \vee \Gamma \vdash \neg p)$ (8.2.4.1)

(#p&(s>p))+(#p&(s>~p)) ; FNFN FNFN FNFN FNFN (8.2.4.1.1)

Incompleteness of formula set:

$\exists p . (\Gamma /\!\vdash p \wedge \Gamma /\!\vdash \neg p)$ (8.2.4.2

(%p&(s<p))&(%p&(s<~p)) ; FFFF FFFF FFFF FFFF (8.2.4.2.1)

Meth8 does not replicate those quantified theorems in Sections 8.2.3 or 8.2.4. Eq. 8.2.4.2.1 is validated as contradictory. In sum, using Meth8-VŁ4 we cannot find tautology in these examples. We conclude that the use of quantified operators by Gödel was mistaken as inconsistent, or not bivalent, or both.

CONCLUSIONS: TIME TO REWRITE HISTORICAL QUESTIONS

Our analysis demonstrates that Gödel first incompleteness theorem and second completeness theorems are not tautologous. The Gödel incompleteness theorem states that sequences of logic symbols can be assigned to strings of natural numbers. However, because it is assumed there are more natural number than sequences of logic symbols, the latter are incomplete as a self-referencing mechanism to describe repeatedly themselves as yet more numbers. In effect, the Gödel completeness theorem states that the sequences of logic symbols may be consistent to form a logic system that is sufficiently complete enough to prove theorems as tautology. The arguments ultimately turn on the mapping of sequences of symbols into strings of natural numbers. The arguments also assume a function to map numbers as a domain into symbols and as an image in a codomain, collectively named a range. The incompleteness theorem states that a function exists and operates where all of its domain is larger than the smaller image existing within all of the co-domain. In turn, the completeness theorem states that all of the image, existing solely for its purposes, is self-sufficient unto the existence of itself. We were interested in mapping the image in the co-domain as sequences of logic symbols back into the originating domain, which is named a preimage. Attempting such an inverse function is not allowed by the one-way definition of a function to an image, as dictated by the

incompleteness theorem. We showed that this reverse approach is not tautologous, by evaluating the misuse of the application of quantified operators in the one-way functional mapping.
Our conclusion is that both Gödel's completeness theorem of a logic system and incompleteness theorem of a formula set are both not workable and not tautologous. Indeed, Gödel's use of the quantifiers is not bi-valent, rather take place on multidimensional manifolds: although this is in keeping with his Square of Opposition, nevertheless it is not compliant with the one corrected by Meth8. Our results stand for the definitive evidence that intuitionistic logic and, subsequently, constructivistic logics are not complete, because they deny the completeness theorem. The most important evidence is that those logics can exist only by ignoring the law of excluded middle (LEM) that "p or not p is a tautology", (p+~p)=(p=p). We argue that any system denying the LEM is not tautologous, and hence unworkable. This leads us to ask the historical question: is David Hilbert vindicated now, 85-years later? Indeed, it seems that incompleteness is something that must not be on the menu of scientific accounts.

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