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A View on Intuitionistic Smarandache Topological Semigroup Structure Spaces

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Abstract

The purpose of this paper is to introduce the concepts of intuitionistic Smarandache topological semigroups, intuitionistic Smarandache topological semigroup structure spaces, intuitionistic $\mathcal{S}_G$ exteriors and intuitionistic $\mathcal{S}_G$ semi exteriors. Characterizations and properties of intuitionistic $\mathcal{S}_G$-exterior space is established.

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Keywords: Intuitionistic Smarandache topological semigroups, Intuitionistic Smarandache topological semigroup structure spaces, Intuitionistic $\mathcal{S}_G$ exteriors, Intuitionistic $\mathcal{S}_G$ semi exteriors and intuitionistic $\mathcal{S}_G$-exterior space.

1. Introduction

Padilla Raul introduced the notion of Smarandache Semigroups in the year 1998 in the paper Smarandache Algebraic Structures. Smarandache semigroups are the analogue in the Smarandache ideologies of the groups. The concept of topological semigroups was introduced by A.D. Wallace [6]. K. Abodayeh and G.J. Murphy discussed the concept of compact topological semigroups. It has many applications in error correcting codes, coding theory, cosmology, physics and in the construction of S-sub-biautomaton.

In this chapter, some new concepts like intuitionistic Smarandache topological semigroups, intuitionistic Smarandache topological semigroup structure spaces and intuitionistic $\mathcal{S}_G$ exterior spaces are introduced and studied. Some interesting properties are established.

2. Preliminaries

Definition 2.1. Let X be a nonempty fixed set.An *intuitionistic set*(IS for short) A is an object having the form $A = \langle x, A^1, A^2\rangle$ for all $x \in X$, where A^1 and A^2 are subsets of X satisfying $A^1 \cap A^2 = \phi$. The set A^1 is called the set of members of A, while A^2 is called the set of nonmembers of A.

Every crisp set A on a non-empty set X is obviously an intuitionistic set having the form $\langle x, A, A^c\rangle$, and one can define several relations and operations between intuitionistic sets as follows:

Definition 2.2. Let X be a nonempty set, $A = \langle x, A^1, A^2\rangle$ for all $x \in X$, $B = \langle x, B^1, B^2\rangle$ for all $x \in X$ be intuitionistic sets on X, and let $\{A_i : i \in J\}$ be an arbitrary family of intuitionistic sets in X, where $A_i = \langle x, A_i^1, A_i^2\rangle$ for all $x \in X$.

(i) $A \subseteq B$ if and only if $A^1 \subseteq B^1$ and $B^2 \subseteq A^2$

(ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

(iii) $\overline{A} = \langle x, A^2, A^1\rangle$

(iv) $\cup A_i = \langle x, \cup A_i^1, \cap A_i^2\rangle$

(v) $\cap A_i = \langle x, \cap A_i^1, \cup A_i^2\rangle$

(vi) $A - B = A \cap \overline{B}$

(vii) $\phi_\sim = \langle x, \phi, X\rangle$ and $X_\sim = \langle x, X, \phi\rangle$.

Definition 2.3. Let X and Y be two nonempty sets and $f : X \to Y$ a function

(i) If $B = \langle x, B^1, B^2\rangle$ for all $x \in X$ is an intuitionistic set in Y,then the *preimage* of B under f,denoted by $f^{-1}(B)$,is an intuitionistic set in X defined by $f^{-1}(B) = \langle x, f^{-1}(B^1), f^{-1}(B^2)\rangle$.

(ii) If $A = \langle x, A^1, A^2\rangle$ for all $x \in X$ is an intuitionistic set in X,then the *image* of A under f,denoted by $f(A)$,is the intuitionistic set in Y defined by $f(A) = \langle y, f(A^1), f_-(A^2)\rangle$ where $f_-(A^2) = (f(A^2)^c)^c$.

Definition 2.4. An *intuitionistic topology* (IT for short) on a nonempty set X is a family τ of intuitionistic sets in X satisfying the following axioms:

(i) $\phi_\sim$ and $X_\sim \in \tau$,

(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$;

(iii) $\cup G_i \in \tau$ for any arbitrary family $\{G_i \mid i \in J\} \subseteq \tau$.

In this case the ordered pair (X, τ) is called an *intuitionistic topological space* (ITS for short) and any intuitionistic set in τ is known as an *intuitionistic open set* (IOS for short) in X. The complement $\overline{A}$ of an intuitionistic open set A is called an *intuitionistic closed set* (ICS for short) in X.

Definition 2.5. Let (X, τ) be an intuitionistic topological space and $A = \langle x, A^1, A^2 \rangle$ be an intuitionistic set in X.Then the intuitionistic interior and intuitionistic closure of A are defined by

$cl(A) = \cap\{K : K$ is an intuitionistic closed set in X and $A \subseteq K\}$,

$int(A) = \cup\{G : G$ is an intuitionistic open set in X and $G \subseteq A\}$.

It can be also shown that $cl(A)$ is an intuitionistic closed set and $int(A)$ is an intuitionistic open set in X and A is an intuitionistic closed set in X if $cl(A) = A$ and A is an intuitionistic open set in X if $int(A) = A$.

Definition 2.6. A non-empty set of elements G is said to form a group if in G there is defined a binary operation, called the product and denoted by "." such that

(i) $a, b \in G$ implies that $a.b \in G$.

(ii) $a, b, c \in G$ implies that $(a.b).c = a.(b.c)$.

(iii) There exists an element $e \in G$ such that $a.e = e.a = a$ for all $a \in G$ (the existence of an identity element in G).

(iv) For every $a \in G$ there exists an element $a^{-1} \in G$ such that $a.a^{-1} = a^{-1}.a = e$ (the existence of an inverse in G).

Definition 2.7. A non-empty set S is said to be a semigroup if there is defined a binary operation, denoted by "." such that

(i) $a, b \in S$ implies that $a.b \in S$.

(ii) $a, b, c \in S$ implies that $(a.b).c = a.(b.c)$.

Definition 2.8. The Smarandache semigroup (S-semigroup) is defined to be a semigroup A such that a proper subset of A is a group (with respect to the same induced operation).

Definition 2.9. Let (A, T_A) and (B, T_B) be any two subspaces of topological spaces (X, T) and (Y, S) respectively. A function $f : (A, T_A) \to (B, T_B)$ is said to be a relatively continuous function if and only if for each open set V' in T_B, the intersection $f^{-1}(V') \cap A$ is open in T_A.

Definition 2.10. Let X be a non-empty set and T be a topology on X. Let S be any semigroup in X and let S be endowed with the subspace topology T_S. Then S is a topological semigroup in X if and only if the mapping $\alpha : (x, y) \to xy$ of $(S, T_S) \times (S, T_S)$ into (S, T_S) is relatively continuous.

3. Intuitionistic Smarandache Topological Semigroup Structure Spaces and Intuitionistic $\mathcal{S}_G$ Exteriors

In this section, the concepts of intuitionistic Smarandache topological semigroups, intuitionistic Smarandache topological semigroup structure spaces, intuitionistic $\mathcal{S}_G$ exteriors and intuitionistic $\mathcal{S}_G$ semi exteriors are introduced and studied.

Notation 3.1. Let (X, T) be an intuitionistic topological space and let $A = \langle x, A^1, A^2\rangle$ be any intuitionistic set in X. Then $a \in A$ denotes $a \in A^1$ and $a \notin A^2$.

Definition 3.2. Let G be a group. An intuitionistic set $\mathcal{G} = \langle x, \mathcal{G}^1, \mathcal{G}^2\rangle$ in G is said to be an intuitionistic group if there is defined a binary operation, denoted by "." such that

(i) $a, b \in \mathcal{G}$ implies that $a.b \in \mathcal{G}$.

(ii) $a, b, c \in \mathcal{G}$ implies that $(a.b).c = a.(b.c)$.

(iii) There exists $e \in \mathcal{G}$ such that $a.e = e.a = a$ for all $a \in \mathcal{G}$.

(iv) For every $a \in \mathcal{G}$ there exists $a^{-1} \in \mathcal{G}$ such that $a.a^{-1} = a^{-1}.a = e$.

Definition 3.3. Let S be a semigroup. An intuitionistic set $\mathcal{S} = \langle x, \mathcal{S}^1, \mathcal{S}^2\rangle$ in S is said to be an intuitionistic semigroup if there is defined a binary operation, denoted by "." such that

(i) $a, b \in \mathcal{S}$ implies that $a.b \in \mathcal{S}$.

(ii) $a, b, c \in \mathcal{S}$ implies that $(a.b).c = a.(b.c)$.

Definition 3.4. An intuitionistic semigroup $\mathcal{S} = \langle x, \mathcal{S}^1, \mathcal{S}^2\rangle$ is called an intuitionistic Smarandache semigroup if there is an intuitionistic set $A \subseteq \mathcal{S}$ which is an intuitionistic group of $\mathcal{S}$ where $A = \langle x, A^1, A^2\rangle$.

Example 3.5. Let $S = \{0, 1, 2\}$ be a set of integers modulo 3 with the binary operation as follows:

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(S, .)$ is a semigroup. Define an intuitionistic semigroup A of S by $A = \langle x, \{0, 1\}, \{2\}\rangle$ and let $B = \langle x, \{0\}, \{1, 2\}\rangle$ be an intuitionistic set. Clearly B is an intuitionistic group of A. Therefore, A is an intuitionistic Smarandache semigroup.

Notation 3.6. Let (X, T) and (Y, S) be any two intuitionistic topological spaces and let $A = \langle x, A^1, A^2\rangle$ be an intuitionistic set in X and $B = \langle y, B^1, B^2\rangle$ be an intuitionistic set in Y, then the intuitionistic set $A \times B$ is denoted by $A \times B = \langle (x, y), A^1 \times B^1, A^2 \times B^2 \rangle$, where $(A^1 \times B^1) \cap (A^2 \times B^2) = \phi$.

Example 3.7. Let $X = \{a, b, c\}$ and $Y = \{a, c\}$ be any two intuitionistic topological spaces and let $A = \langle x, \{a, b\}, \{c\} \rangle$ be an intuitionistic set in X and $B = \langle y, \{a\}, \{c\} \rangle$ be an intuitionistic set in Y.

Now, $A \times B = \langle (x, y), \{(a, a), (b, a)\}, \{(c, c)\} \rangle$ is an intuitionistic set with $(A^1 \times B^1) \cap (A^2 \times B^2) = \phi$.

Definition 3.8. Let (X, T) and (Y, S) be any two intuitionistic topological spaces. The intuitionistic product topology on $X \times Y$ is an intuitionistic topology having as intuitionistic basis the collection $\mathcal{B}$ of all intuitionistic sets of the form $U \times V$, where $U = \langle x, U^1, U^2 \rangle$ is an intuitionistic open set of X and $V = \langle y, V^1, V^2 \rangle$ is an intuitionistic open set of Y.

Definition 3.9. Let (X, T) be an intuitionistic topological space and $\mathcal{G} = \langle x, \mathcal{G}^1, \mathcal{G}^2 \rangle$ be an intuitionistic set of X.

Then $T_{\mathcal{G}} = \{A \cap \mathcal{G} : A = \langle x, A^1, A^2 \rangle \in T\}$ is an intuitionistic topology on $\mathcal{G}$ and is called the induced intuitionistic topology. The pair $(\mathcal{G}, T_{\mathcal{G}})$ is called an intuitionistic subspace of (X, T).

Definition 3.10. Let (X, T) and (Y, S) be any two intuitionistic topological spaces and $A = \langle x, A^1, A^2 \rangle$ and $B = \langle x, B^1, B^2 \rangle$ be any two intuitionistic sets in X and Y respectively. Let (A, T_A) and (B, T_B) be any two intuitionistic subspaces of intuitionistic topological spaces (X, T) and (Y, S) respectively. A function $f : (A, T_A) \to (B, T_B)$ is said to be a relatively intuitionistic continuous function if and only if for each intuitionistic open set V' in T_B, the intersection $f^{-1}(V') \cap A$ is intuitionistic open in T_A.

Definition 3.11. Let S be a semigroup and T be an intuitionistic topology on S. Let $\mathcal{S} = \langle x, \mathcal{S}^1, \mathcal{S}^2 \rangle$ be any intuitionistic Smarandache semigroup in S and let $\mathcal{S}$ be endowed with the intuitionistic subspace topology T_S. Then $\mathcal{S}$ is an intuitionistic Smarandache topological semigroup in S if and only if the mapping $\alpha : (x, y) \to xy$ of $(\mathcal{S}, T_S) \times (\mathcal{S}, T_S)$ into $(\mathcal{S}, T_S)$ is relatively intuitionistic continuous.

Example 3.12. Let $S = \{0, 1, 2\}$ be a set of integers modulo 3 with the binary operation as follows:

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(S, .)$ is a semigroup. Let $T = \{S_\sim, \phi_\sim, A, B\}$ be an intuitionistic topology on S, where the intuitionistic semigroups A and B of S are defined by $A = \langle x, \{0, 1\}, \{2\} \rangle$, $B = \langle x, \{0\}, \{1, 2\} \rangle$ and let $\mathcal{S} = \langle x, \{2, 1\}, \{0\} \rangle$ be an intuitionistic

Smarandache semigroup. Then $T_S = \{\phi_\sim, \mathcal{S}, \mathcal{R}\}$ is an intuitionistic topology on $\mathcal{S}$, where, $\mathcal{R} = \langle x, \{1\}, \{0, 2\}\rangle$. Let $\alpha : (\mathcal{S}, T_S) \times (\mathcal{S}, T_S) \to (\mathcal{S}, T_S)$ be defined by $\alpha(x, y) = xy$. Clearly α is relatively intuitionistic continuous. Hence $\mathcal{S}$ is an intuitionistic Smarandache topological semigroup.

Definition 3.13. Let S be a semigroup and T be an intuitionistic topology on S. A family $\mathcal{S}_G$ of intuitionistic Smarandache topological semigroups in S is said to be intuitionistic Smarandache topological semigroup structure on S if it satisfies the following axioms:

(i) $\phi_\sim, S_\sim \in \mathcal{S}_G$,

(ii) If $A, B \in \mathcal{S}_G$, then $A \cap B \in \mathcal{S}_G$, where $A = \langle x, A^1, A^2\rangle$ and $B = \langle x, B^1, B^2\rangle$,

(iii) If $A_j \in \mathcal{S}_G$ for all $j \in J$, then $\cup_{j \in J} A_j \in \mathcal{S}_G$.

Then the ordered pair $(S, \mathcal{S}_G)$ is called an intuitionistic Smarandache topological semigroup structure space (in short, intuitionistic $\mathcal{S}_G$ structure space). Every member of intuitionistic $\mathcal{S}_G$ structure space is called an intuitionistic open Smarandache topological semigroup. The complement of intuitionistic open Smarandache topological semigroup is an intuitionistic closed Smarandache topological semigroup.

Example 3.14. Let $S = \{0, 1, 2\}$ be a set of integers modulo 3 with the binary operation as follows:

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(S, .)$ is a semigroup. Let $T = \{S_\sim, \phi_\sim, A, B\}$ be an intuitionistic topology on S where, $A = \langle x, \{1, 2\}, \{0\}\rangle$, $B = \langle x, \{2\}, \{0, 1\}\rangle$ and define $\mathcal{S}_G = \{S_\sim, \phi_\sim, A, B, C, D, E\}$, the family of intuitionistic Smarandache topological semigroups in S where, $C = \langle x, \{0, 1\}, \{2\}\rangle$, $D = \langle x, \{0\}, \{1, 2\}\rangle$, and $E = \langle x, \{0, 2\}, \{1\}\rangle$. Clearly, $\mathcal{S}_G$ is an intuitionistic Smarandache topological semigroup structure. Hence, the ordered pair $(S, \mathcal{S}_G)$ is an intuitionistic Smarandache topological semigroup structure space.

Definition 3.15. Let $(S, \mathcal{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Let $A = \langle x, A^1, A^2\rangle$ be any intuitionistic Smarandache topological semigroup. Then the intuitionistic $\mathcal{S}_G$ interior of A is defined by $I\mathcal{S}_G int(A) = \cup\{B = \langle x, B^1, B^2\rangle : B$ is an intuitionistic open Smarandache topological semigroup and $B \subseteq A\}$.

Definition 3.16. Let $(S, \mathcal{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Let $A = \langle x, A^1, A^2\rangle$ be any intuitionistic Smarandache topological semigroup. Then the intuitionistic $\mathcal{S}_G$ closure of A is defined by $I\mathcal{S}_G cl(A) = \cap\{B = \langle x, B^1, B^2\rangle : B$ is an intuitionistic closed Smarandache topological semigroup and $A \subseteq B\}$.

Proposition 3.17. Let $(S, \mathcal{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Let $A = \langle x, A^1, A^2 \rangle$ be any intuitionistic Smarandache topological semigroup. Then the following conditions hold:

(i) $I\mathcal{S}_G int(A) \subseteq A \subseteq I\mathcal{S}_G cl(A)$.

(ii) $\overline{(I\mathcal{S}_G int(A))} = I\mathcal{S}_G cl(\overline{A})$.

(iii) $\overline{(I\mathcal{S}_G cl(A))} = I\mathcal{S}_G int(\overline{A})$.

(iv) $I\mathcal{S}_G int(A \cap B) = I\mathcal{S}_G int(A) \cap I\mathcal{S}_G int(A)$.

Proof. The proof follows from Definition 3.9. and Definition 3.10. ■

Definition 3.18. Let $(S, \mathcal{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. An intuitionistic Smarandache topological semigroup $A = \langle x, A^1, A^2 \rangle$ of S is said to be an intuitionistic semi-open Smarandache topological semigroup if $A \subseteq I\mathcal{S}_G cl(I\mathcal{S}_G int(A))$. The complement of an intuitionistic semi-open Smarandache topological semigroup is called an intuitionistic semi-closed Smarandache topological semigroup.

Definition 3.19. Let $(S, \mathcal{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Let $A = \langle x, A^1, A^2 \rangle$ be any intuitionistic Smarandache topological semigroup. Then the intuitionistic $\mathcal{S}_G$ semi-interior of A is defined by
$I\mathcal{S}_G Sint(A) = \cup\{B = \langle x, B^1, B^2 \rangle : B$ is an intuitionistic semi-open Smarandache topological semigroup and $B \subseteq A\}$.

Definition 3.20. Let $(S, \mathcal{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Let $A = \langle x, A^1, A^2 \rangle$ be any intuitionistic Smarandache topological semigroup. Then the intuitionistic $\mathcal{S}_G$ semi-closure of A is defined by
$I\mathcal{S}_G Scl(A) = \cap\{B = \langle x, B^1, B^2 \rangle : B$ is an intuitionistic semi-closed Smarandache topological semigroup and $A \subseteq B\}$.

Proposition 3.21. Let $(S, \mathcal{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Let $A = \langle x, A^1, A^2 \rangle$ be any intuitionistic Smarandache topological semigroup. Then the following conditions hold:

(i) $I\mathcal{S}_G Sint(A) \subseteq A \subseteq I\mathcal{S}_G Scl(A)$.

(ii) $\overline{(I\mathcal{S}_G Sint(A))} = I\mathcal{S}_G Scl(\overline{A})$.

(iii) $\overline{(I\mathcal{S}_G Scl(A))} = I\mathcal{S}_G Sint(\overline{A})$.

Proof. The proof follows from Definition 3.12. and Definition 3.13. ■

Definition 3.22. Let $(S, \mathcal{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Let $A = \langle x, A^1, A^2 \rangle$ be any intuitionistic Smarandache topological semigroup in S. Then the intuitionistic $\mathcal{S}_G$ semi exterior of A, is denoted and defined as $I\mathcal{S}_G SExt(A) = I\mathcal{S}_G Sint(\overline{A})$.

Definition 3.23. Let $(S, \mathcal{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Let $A = \langle x, A^1, A^2 \rangle$ be any intuitionistic Smarandache topological semigroup in S. Then the intuitionistic $\mathcal{S}_G$ exterior of A, is denoted and defined as $I\mathcal{S}_G Ext(A) = I\mathcal{S}_G int(\overline{A})$.

Proposition 3.24. Let $(S, \mathcal{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Let $A = \langle x, A^1, A^2 \rangle$ be any intuitionistic Smarandache topological semigroup in S. Then the following statements hold:

(i) $I\mathcal{S}_G Ext(A) = \overline{I\mathcal{S}_G cl(A)}$,

(ii) $I\mathcal{S}_G Ext(\overline{A}) = I\mathcal{S}_G int(A)$,

(iii) $I\mathcal{S}_G Ext(I\mathcal{S}_G Ext(A)) = I\mathcal{S}_G int(I\mathcal{S}_G cl(A))$

(iv) $I\mathcal{S}_G Ext(X_\sim) = \phi_\sim$

(v) $I\mathcal{S}_G Ext(\phi_\sim) = X_\sim$.

Proof. The proof is obvious. ■

Proposition 3.25. Let $(S, \mathcal{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Let $A = \langle x, A^1, A^2 \rangle$ and $B = \langle x, B^1, B^2 \rangle$ be any two intuitionistic Smarandache topological semigroups. Then, $I\mathcal{S}_G Ext(A \cup B) = I\mathcal{S}_G Ext(A) \cap I\mathcal{S}_G Ext(B)$.

Proof.
$$\begin{aligned} I\mathcal{S}_G Ext(A \cup B) &= I\mathcal{S}_G int\overline{(A \cup B)} \\ &= I\mathcal{S}_G int(\overline{A} \cap \overline{B}) \\ &= I\mathcal{S}_G int(\overline{A}) \cap I\mathcal{S}_G int(\overline{B}) \\ &= I\mathcal{S}_G Ext(A) \cap I\mathcal{S}_G Ext(B). \end{aligned}$$
Therefore, $I\mathcal{S}_G Ext(A \cup B) = I\mathcal{S}_G Ext(A) \cap I\mathcal{S}_G Ext(B)$. ■

Proposition 3.26. Let $(S, \mathcal{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Let $A = \langle x, A^1, A^2 \rangle$ and $B = \langle x, B^1, B^2 \rangle$ be any two intuitionistic Smarandache topological semigroups. Then, $I\mathcal{S}_G Ext(A \cap B) \subseteq I\mathcal{S}_G Ext(A) \cup I\mathcal{S}_G Ext(B)$.

Proof.
$$\begin{aligned} I\mathcal{S}_G Ext(A \cap B) &= I\mathcal{S}_G int\overline{(A \cap B)} \\ &= I\mathcal{S}_G int(\overline{A} \cup \overline{B}) \\ &\subseteq I\mathcal{S}_G int(\overline{A}) \cup I\mathcal{S}_G int(\overline{B}) \\ &= I\mathcal{S}_G Ext(A) \cup I\mathcal{S}_G Ext(B). \end{aligned}$$
Therefore, $I\mathcal{S}_G Ext(A \cap B) \subseteq I\mathcal{S}_G Ext(A) \cup I\mathcal{S}_G Ext(B)$. ■

Proposition 3.27. Let $(S, \mathbb{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Let $A = \langle x, A^1, A^2\rangle$ be any intuitionistic closed Smarandache topological semigroup in S. Then the following statements hold:

(i) $I\mathbb{S}_G Ext(\overline{I\mathbb{S}_G Ext(A)}) = I\mathbb{S}_G Ext(A)$.

(ii) $I\mathbb{S}_G Ext(A) = \overline{A}$.

Proof. **(i)**
$$\begin{aligned} I\mathbb{S}_G Ext(\overline{I\mathbb{S}_G Ext(A)}) &= I\mathbb{S}_G Ext(\overline{I\mathbb{S}_G int(\overline{A})}) \\ &= I\mathbb{S}_G int(I\mathbb{S}_G int(\overline{A})) \\ &= I\mathbb{S}_G int(\overline{A}) \\ &= I\mathbb{S}_G Ext(A) \end{aligned}$$

(ii) Since A is an intuitionistic closed Smarandache topological semigroup, $I\mathbb{S}_G cl(A) = A$. By Definition of exterior $I\mathbb{S}_G Ext(A) = \overline{I\mathbb{S}_G cl(A)} = \overline{A}$.
Therefore, $I\mathbb{S}_G Ext(A) = \overline{A}$. ■

Proposition 3.28. Let $(S, \mathbb{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Let $A = \langle x, A^1, A^2\rangle$ and $B = \langle x, B^1, B^2\rangle$ be any two intuitionistic Smarandache topological semigroups. If $A \subseteq B$, then $I\mathbb{S}_G Ext(A) \supseteq I\mathbb{S}_G Ext(B)$.

Proof. $A \subseteq B$ implies that
$$\begin{aligned} &I\mathbb{S}_G cl(A) \subseteq I\mathbb{S}_G cl(B) \\ &\overline{I\mathbb{S}_G cl(A)} \supseteq \overline{I\mathbb{S}_G cl(B)} \\ &I\mathbb{S}_G int(\overline{A}) \supseteq I\mathbb{S}_G int(\overline{B}) \\ &I\mathbb{S}_G Ext(A) \supseteq I\mathbb{S}_G Ext(B). \end{aligned}$$
■

Definition 3.29. Let $(S_1, \mathbb{S}_{G_1})$ and $(S_2, \mathbb{S}_{G_2})$ be any two intuitionistic Smarandache topological semigroup structure spaces. Let $f : (S_1, \mathbb{S}_{G_1}) \to (S_2, \mathbb{S}_{G_2})$ be a function. Then f is said to be an intuitionistic $\mathbb{S}_G$-continuous function if $f^{-1}(A)$ is an intuitionistic open Smarandache topological semigroup in $(S_1, \mathbb{S}_{G_1})$, for every intuitionistic open Smarandache topological semigroup $A = \langle x, A^1, A^2\rangle$ in $(S_2, \mathbb{S}_{G_2})$.

Proposition 3.30. Let $(S_1, \mathbb{S}_{G_1})$ and $(S_2, \mathbb{S}_{G_2})$ be any two intuitionistic Smarandache topological semigroup structure spaces. Let $f : (S_1, \mathbb{S}_{G_1}) \to (S_2, \mathbb{S}_{G_2})$ be a function. Then the following statements hold:

(i) f is an intuitionistic $\mathbb{S}_G$-continuous function.

(ii) $f^{-1}(B)$ is an intuitionistic closed Smarandache topological semigroup in $(S_1, \mathbb{S}_{G_1})$, for every intuitionistic closed Smarandache topological semigroup $B = \langle x, B^1, B^2\rangle$ in $(S_2, \mathbb{S}_{G_2})$.

(iii) $I\mathbb{S}_G cl(f^{-1}(A)) \subseteq f^{-1}(I\mathbb{S}_G cl(A))$, for each intuitionistic Smarandache topological semigroup $A = \langle x, A^1, A^2\rangle$ in $(S_2, \mathbb{S}_{G_2})$.

(iv) $f^{-1}(I\mathbb{S}_G int(A)) \subseteq I\mathbb{S}_G int(f^{-1}(A))$, for each intuitionistic Smarandache topological semigroup A in $(S_2, \mathbb{S}_{G_2})$.

(v) $f(I\mathcal{S}_G int(A)) \subseteq I\mathcal{S}_G int(f(A))$, for each intuitionistic Smarandache topological semigroup A in $(S_2, \mathcal{S}_{G_2})$.

Proof. The proof is simple. ■

Definition 3.31. Let $(S, \mathcal{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Let $A = \langle x, A^1, A^2 \rangle$ be any intuitionistic Smarandache topological semigroup in S. Then A is said to be an intuitionistic dense Smarandache topological semigroup in $(S, \mathcal{S}_G)$ if there exists no intuitionistic closed Smarandache topological semigroup $B = \langle x, B^1, B^2 \rangle$ in $(S, \mathcal{S}_G)$ such that $A \subset B \subset S_\sim$.

Proposition 3.32. If a function $f : (S_1, \mathcal{S}_{G_1}) \to (S_2, \mathcal{S}_{G_2})$ from an intuitionistic Smarandache topological semigroup structure space $(S_1, \mathcal{S}_{G_1})$ into another intuitionistic Smarandache topological semigroup structure space $(S_2, \mathcal{S}_{G_2})$ is an intuitionistic $\mathcal{S}_G$-continuous and bijective function. If $A = \langle x, A^1, A^2 \rangle$ is an intuitionistic dense Smarandache topological semigroup in $(S_1, \mathcal{S}_{G_1})$, then $f(A)$ is an intuitionistic dense Smarandache topological semigroup in $(S_2, \mathcal{S}_{G_2})$.

Proof. Suppose that $f(A)$ is not an intuitionistic dense Smarandache topological semigroup in $(S_2, \mathcal{S}_{G_2})$. Then there exists an intuitionistic closed Smarandache topological semigroup $C = \langle x, C^1, C^2 \rangle$ in $(S_2, \mathcal{S}_{G_2})$ such that $f(A) \subset C \subset S_{2\sim}$. Then, $f^{-1}(f(A)) \subset f^{-1}(C) \subset f^{-1}(S_{2\sim})$. Since f is bijective, $f^{-1}(f(A)) = A$. Hence, $A \subset f^{-1}(C) \subset S_{1\sim}$.

Since f is an intuitionistic $\mathcal{S}_G$-continuous function and C is an intuitionistic closed Smarandache topological semigroup in $(S_2, \mathcal{S}_{G_2})$, $f^{-1}(C)$ is an intuitionistic closed Smarandache topological semigroup in $(S_1, \mathcal{S}_{G_1})$. Then, $I\mathcal{S}_G cl(A) \neq S_{1\sim}$, which is a contradiction. Therefore, $f(A)$ is an intuitionistic dense Smarandache topological semigroup in $(S_2, \mathcal{S}_{G_2})$. ■

Definition 3.33. Let $(S_1, \mathcal{S}_{G_1})$ and $(S_2, \mathcal{S}_{G_2})$ be any two intuitionistic Smarandache topological semigroup structure spaces. A function $f : (S_1, \mathcal{S}_{G_1}) \to (S_2, \mathcal{S}_{G_2})$ is called an intuitionistic $\mathcal{S}_G$-irresolute function if $f^{-1}(A)$ is an intuitionistic open Smarandache topological semigroup in $(S_1, \mathcal{S}_{G_1})$, for each intuitionistic open Smarandache topological semigroup $A = \langle x, A^1, A^2 \rangle$ in $(S_2, \mathcal{S}_{G_2})$.

Proposition 3.34. Let $(S_1, \mathcal{S}_{G_1})$ and $(S_2, \mathcal{S}_{G_2})$ be any two intuitionistic Smarandache topological semigroup structure spaces. If
$f : (S_1, \mathcal{S}_{G_1}) \to (S_2, \mathcal{S}_{G_2})$ is an intuitionistic $\mathcal{S}_G$-irresolute and surjective function then $f(I\mathcal{S}_G Ext(A)) \subseteq I\mathcal{S}_G Ext(f(A))$, for each intuitionistic Smarandache topological semigroup $A = \langle x, A^1, A^2 \rangle$ in $(S_1, \mathcal{S}_{G_1})$.

Proof. Suppose that f is an intuitionistic $\mathcal{S}_G$-irresolute function and let A be an intuitionistic Smarandache topological semigroup in $(S_1, \mathcal{S}_{G_1})$.

Then, $I\mathcal{S}_G Ext(f(A))$ is an intuitionistic open Smarandache topological semigroup in $(S_2, \mathcal{S}_{G_2})$. By assumption, $f^{-1}(I\mathcal{S}_G Ext(f(A)))$ is an intuitionistic open Smarandache topological semigroup in $(S_1, \mathcal{S}_{G_1})$. Therefore,

$$I\mathcal{S}_G Ext(\overline{f^{-1}(I\mathcal{S}_G Ext(f(A)))}) = f^{-1}(I\mathcal{S}_G Ext(f(A))).$$

Now, $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{I\mathcal{S}_G Ext(f(A))})$.

$$\begin{aligned} I\mathcal{S}_G Ext(A) &\subseteq I\mathcal{S}_G Ext(f^{-1}(\overline{I\mathcal{S}_G Ext(f(A))})) \\ &= f^{-1}(I\mathcal{S}_G Ext(f(A))). \end{aligned}$$

Hence $f(I\mathcal{S}_G Ext(A)) \subseteq I\mathcal{S}_G Ext(f(A))$. ■

Definition 3.35. Let $(S_1, \mathcal{S}_{G_1})$ and $(S_2, \mathcal{S}_{G_2})$ be any two intuitionistic Smarandache topological semigroup structure spaces. A function $f : (S_1, \mathcal{S}_{G_1}) \to (S_2, \mathcal{S}_{G_2})$ is called an intuitionistic $\mathcal{S}_G$-closed function if $f(A)$ is an intuitionistic closed Smarandache topological semigroup in $(S_2, \mathcal{S}_{G_2})$, for each intuitionistic closed Smarandache topological semigroup $A = \langle x, A^1, A^2\rangle$ in $(S_1, \mathcal{S}_{G_1})$.

Proposition 3.36. Let $(S_1, \mathcal{S}_{G_1})$ and $(S_2, \mathcal{S}_{G_2})$ be any two intuitionistic Smarandache topological semigroup structure spaces. If
$f : (S_1, \mathcal{S}_{G_1}) \to (S_2, \mathcal{S}_{G_2})$ is an intuitionistic $\mathcal{S}_G$-closed and surjective function then $f^{-1}(I\mathcal{S}_G Ext(A)) \supseteq I\mathcal{S}_G Ext(f^{-1}(A))$ for each intuitionistic Smarandache topological semigroup A in $(S_2, \mathcal{S}_{G_2})$.

Proof. Let A be an intuitionistic Smarandache topological semigroup in $(S_2, \mathcal{S}_{G_2})$ and let $B = f^{-1}(A)$. Then, $I\mathcal{S}_G Ext(B) = I\mathcal{S}_G Ext(f^{-1}(A))$ is an intuitionistic open Smarandache topological semigroup in $(S_1, \mathcal{S}_{G_1})$. Now, $\overline{I\mathcal{S}_G Ext(B)} \supseteq B$. Hence, $f(\overline{I\mathcal{S}_G Ext(B)}) \supseteq f(B)$. That is,

$$I\mathcal{S}_G Ext(f(\overline{I\mathcal{S}_G Ext(B)})) \subseteq I\mathcal{S}_G Ext(f(B)). \tag{3.1}$$

Since f is surjective,

$$f(\overline{I\mathcal{S}_G Ext(B)}) = \overline{f(I\mathcal{S}_G Ext(B))} \tag{3.2}$$

Since f is an intuitionistic $\mathcal{S}_G$-closed function, $\overline{f(I\mathcal{S}_G Ext(B))}$ is an intuitionistic closed Smarandache topological semigroup in $(S_2, \mathcal{S}_{G_2})$ and

$$I\mathcal{S}_G Ext(\overline{f(I\mathcal{S}_G Ext(B))}) = f(I\mathcal{S}_G Ext(B)) \tag{3.3}$$

From (3.1), (3.2) and (3.3) it follows that,

$$f(I\mathcal{S}_G Ext(B)) \subseteq I\mathcal{S}_G Ext(f(B)) = I\mathcal{S}_G Ext(A).$$

Hence, $I\mathcal{S}_G Ext(B) \subseteq f^{-1}(I\mathcal{S}_G Ext(A))$. Therefore,

$$I\mathcal{S}_G Ext(f^{-1}(A)) \subseteq f^{-1}(I\mathcal{S}_G Ext(A)).$$

■

4. Characterization of Intuitionistic $\mathcal{S}_G$-Exterior Spaces

In this section, the concept of intuitionistic $\mathcal{S}_G$-exterior space is introduced and studied. Also the characterization of intuitionistic $\mathcal{S}_G$-exterior space is established.

Definition 4.1. Let $(S, \mathcal{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Then $(S, \mathcal{S}_G)$ is said to be an intuitionistic $\mathcal{S}_G$-exterior space if the intuitionistic $\mathcal{S}_G$-exterior of each intuitionistic closed Smarandache topological semigroup is an intuitionistic closed Smarandache topological semigroup.

Example 4.2. Let $S = \{0, 1, 2\}$ be a set of integers modulo 3 with the binary operation as follows:

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(S, .)$ is a semigroup. Let $T = \{S_\sim, \phi_\sim, A, B\}$ be an intuitionistic topology on S where, the intuitionistic semigroups A and B of S are defined by $A = \langle x, \{1, 2\}, \{0\}\rangle$ and $B = \langle x, \{2\}, \{0, 1\}\rangle$. Define $S_G = \{S_\sim, \phi_\sim, A, B, C, D, E\}$, the family of intuitionistic Smarandache topological semigroups in S, where $C = \langle x, \{0, 1\}, \{2\}\rangle$, $D = \langle x, \{0\}, \{1, 2\}\rangle$ and $E = \langle x, \{0, 2\}, \{1\}\rangle$. Clearly, the ordered pair $(S, \mathcal{S}_G)$ is an intuitionistic $\mathcal{S}_G$-exterior space.

Notation 4.3. $I\mathcal{S}_G CoExt(A)$ denotes $\overline{I\mathcal{S}_G int(A)}$.

Proposition 4.4. Let $(S, \mathcal{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. Then the following statements are equivalent:

(i) $(S, \mathcal{S}_G)$ is an intuitionistic $\mathcal{S}_G$-exterior space.

(ii) For each intuitionistic open Smarandache topological semigroup $A = \langle x, A^1, A^2\rangle$, $I\mathcal{S}_G CoExt(A)$ is an intuitionistic open Smarandache topological semigroup.

(iii) For each intuitionistic open Smarandache topological semigroup A, we have, $I\mathcal{S}_G Ext(\overline{I\mathcal{S}_G CoExt(A)}) = I\mathcal{S}_G CoExt(A)$.

(iv) For every pair of intuitionistic open Smarandache topological semigroups A and $B = \langle x, B^1, B^2\rangle$ with $I\mathcal{S}_G CoExt(A) = \overline{B}$, $I\mathcal{S}_G Ext(B) = I\mathcal{S}_G CoExt(A)$.

Proof. **(i)⇒(ii)**
Let A be an intuitionistic open Smarandache topological semigroup in $(S, \mathcal{S}_G)$. Then $\overline{A}$ is an intuitionistic closed Smarandache topological semigroup in $(S, \mathcal{S}_G)$. Then by assumption, $I\mathcal{S}_G Ext(\overline{A})$ is intuitionistic closed Smarandache topological semigroup in $(S, \mathcal{S}_G)$.

Now, $\overline{I\mathcal{S}_G Ext(\overline{A})}$ is an intuitionistic open Smarandache topological semigroup in $(S, \mathcal{S}_G)$. Hence, $I\mathcal{S}_G CoExt(A)$ is intuitionistic open Smarandache topological semigroup in $(S, \mathcal{S}_G)$. Hence (i)⇒(ii).

(ii)⇒(iii)
Let A be an intuitionistic open Smarandache topological semigroup in $(S, \mathcal{S}_G)$. By assumption, $I\mathcal{S}_G CoExt(A)$ is an intuitionistic open Smarandache topological semigroup in $(S, \mathcal{S}_G)$. Therefore,

$$I\mathcal{S}_G int(I\mathcal{S}_G CoExt(A)) = I\mathcal{S}_G CoExt(A).$$

Thus, $I\mathcal{S}_G Ext(\overline{I\mathcal{S}_G CoExt(A)}) = I\mathcal{S}_G CoExt(A)$. Hence (ii)⇒(iii).

(iii)⇒(iv)
Let A and B be any two intuitionistic open Smarandache topological semigroups in $(S, \mathcal{S}_G)$ such that $I\mathcal{S}_G CoExt(A) = \overline{B}$. By (iii),

$$I\mathcal{S}_G Ext(\overline{I\mathcal{S}_G CoExt(A)}) = I\mathcal{S}_G CoExt(A).$$

This implies that $I\mathcal{S}_G Ext(B) = I\mathcal{S}_G CoExt(A)$. Hence (iii)⇒(iv).

(iv)⇒(i)
Let A be an intuitionistic open Smarandache topological semigroup in $(S, \mathcal{S}_G)$ and define B such that $I\mathcal{S}_G CoExt(A) = \overline{B}$. By (iv), it follows that $I\mathcal{S}_G Ext(B) = I\mathcal{S}_G CoExt(A)$. That is, $I\mathcal{S}_G CoExt(A)$ is an intuitionistic open Smarandache topological semigroup in $(S, \mathcal{S}_G)$.

This implies that, $I\mathcal{S}_G Ext(\overline{A})$ is an intuitionistic closed Smarandache topological semigroup in $(S, \mathcal{S}_G)$. Thus, $(S, \mathcal{S}_G)$ is an intuitionistic $\mathcal{S}_G$-exterior space. Hence, (iv)⇒(i). Hence the proof. ■

Proposition 4.5. Let $(S, \mathcal{S}_G)$ be an intuitionistic $\mathcal{S}_G$-exterior space. Let R be an intuitionistic open Smarandache topological semigroup subspace in S. Then $(R, \mathcal{S}_{G_R})$ is an intuitionistic $\mathcal{S}_G$-exterior space.

Proof. Let $(S, \mathcal{S}_G)$ be an intuitionistic $\mathcal{S}_G$-exterior space. Let R be an intuitionistic open Smarandache topological semigroup subspace in $(S, \mathcal{S}_G)$ with the induced intuitionistic open Smarandache topological semigroup subspace topology $\mathcal{S}_{G_R}$. Let A be an intuitionistic open Smarandache topological semigroup in $(R, \mathcal{S}_{G_R})$. Then, $A = B \cap R$, where B is an intuitionistic open Smarandache topological semigroup in $(S, \mathcal{S}_G)$. Now,

$$\begin{aligned} I\mathcal{S}_G CoExt(A) &= I\mathcal{S}_G CoExt(B \cap R) \\ &= \overline{I\mathcal{S}_G int(B \cap R)} \\ &= \overline{I\mathcal{S}_G int(B) \cap I\mathcal{S}_G int(R)} \\ &= \overline{I\mathcal{S}_G int(B)} \cup \overline{I\mathcal{S}_G int(R)} \\ &= I\mathcal{S}_G CoExt(B) \cup I\mathcal{S}_G CoExt(R) \end{aligned}$$

Since $(S, \mathbb{S}_G)$ is an intuitionistic $\mathbb{S}_G$-exterior space, $I\mathbb{S}_G CoExt(B)$ and $I\mathbb{S}_G CoExt(R)$ are intuitionistic open Smarandache topological semigroups.

Therefore, $I\mathbb{S}_G CoExt(B) \cup I\mathbb{S}_G CoExt(R)$ is an intuitionistic open Smarandache topological semigroup. Thus, $I\mathbb{S}_G CoExt(A)$ is an intuitionistic open Smarandache topological semigroup. Hence, by Proposition 4.1., $(R, \mathbb{S}_{G_R})$ is an intuitionistic $\mathbb{S}_G$-exterior space.

Proposition 4.6. Let $(S_1, \mathbb{S}_{G_1})$ be any intuitionistic $\mathbb{S}_G$-exterior space and $(S_2, \mathbb{S}_{G_2})$ be any intuitionistic Smarandache topological semigroup structure space. Let $f : (S_1, \mathbb{S}_{G_1}) \to (S_2, \mathbb{S}_{G_2})$ be an intuitionistic $\mathbb{S}_G$-irresolute, intuitionistic $\mathbb{S}_G$-closed, surjective function. Then $(S_2, \mathbb{S}_{G_2})$ is an intuitionistic $\mathbb{S}_G$-exterior space.

Proof. Let $A = \langle x, A^1, A^2 \rangle$ be an intuitionistic closed Smarandache topological semigroup in $(S_2, \mathbb{S}_{G_2})$. Since f is an intuitionistic $\mathbb{S}_G$-irresolute function, $f^{-1}(A)$ is an intuitionistic closed Smarandache topological semigroup in $(S_1, \mathbb{S}_{G_1})$. Since $(S_1, \mathbb{S}_{G_1})$ is an intuitionistic $\mathbb{S}_G$-exterior space, $I\mathbb{S}_G Ext(f^{-1}(A))$ is an intuitionistic closed Smarandache topological semigroup in $(S_1, \mathbb{S}_{G_1})$. As f is an intuitionistic $\mathbb{S}_G$-closed function, $f(I\mathbb{S}_G Ext(f^{-1}(A))$ is an intuitionistic closed Smarandache topological semigroup in $(S_2, \mathbb{S}_{G_2})$. Since $f^{-1}(A)$ is an intuitionistic closed Smarandache topological semigroup,

$$f(I\mathbb{S}_G Ext(f^{-1}(A))) = f(\overline{f^{-1}(A)}) = f(f^{-1}\overline{(A)}) = \overline{A} \tag{4.4}$$

Since A is an intuitionistic closed Smarandache topological semigroup,

$$I\mathbb{S}_G Ext(A) = \overline{A} \tag{4.5}$$

From (4.1) and (4.2), it follows that, $f(I\mathbb{S}_G Ext(f^{-1}(A))) = I\mathbb{S}_G Ext(A)$. Since $f(I\mathbb{S}_G Ext(f^{-1}(A)))$ is an intuitionistic closed Smarandache topological semigroup, $I\mathbb{S}_G Ext(A)$ is an intuitionistic closed Smarandache topological semigroup. Hence, $(S_2, \mathbb{S}_{G_2})$ is an intuitionistic $\mathbb{S}_G$-exterior space. ■

Proposition 4.7. Let $(S_1, \mathbb{S}_{G_1})$ be any intuitionistic Smarandache topological semigroup structure space and $(S_2, \mathbb{S}_{G_2})$ be any intuitionistic $\mathbb{S}_G$-exterior space. Let $f : (S_1, \mathbb{S}_{G_1}) \to (S_2, \mathbb{S}_{G_2})$ be an intuitionistic $\mathbb{S}_G$-irresolute, intuitionistic $\mathbb{S}_G$-closed, surjective function. Then $(S_1, \mathbb{S}_{G_1})$ is an intuitionistic $\mathbb{S}_G$-exterior space.

Proof. Let $A = \langle x, A^1, A^2 \rangle$ be an intuitionistic closed Smarandache topological semigroup in $(S_1, \mathbb{S}_{G_1})$. Since f is an intuitionistic $\mathbb{S}_G$-closed function, $f(A)$ is an intuitionistic closed Smarandache topological semigroup in $(S_2, \mathbb{S}_{G_2})$. Since $(S_2, \mathbb{S}_{G_2})$ is an intuitionistic $\mathbb{S}_G$-exterior space, $I\mathbb{S}_G Ext(f(A))$ is an intuitionistic closed Smarandache topological semigroup in $(S_2, \mathbb{S}_{G_2})$. As f is an intuitionistic $\mathbb{S}_G$-irresolute function, $f^{-1}(I\mathbb{S}_G Ext(f(A))$ is an intuitionistic closed Smarandache topological semigroup in $(S_1, \mathbb{S}_{G_1})$. Since $f(A)$ is an intuitionistic closed Smarandache topological semigroup, $I\mathbb{S}_G Ext(f(A)) = \overline{f(A)}$. Now,

$$f^{-1}(I\mathbb{S}_G Ext(f(A))) = f^{-1}(\overline{f(A)}) = f^{-1}(f(\overline{A})) = \overline{A}. \tag{4.6}$$

Since A is an intuitionistic closed Smarandache topological semigroup,

$$I\mathbb{S}_G Ext(A) = \overline{A} \tag{4.7}$$

From (4.3) and (4.4), it follows that, $f^{-1}(I\mathbb{S}_G Ext(f(A))) = I\mathbb{S}_G Ext(A)$. Since $f^{-1}(I\mathbb{S}_G Ext(f(A)))$ is an intuitionistic closed Smarandache topological semigroup, $I\mathbb{S}_G Ext(A)$ is an intuitionistic closed Smarandache topological semigroup. Hence, $(S_1, \mathbb{S}_{G_1})$ is an intuitionistic $\mathbb{S}_G$-exterior space. ■

Definition 4.8. Let $(S, \mathbb{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. An intuitionistic Smarandache topological semigroup $A = \langle x, A^1, A^2\rangle$ of S is said to be an intuitionistic pre-open Smarandache topological semigroup if $A \subseteq I\mathbb{S}_G int(I\mathbb{S}_G cl(A))$. The complement of an intuitionistic pre-open Smarandache topological semigroup is called an intuitionistic pre-closed Smarandache topological semigroup.

Definition 4.9. Let $(S, \mathbb{S}_G)$ be an intuitionistic Smarandache topological semigroup structure space. An intuitionistic Smarandache topological semigroup $A = \langle x, A^1, A^2\rangle$ of S is said to be an intuitionistic β-open Smarandache topological semigroup if $A \subseteq I\mathbb{S}_G cl(I\mathbb{S}_G int(I\mathbb{S}_G cl(A)))$. The complement of an intuitionistic β-open Smarandache topological semigroup is called an intuitionistic β-closed Smarandache topological semigroup.

Proposition 4.10. Let $(S, \mathbb{S}_G)$ be an intuitionistic $\mathbb{S}_G$-exterior space. If $A = \langle x, A^1, A^2\rangle$ is an intuitionistic β-open Smarandache topological semigroup, then it is an intuitionistic pre-open Smarandache topological semigroup in $(S, \mathbb{S}_G)$.

Proof. Let A be an intuitionistic β-open Smarandache topological semigroup in $(S_2, \mathbb{S}_{G_2})$. Therefore, $A \subseteq I\mathbb{S}_G cl(I\mathbb{S}_G int(I\mathbb{S}_G cl(A)))$. Let $I\mathbb{S}_G int(I\mathbb{S}_G cl(A))$ be an intuitionistic open Smarandache topological semigroup. Since $(S, \mathbb{S}_G)$ is an intuitionistic $\mathbb{S}_G$-exterior space, $I\mathbb{S}_G CoExt(\overline{I\mathbb{S}_G int(I\mathbb{S}_G cl(A)))}$ is an intuitionistic open Smarandache topological semigroup. Then $\overline{I\mathbb{S}_G int(I\mathbb{S}_G int(I\mathbb{S}_G cl(A)))}$ is an intuitionistic open Smarandache topological semigroup.

That is, $\overline{I\mathbb{S}_G int(I\mathbb{S}_G cl(A))}$ is an intuitionistic open Smarandache topological semigroup. Now, $I\mathbb{S}_G int(I\mathbb{S}_G cl(A))$ is an intuitionistic closed Smarandache topological semigroup. This implies that,

$$I\mathbb{S}_G cl(I\mathbb{S}_G int(I\mathbb{S}_G cl(A))) = I\mathbb{S}_G int(I\mathbb{S}_G cl(A)).$$

Therefore, $A \subseteq I\mathbb{S}_G int(I\mathbb{S}_G cl(A))$. Hence A is an intuitionistic pre-open Smarandache topological semigroup in $(S, \mathbb{S}_G)$. ■

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