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Smarandache curves According to Sabban Frame for Darboux vector of Mannheim Partner Curve

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Abstract. In this paper, we investigated special Smarandache curves belonging to Sabban frame drawn on the surface of the sphere by Darboux vector of Mannheim partner curve. We created Sabban frame belonging to this curve. It was explained Smarandache curves position vector is consisted by Sabban vectors belonging to this curve. Then, we calculated geodesic curvatures of this Smarandache curves. Found results were expressed depending on the Mannheim curve.

Introduction and Preliminaries

A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [8]. K. Taşköprü, M. Tosun studied special Smarandache curves according to Sabban frame on S^2 [9]. Şenyurt and Çalışkan investigated special Smarandache curves in terms of Sabban frame for fixed pole curve and spherical indicatrix and they gave some characterization of Smarandache curves [1, 2]. Şenyurt et al. investigated special Smarandache curves according to Sabban frame for the curve drawn on the surface of the sphere by the unit Darboux vector of Bertrand partner curve and this gave some characterization of Smarandache curves [10]. Let $\alpha : I \to E^3$ be a unit speed curve denote by $\{T, N, B\}$ the moving Frenet frame. For an arbitrary curve $\alpha \in E^3$, with first and second curvature, κ and τ respectively, the Frenet formulae is given by [5]

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N \tag{1}$$

the vector W is called Darboux vector defined by

$$W = \tau T + \kappa B.$$

If we consider the normalization of the Darboux $C = \dfrac{1}{\|W\|}W$ we have, $\sin\varphi = \dfrac{\tau}{\|W\|}$ and $\cos\varphi = \dfrac{\kappa}{\|W\|}$

and

$$C = \sin\varphi T + \cos\varphi B \tag{2}$$

where $\angle(W, B) = \varphi$, [4]. Let α and α_1 be the C^2-class differentiable two curves and $T_1(s)$, $N_1(s)$, $B_1(s)$ be the Frenet vectors of α_1. If the binormal vector of the curve α_1 is linearly dependent on the principal normal vector of the curve α, then (α) is defined a Mannheim curve and (α_1) a Mannheim partner curve of (α), [3, 6, 7]. The relations between the Frenet vectors we can write

$$T_1 = \sin\varphi T + \cos\varphi B, \;\; N_1 = -\cos\varphi T + \sin\varphi B, \;\; B_1 = N \tag{3}$$

and the curvature and torsion we get

$$\kappa_1 = \varphi' \frac{\kappa}{\eta\tau\|W\|}, \;\; \tau_1 = \frac{\kappa}{\eta\tau}. \tag{4}$$

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where $\eta = \dfrac{\kappa}{\kappa^2 + \tau^2} = constant$. Let $\gamma : I \to S^2$ be a unit speed spherical curve. We can write

$$\gamma(s) = \gamma(s), \quad t(s) = \gamma'(s), \quad d(s) = \gamma(s) \wedge t(s), \;[9] \tag{5}$$

$\{\gamma(s), t(s), d(s)\}$ frame is expressed the Sabban frame of γ on S^2. Then we have equations,

$$\gamma'(s) = t(s), \quad t'(s) = -\gamma(s) + \kappa_g(s) d(s), \quad d'(s) = -\kappa_g(s) t(s), \;[9]. \tag{6}$$

where κ_g is expressed the geodesic curvature of the curve γ on S^2 which is

$$\kappa_g(s) = \langle t'(s), d(s) \rangle \;[9]. \tag{7}$$

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Let (C_1) be a unit speed spherical curve on S^2. Then we can write

$$C_1 = \sin\varphi_1 T_1 + \cos\varphi_1 B_1, \quad T_{C_1} = \cos\varphi_1 T_1 - \sin\varphi_1 B_1, \quad C_1 \wedge T_{C_1} = N_1. \tag{8}$$

where $\angle(C_1, B_1) = \varphi_1$. Then from the equation 6 we have the following equations of (C_1) are

$$C_1{}' = T_{C_1}, \quad T'_{C_1} = -C_1 + \frac{\|W_1\|}{\varphi_1{}'} C_1 \wedge T_{C_1}, \quad (C_1 \wedge T_{C_1})' = -\frac{\|W_1\|}{\varphi_1{}'} T_{C_1}. \tag{9}$$

From the equation 7, we have the following geodesic curvature of (C_1) is

$$\kappa_g = \langle T'_{C_1}, C_1 \wedge T_{C_1} \rangle \Longrightarrow \kappa_g = \frac{\|W_1\|}{\varphi_1{}'}. \tag{10}$$

Definition 1 *Let (C_1) be a spherical curve on S^2. C_1 and T_{C_1} be unit vector belonging to (C_1). In this case, β_1-Smarandache curve can be defined by*

$$\beta_1(s) = \frac{1}{\sqrt{2}}(C_1 + T_{C_1}). \tag{11}$$

Theorem 2 *The expression according to the Mannheim curve of geodesic curvature belonging to β_1-Smarandache curve is*

$$\kappa_g^{\beta_1} = \frac{1}{\left(2 + \frac{1}{\sigma^2}\right)^{\frac{5}{2}}} \left(\frac{1}{\sigma}\overline{\lambda}_1 - \frac{1}{\sigma}\overline{\lambda}_2 + 2\overline{\lambda}_3\right), \tag{12}$$

where coefficients are

$$\frac{1}{\sigma} = \left(\frac{\|W\|}{\sqrt{\varphi'^2 + \|W\|^2}}\right)' \frac{\eta\tau\sqrt{\kappa^2 + \tau^2}}{\varphi'} \tag{13}$$

and

$$\overline{\lambda}_1 = -2 - \frac{1}{\sigma^2} + \frac{1}{\sigma'}\frac{1}{\sigma}, \quad \overline{\lambda}_2 = -2 - 3\frac{1}{\sigma^2} - \frac{1}{\sigma^4} - \frac{1}{\sigma'}\frac{1}{\sigma}, \quad \overline{\lambda}_3 = 2\frac{1}{\sigma} + \frac{1}{\sigma^3} + 2\frac{1}{\sigma'} \tag{14}$$

Proof. Substituting the equation 8 into equation 11 we obtain

$$\beta_1(s) = \frac{1}{\sqrt{2}}\Big((\sin\varphi_1 + \cos\varphi_1)T_1 + (\cos\varphi_1 - \sin\varphi_1)B_1\Big). \tag{15}$$

If equation 11 derivative is taken, we can write

$$T_{\beta_1} = \frac{\varphi_1{}'(\cos\varphi_1 - \sin\varphi_1)}{\sqrt{2\varphi_1{}'^2 + \|W_1\|^2}} T_1 + \frac{\|W_1\|}{\sqrt{2\varphi_1{}'^2 + \|W_1\|^2}} N_1 - \frac{\varphi_1{}'(\cos\varphi_1 + \sin\varphi_1)}{\sqrt{2\varphi_1{}'^2 + \|W_1\|^2}} B_1. \tag{16}$$

Considering the equations 15 and 16 we get

$$\beta_1 \wedge T_{\beta_1} = \frac{\|W_1\|(\cos\varphi_1 + \sin\varphi_1)}{\sqrt{2\|W_1\|^2 + 4\left(\varphi_1{}'\right)^2}} T_1 - \frac{\varphi_1{}'}{\sqrt{2\|W_1\|^2 + 4\left(\varphi_1{}'\right)^2}} N_1 + \frac{\|W_1\|(\cos\varphi_1 + \sin\varphi_1)}{\sqrt{2\|W_1\|^2 + 4\left(\varphi_1{}'\right)^2}} B_1. \tag{17}$$

If equation 16 derivative is taken, where coefficients are

$$\lambda_1 = -2 - \left(\frac{\|W_1\|}{\varphi_1{}'}\right)^2 + \left(\frac{\|W_1\|}{\varphi_1{}'}\right)'\left(\frac{\|W_1\|}{\varphi_1{}'}\right), \ \lambda_2 = -2 - 3\left(\frac{\|W_1\|}{\varphi_1{}'}\right)^2 - \left(\frac{\|W_1\|}{\varphi_1{}'}\right)^4 - \left(\frac{\|W_1\|}{\varphi_1{}'}\right)'\left(\frac{\|W_1\|}{\varphi_1{}'}\right), \ \lambda_3 = 2\left(\frac{\|W_1\|}{\varphi_1{}'}\right) + \left(\frac{\|W_1\|}{\varphi_1{}'}\right)^3 + \left(\frac{\|W_1\|}{\varphi_1{}'}\right)' \tag{18}$$

impending T'_{β_1} is, we reach

$$T'_{\beta_1} = \frac{\left(\varphi_1{}'\right)^4 \sqrt{2}(\lambda_1 \sin\varphi_1 + \lambda_2 \cos\varphi_1)}{\left(\|W_1\|^2 + \left(\varphi_1{}'\right)^2\right)^2} T_1 + \frac{\lambda_3\left(\varphi_1{}'\right)^4 \sqrt{2}}{\left(\|W_1\|^2 + \left(\varphi_1{}'\right)^2\right)^2} N_1 + \frac{\left(\varphi_1{}'\right)^4 \sqrt{2}(\lambda_1 \cos\varphi_1 - \lambda_2 \sin\varphi_1)}{\left(\|W_1\|^2 + \left(\varphi_1{}'\right)^2\right)^2} B_1 \tag{19}$$

From the equations 17 and 19, $\kappa_g^{\beta_1}$ geodesic curvature according to Mannheim partner curve of the β_1 is

$$\kappa_g^{\beta_1} = \frac{1}{\left(2 + \left(\frac{\|W_1\|}{\varphi_1{}'}\right)^2\right)^{\frac{5}{2}}}\left(\frac{\|W_1\|}{\varphi_1{}'}\lambda_1 - \frac{\|W_1\|}{\varphi_1{}'}\lambda_2 + 2\lambda_3\right). \tag{20}$$

From the equations 3 and 4, Sabban apparatuses according to Mannheim curve of the β_1-Smarandache curve are

$$\begin{aligned}
\beta_1(s) &= \frac{(\varphi' + \|W\|)\sin\varphi}{\sqrt{2\varphi'^2 + 2\|W\|^2}} T - \frac{\varphi' - \|W\|}{\sqrt{2\varphi'^2 + 2\|W\|^2}} N + \frac{(\varphi' + \|W\|)\cos\varphi}{\sqrt{2\varphi'^2 + 2\|W\|^2}} B, \\
T_{\beta_1}(s) &= \frac{(\varphi' - \|W\|)\sigma\sin\varphi - \sqrt{\|W\|^2 + \varphi'^2}\cos\varphi}{\sqrt{\|W\|^2 + \varphi'^2}\sqrt{1 + 2\sigma^2}} T - \frac{\sigma(\varphi' + \|W\|)}{\sqrt{\|W\|^2 + \varphi'^2}\sqrt{1 + 2\sigma^2}} N \\
&\quad + \frac{\sqrt{\|W\|^2 + \varphi'^2}\sin\varphi - (\|W\| - \varphi')\sigma\cos\varphi}{\sqrt{\|W\|^2 + \varphi'^2}\sqrt{1 + 2\sigma^2}} B, \\
\beta_1 \wedge T_{\beta_1}(s) &= \frac{(\|W\| - \varphi')\sin\varphi - 2\sigma\sqrt{\|W\|^2 + \varphi'^2}\cos\varphi}{\sqrt{2 + 4\sigma^2}\sqrt{\|W\|^2 + \varphi'^2}} T + \frac{\varphi' + \sigma\|W\|}{\sqrt{2 + 4\sigma^2}\sqrt{\|W\|^2 + \varphi'^2}} N \\
&\quad + \frac{(\|W\| + \varphi')\cos\varphi + 2\sigma\sqrt{\|W\|^2 + \varphi'^2}\sin\varphi}{\sqrt{2 + 4\sigma^2}\sqrt{\|W\|^2 + \varphi'^2}} B, \\
\kappa_g^{\beta_1} &= \frac{1}{\left(2 + \frac{1}{\sigma^2}\right)^{\frac{5}{2}}}\left(\frac{1}{\sigma}\overline{\lambda}_1 - \frac{1}{\sigma}\overline{\lambda}_2 + 2\overline{\lambda}_3\right),
\end{aligned}$$

where

$$\frac{1}{\sigma} = \frac{\varphi_1{}'}{\|W_1\|} = \left(\frac{\|W\|}{\sqrt{\varphi'^2 + \|W\|^2}}\right)' \frac{\lambda\tau\sqrt{\kappa^2 + \tau^2}}{\varphi'}$$

impending coefficients are

$$\overline{\lambda}_1 = -2 - \frac{1}{\sigma^2} + \frac{1}{\sigma'}\frac{1}{\sigma}, \ \overline{\lambda}_2 = -2 - 3\frac{1}{\sigma^2} - \frac{1}{\sigma^4} - \frac{1}{\sigma'}\frac{1}{\sigma}, \ \overline{\lambda}_3 = 2\frac{1}{\sigma} + \frac{1}{\sigma^3} + 2\frac{1}{\sigma'}$$

Definition 3 *Let (C_1) be a spherical curve on S^2. C_1 and $C_1 \wedge T_{C_1}$ be unit vector of (C_1). In this case, β_2- Smarandache curve can be defined by*

$$\beta_2(s) = \frac{1}{\sqrt{2}}(C_1 + C_1 \wedge T_{C_1}). \tag{21}$$

Theorem 4 *The expressions according to the Mannheim curve of Sabban apparatuses belonging to β_2-Smarandache curve are*

$$\begin{aligned}
\beta_2(s) &= \frac{\|W\|\sin\varphi - \sqrt{\varphi'^2+\|W\|^2}\cos\varphi}{\sqrt{2\varphi'^2+2\|W\|^2}}T + \frac{\varphi'}{\sqrt{2\varphi'^2+2\|W\|^2}}N + \frac{\sqrt{\varphi'^2+\|W\|^2}\sin\varphi + \|W\|\cos\varphi}{\sqrt{2\varphi'^2+2\|W\|^2}}B,\\
T_{\beta_2}(s) &= \frac{\varphi'\sin\varphi}{\sqrt{\|W\|^2+\varphi'^2}}T - \frac{\|W\|}{\sqrt{\|W\|^2+\varphi'^2}}N + \frac{\varphi'\cos\varphi}{\sqrt{\|W\|^2+\varphi'^2}}B,\\
\beta_2 \wedge T_{\beta_2}(s) &= \frac{-\|W\|\sin\varphi - \sqrt{\|W\|^2+\varphi'^2}\cos\varphi}{\sqrt{2\|W\|^2+2\varphi'^2}}T - \frac{\varphi'}{\sqrt{2\|W\|^2+2\varphi'^2}}N + \frac{\sqrt{\|W\|^2+\varphi'^2}\sin\varphi - \|W\|\cos\varphi}{\sqrt{2\|W\|^2+2\varphi'^2}}B,\\
\kappa_g^{\beta_2}(s) &= \frac{1+\sigma}{1-\sigma},
\end{aligned} \tag{22}$$

Proof. The proof is similar to the proof of Theorem 2.

Definition 5 *Let (C_1) be a spherical curve on S^2. T_{C_1} and $C_1 \wedge T_{C_1}$ be unit vector of (C_1). In this case, β_3-Smarandache curve can be defined by*

$$\beta_3(s) = \frac{1}{\sqrt{2}}(T_{C_1} + C_1 \wedge T_{C_1}). \tag{23}$$

Theorem 6 *The expressions according to the Mannheim curve of Sabban apparatuses belonging to β_3-Smarandache curve are*

$$\begin{aligned}
\beta_3(s) &= \frac{\varphi'\sin\varphi - \sqrt{\|W\|^2+\varphi'^2}\cos\varphi}{\sqrt{2\varphi'^2+2\|W\|^2}}T + \frac{\|W\|}{\sqrt{2\varphi'^2+2\|W\|^2}}N + \frac{\sqrt{\|W\|^2+\varphi'^2}\sin\varphi + \varphi'\cos\varphi}{\sqrt{2\varphi'^2+2\|W\|^2}}B,\\
T_{\beta_3}(s) &= \frac{-\sqrt{\|W\|^2+\varphi'^2}\cos\varphi - (\varphi'+\sigma\|W\|)\sin\varphi}{\sqrt{2+\sigma^2}\sqrt{\|W\|^2+\varphi'^2}}T + \frac{\|W\|-\sigma\varphi'}{\sqrt{2+\sigma^2}\sqrt{\|W\|^2+\varphi'^2}}N\\
&\quad + \frac{\sqrt{\|W\|^2+\varphi'^2}\sin\varphi - (\sigma\|W\|+\varphi')\cos\varphi}{\sqrt{2+\sigma^2}\sqrt{\|W\|^2+\varphi'^2}}B,\\
\beta_3 \wedge T_{\beta_3}(s) &= \frac{(2\|W\|-\sigma\varphi')\sin\varphi - \sigma\sqrt{\|W\|^2+\varphi'^2}\cos\varphi}{\sqrt{4+2\sigma^2}\sqrt{\|W\|^2+\varphi'^2}}T + \frac{2\varphi'-\sigma\|W\|}{\sqrt{4+2\sigma^2}\sqrt{\|W\|^2+\varphi'^2}}N\\
&\quad + \frac{(2\|W\|-\sigma\varphi')\cos\varphi + \sigma\sqrt{\|W\|^2+\varphi'^2}\sin\varphi}{\sqrt{4+2\sigma^2}\sqrt{\|W\|^2+\varphi'^2}}B,\\
\kappa_g^{\beta_3}(s) &= \frac{1}{(2+\sigma^2)^{\frac{5}{2}}}\left(2\sigma^5\triangle_1 - \sigma^4\triangle_2 + \sigma^4\triangle_3\right),
\end{aligned} \tag{24}$$

where coefficients are

$$\triangle_1 = \frac{1}{\sigma} + 2\frac{1}{\sigma^3} + 2\frac{1}{\sigma'}\frac{1}{\sigma},\ \ \triangle_2 = -1 - 3\frac{1}{\sigma^2} - 2\frac{1}{\sigma^4} - \frac{1}{\sigma'},\ \ \triangle_3 = -\frac{1}{\sigma^2} - 2\frac{1}{\sigma^4} + \frac{1}{\sigma'}. \tag{25}$$

Proof. The proof is similar to the proof of Theorem 2.

Definition 7 *Let (C_1) be spherical curve on S^2. C_1, T_{C_1} and $C_1 \wedge T_{C_1}$ be unit vector of (C_1). In this case, β_4-Smarandache curve can be defined by*

$$\beta_4(s) = \frac{1}{3}(C_1 + T_{C_1} + C_1 \wedge T_{C_1}). \tag{26}$$

Theorem 8 *The expressions according to the Mannheim curve of Sabban apparatuses belonging to* β_4-*Smarandache curve are*

$$
\begin{aligned}
\beta_4(s) &= \frac{(\varphi' + \|W\|)\sin\varphi - \sqrt{\varphi'^2 + \|W\|^2}\cos\varphi}{\sqrt{3\varphi'^2 + 3\|W\|^2}}T + \frac{\varphi' - \|W\|}{\sqrt{3\varphi'^2 + 3\|W\|^2}}N + \frac{\sqrt{\varphi'^2 + \|W\|^2}\sin\varphi + (\varphi' + \|W\|)\cos\varphi}{\sqrt{3\varphi'^2 + 3\|W\|^2}}B,\\
T_{\beta_4}(s) &= \frac{\left((\sigma - 1)\varphi' - \sigma\|W\|\right)\sin\varphi - \sqrt{\|W\|^2 + \varphi'^2}\cos\varphi}{\sqrt{2(1 - \sigma + \sigma^2)}\sqrt{\|W\|^2 + \varphi'^2}}T + \frac{\sigma\varphi' + (\sigma - 1)\|W\|}{\sqrt{2(1 - \sigma + \sigma^2)}\sqrt{\|W\|^2 + \varphi'^2}}N\\
&\quad + \frac{\sqrt{\|W\|^2 + \varphi'^2}\sin\varphi - \left(\sigma\|W\| + (1 - \sigma)\varphi'\right)\cos\varphi}{\sqrt{2(1 - \sigma + \sigma^2)}\sqrt{\|W\|^2 + \varphi'^2}}B,\\
\beta_4 \wedge T_{\beta_4}(s) &= \frac{((2 - \sigma)\|W\| - (1 + \sigma)\varphi')\sin\varphi - (2\sigma 1)\sqrt{\|W\|^2 + \varphi'^2}\cos\varphi}{\sqrt{6 - 6\sigma + 6\sigma^2}\sqrt{\|W\|^2 + \varphi'^2}}T + \frac{(2 - \sigma)\varphi' + (1 + \sigma)\|W\|}{\sqrt{6 - 6\sigma + 6\sigma^2}\sqrt{\|W\|^2 + \varphi'^2}}N\\
&\quad + \frac{((2 - \sigma)\|W\| + (1 + \sigma)\varphi')\cos\varphi + (2\sigma - 1)\sqrt{\|W\|^2 + \varphi'^2}\sin\varphi}{\sqrt{6 - 6\sigma + 6\sigma^2}\sqrt{\|W\|^2 + \varphi'^2}}B,\\
\kappa_g^{\beta_4}(s) &= \frac{(2\sigma^4 - \sigma^5)\varkappa_1 - (\sigma^5 + \sigma^4)\varkappa_2 + (2\sigma^5 - \sigma^4)\varkappa_3}{4\sqrt{2}(1 - \sigma + \sigma^2)^{\frac{5}{2}}},
\end{aligned}
\tag{27}
$$

where coefficients are

$$
\varkappa_1 = -2+4\frac{1}{\sigma}-4\frac{1}{\sigma^2}+2\frac{1}{\sigma^3}+2\frac{1}{\sigma'}\left(2\frac{1}{\sigma}-1\right),\ \varkappa_2 = -2+2\frac{1}{\sigma}-4\frac{1}{\sigma^2}+2\frac{1}{\sigma^3}-2\frac{1}{\sigma^4}-\frac{1}{\sigma'}\left(1+\frac{1}{\sigma}\right),\ \varkappa_3 = 2\frac{1}{\sigma}-4\frac{1}{\sigma^2}+4\frac{1}{\sigma^3}-2\frac{1}{\sigma^4}+\frac{1}{\sigma'}\left(2-\frac{1}{\sigma}\right)
\tag{28}
$$

Proof. The proof is similar to the proof of Theorem 2.

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