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Smarandache fuzzy BCI-algebras

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Abstract. The notions of a Smarandache fuzzy subalgebra (ideal) of a Smarandache BCI-algebra, a Smarandache fuzzy clean(fresh) ideal of a Smarandache BCI-algebra are introduced. Examples are given, and several related properties are investigated.

1. Introduction

Generally, in any human field, a Smarandache structure on a set A means a weak structure W on A such that there exists a proper subset B of A with a strong structure S which is embedded in A. In [4], R. Padilla showed that Smarandache semigroups are very important for the study of congruences. Y. B. Jun ([1,2]) introduced the notion of Smarandache BCI-algebras, Smarandache fresh and clean ideals of Smarandache BCI-algebras, and obtained many interesting results about them.

In this paper, we discuss a Smarandache fuzzy structure on BCI-algebras and introduce the notions of a Smarandache fuzzy subalgebra (ideal) of a Smarandache BCI-algebra, a Smarandache fuzzy clean (fresh) ideal of a Smarandache BCI-algebra are introduced, and we investigate their properties.

2. Preliminaries

An algebra $(X; *, 0)$ of type (2,0) is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)(((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)((x * (x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)((x * x = 0)$,
(IV) $(\forall x, y \in X)(x * y = 0$ and $y * x = 0$ imply $x = y)$.

If a BCI-algebra X satisfies the following identity;

(V) $(\forall x \in X)(0 * x = 0)$,

then X is said to be a *BCK-algebra*. We can define a partial order " $\leq$ " on X by $x \leq y$ if and only if $x * y = 0$.

Every BCI-algebra X has the following properties:

(a_1) $(\forall x \in X)(x * 0 = x)$,
(a_1) $(\forall x, y, z \in X)(x \leq y$ implies $x * z \leq y * z, z * y \leq z * x)$.

A non-empty subset I of a BCI-algebra X is called an *ideal* of X if it satisfies the following conditions:

(i) $0 \in I$,
(ii) $(\forall x \in X)(\forall y \in I)(x * y \in I$ implies $x \in I)$.

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Definition 2.1. ([1]) A *Smarandache BCI-algebra* is defined to be a BCI-algebra X in which there exists a proper subset Q of X such that

(i) $0 \in Q$ and $|Q| \geq 2$,
(ii) Q is a BCK-algebra under the same operation of X.

By a *Smarandache positive implicative* (resp. *commutative* and *implicative*) *BCI-algebra*, we mean a BCI-algebra X which has a proper subset Q of X such that

(i) $0 \in Q$ and $|Q| \geq 2$,
(ii) Q is a positive implicative (resp. commutative and implicative) BCK-algebra under the same operation of X.

Let $(X; *, 0)$ be a Smarandache BCI-algebra and H be a subset of X such that $0 \in H$ and $|H| \geq 2$. Then H is called a *Smarandache subalgebra* of X if $(H; *, 0)$ is a Smarandache BCI-algebra.

A non-empty subset I of X is called a *Smarandache ideal* of X related to Q if it satisfies:

(i) $0 \in I$,
(ii) $(\forall x \in Q)(\forall y \in I)(x * y \in I$ implies $x \in I)$,

where Q is a BCK-algebra contained in X. If I is a Smarandache ideal of X related to every BCK-algebra contained in X, we simply say that I is a Smarandache ideal of X.

In what follows, let X and Q denote a Smarandache BCI-algebra and a BCK-algebra which is properly contained in X, respectively.

Definition 2.2. ([2]) A non-empty subset I of X is called a *Smarandache ideal* of X related to Q (or briefly, a *Q-Smarandache ideal*) of X if it satisfies:

(c_1) $0 \in I$,
(c_2) $(\forall x \in Q)(\forall y \in I)(x * y \in I$ implies $x \in I)$.

If I is a Smarandache ideal of X related to every BCK-algebra contained in X, we simply say that I is a Smarandache ideal of X.

Definition 2.3. ([2]) A non-empty subset I of X is called a *Smarandache fresh ideal* of X related to Q (or briefly, a *Q-Smarandache fresh ideal* of X) if it satisfies the conditions (c_1) and

(c_3) $(\forall x, y, z \in Q)(((x * y) * z) \in I$ and $y * z \in I$ imply $x * z \in I)$.

Theorem 2.4. ([2]) *Every Q-Smarandache fresh ideal which is contained in Q is a Q-Smarandache ideal.*

The converse of Theorem 2.4 need not be true in general.

Theorem 2.5. ([2]) *Let I and J be Q-Smarandache ideals of X and $I \subset J$. If I is a Q-Smarandache fresh ideal of X, then so is J.*

Definition 2.6. ([2]) A non-empty subset I of X is called a *Smarandache clean ideal* of X related to Q (or briefly, a *Q-Smarandache clean ideal* of X) if it satisfies the conditions (c_1) and

(c_4) $(\forall x, y \in Q)(z \in I)((x * (y * x)) * z \in I$ implies $x \in I)$.

Theorem 2.7. ([2]) *Every Q-Smarandache clean ideal of X is a Q-Smarandache ideal.*

The converse of Theorem 2.7 need not be true in general.

Theorem 2.8. ([2]) *Every Q-Smarandache clean ideal of X is a Q-Smarandache fresh ideal.*

Theorem 2.9. ([2]) *Let I and J be Q-Smarandache ideals of X and $I \subset J$. If I is a Q-Smarandache clean ideal of X, then so is J.*

A fuzzy set μ in X is called a *fuzzy subalgebra* of a BCI-algebra X if $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. A fuzzy set μ in X is called a *fuzzy ideal* of X if

(F_1) $\mu(0) \geq \mu(x)$ for all $x \in X$,
(F_2) $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$ for all $x, y \in X$.

Let μ be a fuzzy set in a set X. For $t \in [0,1]$, the set $\mu_t := \{x \in X | \mu(x) \geq t\}$ is called a *level subset* of μ.

3. Smarandache fuzzy ideals

Definition 3.1. Let X be a Smarandache BCI-algebra. A map $\mu : X \to [0,1]$ is called a *Smarandache fuzzy subalgebra* of X if it satisfies

(SF_1) $\mu(0) \geq \mu(x)$ for all $x \in P$,
(SF_2) $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in P$,

where $P \subsetneq X$, P is a BCK-algebra with $|P| \geq 2$.
A map $\mu : X \to [0,1]$ is called a *Smarandache fuzzy ideal* of X if it satisfies (SF_1) and

(SF_2) $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$ for all $x, y \in P$,

where $P \subsetneq X$, P is a BCK-algebra with $|P| \geq 2$. This Smarandache fuzzy subalgebra (ideal) is denoted by μ_P, i.e., $\mu_P : P \to [0,1]$ is a fuzzy subalgebra(ideal) of X.

Example 3.2. Let $X := \{0,1,2,3,4,5\}$ be a Smarandache BCI-algebra ([1]) with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	1	3	3	3
2	2	2	0	3	3	3
3	3	3	3	0	0	0
4	4	3	4	1	0	0
5	5	3	5	1	1	0

Define a map $\mu : X \to [0,1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0,1,2,3\}, \\ 0.7 & \text{otherwise} \end{cases}$$

Clearly μ is a Samrandache fuzzy subalgebra of X. It is verified that μ restricted to a subset $\{0,1,2,3\}$ which is a subalgebra of X is a fuzzy subalgebra of X, i.e., $\mu_{\{0,1,2,3\}} : \{0,1,2,3\} \to [0,1]$ is a fuzzy subalgebra of X. Thus $\mu : X \to [0,1]$ is a Smarandache fuzzy subalgebra of X. Note that $\mu : X \to [0,1]$ is not a fuzzy subalgebra of X, since $\mu(5*4) = \mu(0) = 0.5 \not\geq \min\{\mu(5), \mu(4)\} = 0.7$.

Example 3.3. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra ([1]) with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	4	4
1	1	0	0	1	4	4
2	2	2	0	2	4	4
3	3	3	3	0	4	4
4	4	44	4	0	0	
5	5	4	4	5	1	0

Define a map $\mu : X \to [0, 1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0, 1, 2\} \\ 0.7 & \text{otherwise} \end{cases}$$

Clearly μ is a Samrandache fuzzy ideal of X. It is verified that μ restricted to a subset $\{0, 1, 2\}$ which is an ideal of X is a fuzzy ideal of X, i.e., $\mu_{\{0,1,2\}} : \{0, 1, 2\} \to [0, 1]$ is a fuzzy ideal of X. Thus $\mu : X \to [0, 1]$ is a Smarandache fuzzy ideal of X. Note that $\mu : X \to [0, 1]$ is not a fuzzy ideal of X, since $\mu(2) = 0.5 \not\geq \min\{\mu(2*4) = \mu(4), \mu(4)\} = \mu(4) = 0.7$.

Lemma 3.4. *Every Smarandache fuzzy ideal μ_P of a Smarandache BCI-algebra X is order reversing.*

Proof. Let P be a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. If $x, y \in P$ with $x \leq y$, then $x * y = 0$. Hence we have $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} = \min\{\mu(0), \mu(y)\} = \mu(y)$. □

Theorem 3.5. *Any Smarandache fuzzy ideal μ_P of a Smarandache BCI-algebra X must be a Smarandache fuzzy subalgebra of X.*

Proof. Let P be a BCK-algebra with $P \subsetneq X$ and $|X| \geq 2$. Since $x * y \leq x$ for any $x, y \in P$, it follows from Lemma 3.4 that $\mu(x) \leq \mu(x * y)$, so by (SF_2) we obtain $\mu(x * y) \geq \mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq \min\{\mu(x), \mu(y)\}$. This shows that μ is a Smarandache fuzzy subalgebra of X, proving the theorem. □

Proposition 3.6. *Let μ_P be a Smarandache fuzzy ideal of a Smarandache BCI-algebra X. If the inequality $x * y \leq z$ holds in P, then $\mu(x) \geq \min\{\mu(x), \mu(z)\}$ for all $x, y, z \in P$.*

Proof. Let P be a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. If $x * y \leq z$ in P, then $(x * y) * z = 0$. Hence we have $\mu(x * y) \geq \min\{\mu((x * y) * z), \mu(z)\} = \min\{\mu(0), \mu(z)\} = \mu(z)$. It follows that $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq \min\{\mu(y), \mu(z)\}$. □

Theorem 3.7. *Let X be a Smarandache BCI-algebra. A Smarandache fuzzy subalgebra μ_P of X is a Smarandache fuzzy ideal of X if and only if for all $x, y \in P$, the inequality $x * y \leq z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$.*

Proof. Suppose that μ_P is a Smarandache fuzzy subalgebra of X satisfying the condition $x * y \leq z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$. Since $x * (x * y) \leq y$ for all $x, y \in P$, it follows that $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. Hence μ_P is a Smarandache fuzzy ideal of X. The converse follows from Proposition 3.6. □

Definition 3.8. Let X be a Smarandache BCI-algebra. A map $\mu : X \to [0, 1]$ is called a *Smarandache fuzzy clean ideal* of X if it satisfies (SF_1) and

(SF_3) $\mu(x) \geq \min\{\mu(x * (y * x)) * z), \mu(z)\}$ for all $x, y, z \in P$, where $P \subsetneq X$ and P is a BCK-algebra with $|P| \geq 2$. This Smarandache fuzzy clean ideal is denoted by μ_P, i.e., $\mu_P : P \to [0, 1]$ is a Smarandache fuzzy clean ideal of X.

Example 3.9. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra ([2]) with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	0	5
2	2	1	0	1	0	5
3	3	4	4	4	0	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Define a map $\mu : X \to [0, 1]$ by

$$\mu(x) := \begin{cases} 0.4 & \text{if } x \in \{0, 1, 2, 3\} \\ 0.8 & \text{otherwise} \end{cases}$$

Clearly μ is a Samrandache fuzzy clean ideal of X, but μ is not a fuzzy clean ideal of X, since $\mu(3) = 0.4 \not\geq \min\{\mu((3 * (0 * 3)) * 5), \mu(5)\} = \min\{\mu(5), \mu(5)\} = \mu(5) = 0.8$.

Theorem 3.10. *Let X be a Smarandache BCI-algebra. Any Smarandache fuzzy clean ideal μ_P of X must be a Smarandache fuzzy ideal of X.*

Proof. Let X be a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. Let $\mu_P : P \to [0, 1]$ be a Smarndache fuzzy clean ideal of X. If we let $y := x$ in (SF_3), then $\mu(x) \geq \min\{\mu((x * (x * x)) * z), \mu(z)\} = \min\{\mu((x * 0) * z), \mu(z)\} = \min\{\mu(x * z), \mu(z)\}$, for all $x, y, z \in P$. This shows that μ satisfies (SF_2). Combining (SF_1), μ_P is a Smarandache fuzzy ideal of X, proving the theorem. □

Corollary 3.11. *Every Smarandache fuzzy clean ideal μ_P of a Smarndache BCI-algebra X must be a Smarandache fuzzy subalgebra of X.*

Proof. It follows from Theorem 3.5 and Theorem 3.10. □

The converse of Theorem 3.10 may not be true as shown in the following example.

Example 3.12. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	0	5
2	2	2	0	0	0	5
3	3	3	3	0	0	5
4	4	3	4	1	0	5
5	5	5	5	5	5	0

Let μ_P be a fuzzy set in $P = \{0, 1, 2, 3, 4\}$ defined by $\mu(0) = \mu(2) = 0.8$ and $\mu(1) = \mu(3) = \mu(4) = 0.3$. It is easy to check that μ_P is a fuzzy ideal of X. Hence $\mu : X \to [0, 1]$ is a Smarandache fuzzy ideal of X. But it is not a Smarandache fuzzy clean ideal of X since $\mu(1) = 0.3 \not\geq \min\{\mu((1 * (3 * 1)) * 2), \mu(2)\} = \min\{\mu(0), \mu(2)\} = 0.8$.

Theorem 3.13. *Let X be a Smarandache implicative BCI-algebra. Every Smarandache fuzzy ideal μ_P of X is a Smarandache fuzzy clean ideal of X.*

Proof. Let P be a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. Since X is a Smarandache implicative BCI-algebra, we have $x = x * (y * x)$ for all $x, y \in P$. Let μ_P be a Smarandache fuzzy ideal of X. It follows from (SF_2) that $\mu(x) \geq \min\{\mu(x * z), \mu(z)\} \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$, for all $x, y, z \in P$. Hence μ_P is a Smarandache clean ideal of X. The proof is complete. □

In what follows, we give characterizations of fuzzy implicative ideals.

Theorem 3.14. *Let X be a Smarandache BCI-algebra. Suppose that μ_P is a Smarandache fuzzy ideal of X. Then the following equivalent:*

(i) *μ_P is Smarandache fuzzy clean,*
(ii) *$\mu(x) \geq \mu(x * (y * x))$ for all $x, y \in P$,*
(iii) *$\mu(x) = \mu(x * (y * x))$ for all $x, y \in P$.*

Proof. (i) ⇒ (ii): Let μ_P be a Smarandache fuzzy clean ideal of X. It follows from (SF_3) that $\mu(x) \geq \min\{\mu((x * (y * x)) * 0), \mu(0)\} = \min\{\mu(x * (y * x)), \mu(0)\} = \mu(x * (y * x)),\ \forall x, y \in P$. Hence the condition (ii) holds.
(ii) ⇒ (iii): Since X is a Smarnadache BCI-algebra, we have $x * (y * x) \leq x$ for all $x, y \in P$. It follows from Lemma 3.4 that $\mu(x) \leq \mu(x * (y * x))$. By (ii), $\mu(x) \geq \mu(x * (y * x))$. Thus the condition (iii) holds.
(iii) ⇒ (i): Suppose that the condition (iii) holds. Since μ_P is a Smarandache fuzzy ideal, by (SF_2), we have $\mu(x * (y * x)) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$. Combining (iii), we obtain $\mu(x) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$. Hence μ satisfies the condition (SF_3). Obviously, μ satisfies (SF_1). Therefore μ is a fuzzy clean ideal of X. Hence the condition (i) holds. The proof is complete. □

For any fuzzy sets μ and ν in X, we write $\mu \leq \nu$ if and only if $\mu(x) \leq \nu(x)$ for any $x \in X$.

Definition 3.15. Let X be a Smarandache BCI-algebra and let $\mu_P : P \to [0, 1]$ be a Smarandache fuzzy BCI-algebra of X. For $t \leq \mu(0)$, the set $\mu_t := \{x \in P | \mu(x) \geq t\}$ is called a *level subset* of μ_P.

Theorem 3.16. *A fuzzy set μ in P is a Smarandache fuzzy clean ideal of X if and only if, for all $t \in [0, 1]$, μ_t is either empty or a Smarandache clean ideal of X.*

Proof. Suppose that μ_P is a Smarandache fuzzy clean ideal of X and $\mu_t \neq \emptyset$ for any $t \in [0, 1]$. It is clear that $0 \in \mu_t$ since $\mu(0) \geq t$. Let $\mu((x * (y * x)) * z) \geq t$ and $\mu(z) \geq t$. It follows from (SF_3) that $\mu(x) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\} \geq t$, namely, $x \in \mu_t$. This shows that μ_t is a Smarandache clean ideal of X.

Conversely, assume that for each $t \in [0, 1]$, μ_t is either empty or a Smaranadche clean ideal of X. For any $x \in P$, let $\mu(x) = t$. Then $x \in \mu_t$. Since $\mu_t (\neq \emptyset)$ is a Smarandache clean ideal of X, therefore $0 \in \mu_t$ and hence $\mu(0) \geq \mu(x) = t$. Thus $\mu(0) \geq \mu(x)$ for all $x \in P$. Now we show that μ satisfies (SF_3). If not, then there exist $x', y', z' \in P$ such that $\mu(x') < \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}$. Taking $t_0 := \frac{1}{2}\{\mu(x') + \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}\}$, we have $\mu(x') < t_0 < \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}$. Hence $x' \notin \mu_{t_0}$, $(x' * (y' * x')) * z \in \mu_{t_0}$, and $z' \in \mu_{t_0}$, i.e., μ_{t_0} is not a Smaraqndache clean of X, which is a contradiction. Therefore, μ_P is a Smarnadche fuzzy clean ideal, completing the proof. □

Theorem 3.17. ([2]) (Extension Property) *Let X be a Smarandache BCI-algebra. Let I and J be Q-Smarandache ideals of X and $I \subseteq J \subseteq Q$. If I is a Q-Smarandache clean ideal of X, then so is J.*

Next we give the extension theorem of Smarandache fuzzy clean ideals.

Theorem 3.18. *Let X be a Smarandache BCI-algebra. Let μ and ν be Smarandache fuzzy ideals of X such that $\mu \leq \nu$ and $\mu(0) = \nu(0)$. If μ is a Smarndache fuzzy clean ideal of X, then so is ν.*

Proof. It suffices to show that for any $t \in [0,1]$, ν_t is either empty or a Smarandache clean ideal of X. If the level subset ν_t is non-empty, then $\mu_t \neq \emptyset$ and $\mu_t \subseteq \nu_t$. In fact, if $x \in \mu_t$, then $t \leq \mu(x)$; hence $t \leq \nu(x)$, i.e, $x \in \nu_t$. So $\mu_t \subseteq \nu_t$. By the hypothesis, since μ is a Smarandache fuzzy clean ideal of X, μ_t is a Smarandache clean of X by Theorem 3.16. It follows from Theorem 3.17 that ν_t is a Smarandache clean ideal of X. Hence ν is a Smarandache fuzzy clean of X. The proof is complete. □

Definition 3.19. Let X be a Smarandache BCI-algebra. A map $\mu : X \to [0,1]$ is called a *Smarandache fuzzy fresh ideal* of X if it satisfies (SF_1) and

(SF_4) $\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\}$ for all $x, y, z \in P$,

where P is a BCK-algebra with $P \subsetneq X$ and $|P| \geq 2$. This Smarandache fuzzy ideal is denoted by μ_P, i.e., $\mu_P : P \to [0,1]$ is a Smarandache fuzzy fresh ideal of X.

Example 3.20. Let $X := \{0,1,2,3,4,5\}$ be a Smarandache BCI-algebra ([2]) with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	1	5
2	2	2	0	2	0	5
3	3	1	3	0	3	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Define a map $\mu : X \to [0,1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0,1,3\}, \\ 0.9 & \text{otherwise} \end{cases}$$

Clearly μ is a Samrandache fuzzy fresh ideal of X. But it is not a fuzzy fresh ideal of X, since $\mu(2 * 4) = \mu(0) = 0.5 \not\geq \min\{\mu((2 * 5) * 4), \mu(5 * 4)\} = \mu(5) = 0.9$.

Theorem 3.21. *Any Smarandache fuzzy fresh ideal of a Smarandache BCI-algebra X must be a Smarandache fuzzy ideal of X.*

Proof. Taking $z := 0$ in (SF_4) and $x * 0 = x$, we have $\mu(x * 0) \geq \min\{\mu((x * y) * 0), \mu(y * 0)\}$. Hence $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. Thus (SF_2) holds. □

The converse of Theorem 3.21 may not be true as show in the following example.

Example 3.22. Let $X := \{0,1,2,3,4,5\}$ be a Smarandache BCI-algebra ([2]) with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	1	5
2	2	1	0	1	2	5
3	3	1	1	0	3	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Define a map $\mu : X \to [0,1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0,4\}, \\ 0.4 & \text{otherwise} \end{cases}$$

Clearly $\mu(x)$ is a Samrandache fuzzy ideal of X. But $\mu(x)$ is not a Samrandache fuzzy fresh ideal of X, since $\mu(2*3) = \mu(1) = 0.4 \not\geq \min\{\mu((2*1)*3), \mu(1*3)\} = \min\{\mu(1*3), \mu(0)\} = \mu(0) = 0.5$.

Proposition 3.23. *Let X be a Smarandache BCI-algebra. A Smarandache fuzzy ideal μ_P of X is a Smarandache fuzzy fresh ideal of X if and only if it satisfies the condition $\mu(x*y) \geq \mu((x*y)*y)$ for all $x, y \in P$.*

Proof. Assume that μ_P is a Smarandache fuzzy fresh ideal of X. Putting $z := y$ in (SF_4), we have $\mu(x*y) \geq \min\{\mu((x*y)*y), \mu(y*y)\} = \min\{\mu((x*y)*y), \mu(0)\} = \mu((x*y)*y),\ \forall x, y \in P$.

Conversely, let μ_P be Smarandache fuzzy ideal of X such that $\mu(x*y) \geq \mu((x*y)*y)$. Since, for all $x, y, z \in P$, $((x*z)*z)*(y*z) \leq (x*z)*y = (x*y)*z$, we have $\mu((x*y)*z) \leq \mu(((x*z)*z)*(y*z))$. Hence $\mu(x*z) \geq \mu((x*z)*z) \geq \min\{\mu(((x*z)*z)*(y*z)), \mu(y*z)\} \geq \min\{\mu((x*y)*z), \mu(y*z)\}$. This completes the proof. □

Since $(x*y)*y \leq x*y$, it follows from Lemma 3.4 that $\mu(x*y) \leq \mu((x*y)*y)$. Thus we have the following theorem.

Theorem 3.24. *Let X be a Smarandache BCI-algebra. A Smarandache fuzzy ideal μ_P of X is a Smarandache fuzzy fresh if and only if it satisfies the identity*

$$\mu(x*y) = \mu((x*y)*y),\ textfor\ all\ x, y \in X.$$

We give an equivalent condition for which a Smarandache fuzzy subalgebra of a Smarandache BCI-algebra to be a Smarandache fuzzy clean ideal of X.

Theorem 3.25. *A Smarandache fuzzy subalghebra μ_P of X is a Smarandache fuzzy clean ideal of X if and only if it satisfies*

$$(x*(y*x))*z \leq u \text{ implies } \mu(x) \geq \min\{\mu(z), \mu(u)\} \text{ for all} x, y, z, u \in P. \quad (*)$$

Proof. Assume that μ_P is a Smarandache fuzzy clean ideal of X. Let $x, y, z, u \in P$ be such that $(x*(y*x))*z \leq u$. Since μ is a Smarandache fuzzy ideal of X, we have $\mu(x*(y*x)) \geq \min\{\mu(z), \mu(u)\}$ by Theorem 3.7. By Theorem 3.14-(iii), we obtain $\mu(x) \geq \min\{\mu(z), \mu(u)\}$.

Conversely, suppose that μ_P satisfies $(*)$. Obviously, μ_P satisfies (SF_1), since $(x*(y*x))*((x*(y*x))*z) \leq z$, by $(*)$, we obtain $\mu(x) \geq \min\{\mu((x*(y*x))*z), \mu(z)\}$, which shows that μ_P satisfies (SF_3). Hence μ_P is a Smarandache fuzzy clean ideal of X. The proof is complete. □

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