Smarandache fuzzy $BCI$-algebras

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Abstract. The notions of a Smarandache fuzzy subalgebra (ideal) of a Smarandache $BCI$-algebra, a Smarandache fuzzy clean (fresh) ideal of a Smarandache $BCI$-algebra are introduced. Examples are given, and several related properties are investigated.

1. Introduction

Generally, in any human field, a Smarandache structure on a set $A$ means a weak structure $W$ on $A$ such that there exists a proper subset $B$ of $A$ with a strong structure $S$ which is embedded in $A$. In [4], R. Padilla showed that Smarandache semigroups are very important for the study of congruences. Y. B. Jun ([1,2]) introduced the notion of Smarandache $BCI$-algebras, Smarandache fresh and clean ideals of Smarandache $BCI$-algebras, and obtained many interesting results about them.

In this paper, we discuss a Smarandache fuzzy structure on $BCI$-algebras and introduce the notions of a Smarandache fuzzy subalgebra (ideal) of a Smarandache $BCI$-algebra, a Smarandache fuzzy clean (fresh) ideal of a Smarandache $BCI$-algebra are introduced, and we investigate their properties.

2. Preliminaries

An algebra $(X; *, 0)$ of type $(2,0)$ is called a $BCI$-algebra if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)((x * y) * (x * z)) * (z * y) = 0$,
(II) $(\forall x, y \in X)((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)((x * x = 0)$,
(IV) $(\forall x, y \in X)(x * y = 0$ and $y * x = 0$ imply $x = y$).

If a $BCI$-algebra $X$ satisfies the following identity:

(V) $(\forall x \in X)(0 * x = 0)$,

then $X$ is said to be a $BCK$-algebra. We can define a partial order “$\leq$” on $X$ by $x \leq y$ if and only if $x * y = 0$.

Every $BCI$-algebra $X$ has the following properties:

$(a_1)$ $(\forall x \in X)(x * 0 = x)$,
$(a_1)$ $(\forall x, y, z \in X)(x \leq y$ implies $x * z \leq y * z, z * y \leq z * x)$.

A non-empty subset $I$ of a $BCI$-algebra $X$ is called an ideal of $X$ if it satisfies the following conditions:

(i) $0 \in I$,
(ii) $(\forall x \in X)(\forall y \in I)(x * y \in I$ implies $x \in I)$.
**Definition 2.1.** ([1]) A *Smarandache BCI-algebra* is defined to be a *BCI*-algebra *X* in which there exists a proper subset *Q* of *X* such that

(i) $0 \in Q$ and $|Q| \geq 2$,
(ii) *Q* is a *BCK*-algebra under the same operation of *X*.

By a *Smarandache positive implicative* (resp. *commutative and implicative*) *BCI*-algebra, we mean a *BCI*-algebra *X* which has a proper subset *Q* of *X* such that

(i) $0 \in Q$ and $|Q| \geq 2$,
(ii) *Q* is a positive implicative (resp. commutative and implicative) *BCK*-algebra under the same operation of *X*.

Let $(X; *, 0)$ be a Smarandache *BCI*-algebra and *H* be a subset of *X* such that $0 \in H$ and $|H| \geq 2$. Then *H* is called a *Smarandache subalgebra* of *X* if $(H; *, 0)$ is a Smarandache *BCI*-algebra.

A non-empty subset *I* of *X* is called a *Smarandache ideal* of *X* related to *Q* if it satisfies:

(i) $0 \in I$,
(ii) $(\forall x \in Q)(\forall y \in I)(x \ast y \in I$ implies $x \in I)$,

where *Q* is a *BCK*-algebra contained in *X*. If *I* is a Smarandache ideal of *X* related to every *BCK*-algebra contained in *X*, we simply say that *I* is a Smarandache ideal of *X*.

In what follows, let *X* and *Q* denote a Smarandache *BCI*-algebra and a *BCK*-algebra which is properly contained in *X*, respectively.

**Definition 2.2.** ([2]) A non-empty subset *I* of *X* is called a *Smarandache ideal* of *X* related to *Q* (or briefly, a *Q*-Smarandache ideal) of *X* if it satisfies:

(i) $0 \in I$,
(ii) $(\forall x \in Q)(\forall y \in I)(x \ast y \in I$ implies $x \in I)$,

where *Q* is a *BCK*-algebra contained in *X*. If *I* is a Smarandache ideal of *X* related to every *BCK*-algebra contained in *X*, we simply say that *I* is a Smarandache ideal of *X*.

**Definition 2.3.** ([2]) A non-empty subset *I* of *X* is called a *Smarandache fresh ideal* of *X* related to *Q* (or briefly, a *Q*-Smarandache fresh ideal) of *X* if it satisfies the conditions

(i) $0 \in I$,
(ii) $(\forall x \ast ((y \ast x)) \ast z \in I$ implies $x \ast z \in I)$.

**Theorem 2.4.** ([2]) Every *Q*-Smarandache fresh ideal which is contained in *Q* is a *Q*-Smarandache ideal.

The converse of Theorem 2.4 need not be true in general.

**Theorem 2.5.** ([2]) Let *I* and *J* be *Q*-Smarandache ideals of *X* and $I \subset J$. If *I* is a *Q*-Smarandache fresh ideal of *X*, then so is *J*.

**Definition 2.6.** ([2]) A non-empty subset *I* of *X* is called a *Smarandache clean ideal* of *X* related to *Q* (or briefly, a *Q*-Smarandache clean ideal) of *X* if it satisfies the conditions

(i) $0 \in I$,
(ii) $(\forall x \ast ((y \ast x)) \ast z \in I$ implies $x \ast z \in I)$.
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**Theorem 2.7.** ([2]) Every Q-Smarandache clean ideal of $X$ is a Q-Smarandache ideal.

The converse of Theorem 2.7 need not be true in general.

**Theorem 2.8.** ([2]) Every Q-Smarandache clean ideal of $X$ is a Q-Smarandache fresh ideal.

**Theorem 2.9.** ([2]) Let $I$ and $J$ be Q-Smarandache ideals of $X$ and $I \subset J$. If $I$ is a Q-Smarandache clean ideal of $X$, then so is $J$.

A fuzzy set $\mu$ in $X$ is called a **fuzzy subalgebra** of a BCI-algebra $X$ if $\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

A fuzzy set $\mu$ in $X$ is called a **fuzzy ideal** of $X$ if

- $(F_1)$ $\mu(0) \geq \mu(x)$ for all $x \in X$,
- $(F_2)$ $\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\}$ for all $x, y \in X$.

Let $\mu$ be a fuzzy set in a set $X$. For $t \in [0, 1]$, the set $\mu_t := \{x \in X | \mu(x) \geq t\}$ is called a **level subset** of $\mu$.

3. **Smarandache fuzzy ideals**

**Definition 3.1.** Let $X$ be a Smarandache BCI-algebra. A map $\mu : X \rightarrow [0, 1]$ is called a **Smarandache fuzzy subalgebra** of $X$ if it satisfies

- $(SF_1)$ $\mu(0) \geq \mu(x)$ for all $x \in P$,
- $(SF_2)$ $\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in P$,

where $P \subseteq X$, $P$ is a BCK-algebra with $|P| \geq 2$.

A map $\mu : X \rightarrow [0, 1]$ is called a **Smarandache fuzzy ideal** of $X$ if it satisfies $(SF_1)$ and

- $(SF_2)$ $\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\}$ for all $x, y \in P$,

where $P \subseteq X$, $P$ is a BCK-algebra with $|P| \geq 2$. This Smarandache fuzzy subalgebra (ideal) is denoted by $\mu_P$, i.e., $\mu_P : P \rightarrow [0, 1]$ is a fuzzy subalgebra (ideal) of $X$.

**Example 3.2.** Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra ([1]) with the following Cayley table:

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Define a map $\mu : X \rightarrow [0, 1]$ by

\[
\mu(x) := \begin{cases} 
0.5 & \text{if } x \in \{0, 1, 2, 3\}, \\
0.7 & \text{otherwise}
\end{cases}
\]

Clearly $\mu$ is a Samrandache fuzzy subalgebra of $X$. It is verified that $\mu$ restricted to a subset $\{0, 1, 2, 3\}$ which is a subalgebra of $X$ is a fuzzy subalgebra of $X$, i.e., $\mu_{\{0,1,2,3\}} : \{0, 1, 2, 3\} \rightarrow [0, 1]$ is a fuzzy subalgebra of $X$. Thus $\mu : X \rightarrow [0, 1]$ is a Smarandache fuzzy subalgebra of $X$. Note that $\mu : X \rightarrow [0, 1]$ is not a fuzzy subalgebra of $X$, since $\mu(5 \ast 4) = \mu(0) = 0.5 \geq \min\{\mu(5), \mu(4)\} = 0.7$. 

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Example 3.3. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra ([1]) with the following Cayley table:

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</table>

Define a map $\mu : X \to [0, 1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0, 1, 2\} \\ 0.7 & \text{otherwise} \end{cases}$$

Clearly $\mu$ is a Samrandache fuzzy ideal of $X$. It is verified that $\mu$ restricted to a subset $\{0, 1, 2\}$ which is an ideal of $X$ is a fuzzy ideal of $X$, i.e., $\mu(\{0, 1, 2\}) : \{0, 1, 2\} \to [0, 1]$ is a fuzzy ideal of $X$. Thus $\mu : X \to [0, 1]$ is a Smarandache fuzzy ideal of $X$. Note that $\mu : X \to [0, 1]$ is not a fuzzy ideal of $X$, since $\mu(2) = 0.5 \not\geq \min\{\mu(2*4) = \mu(4), \mu(4)\} = \mu(4) = 0.7$.

Lemma 3.4. Every Smarandache fuzzy ideal $\mu_P$ of a Smarandache BCI-algebra $X$ is order reversing.

Proof. Let $P$ be a BCK-algebra with $P \subseteq X$ and $|P| \geq 2$. If $x, y \in P$ with $x \leq y$, then $x * y = 0$. Hence we have $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} = \min\{\mu(0), \mu(y)\} = \mu(y)$. □

Theorem 3.5. Any Smarandache fuzzy ideal $\mu_P$ of a Smarandache BCI-algebra $X$ must be a Smarandache fuzzy subalgebra of $X$.

Proof. Let $P$ be a BCK-algebra with $P \subseteq X$ and $|X| \geq 2$. Since $x * y \leq x$ for any $x, y \in P$, it follows from Lemma 3.4 that $\mu(x) \leq \mu(x * y)$, so by $(SF_2)$ we obtain $\mu(x * y) \geq \mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq \min\{\mu(x), \mu(y)\}$. This shows that $\mu$ is a Smarandache fuzzy subalgebra of $X$, proving the theorem. □

Proposition 3.6. Let $\mu_P$ be a Smarandache fuzzy ideal of a Smarandache BCI-algebra $X$. If the inequality $x * y \leq z$ holds in $P$, then $\mu(x) \geq \min\{\mu(x), \mu(z)\}$ for all $x, y, z \in P$.

Proof. Let $P$ be a BCK-algebra with $P \subseteq X$ and $|P| \geq 2$. If $x * y \leq z$ in $P$, then $(x * y) * z = 0$. Hence we have $\mu(x * y) \geq \min\{\mu((x * y) * z), \mu(z)\} = \min\{\mu(0), \mu(z)\} = \mu(z)$. It follows that $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq \min\{\mu(y), \mu(z)\}$. □

Theorem 3.7. Let $X$ be a Smarandache BCI-algebra. A Smarandache fuzzy subalgebra $\mu_P$ of $X$ is a Smarandache fuzzy ideal of $X$ if and only if for all $x, y \in P$, the inequality $x * y \leq z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$.

Proof. Suppose that $\mu_P$ is a Smarandache fuzzy subalgebra of $X$ satisfying the condition $x * y \leq z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$. Since $x * (x * y) \leq y$ for all $x, y \in P$, it follows that $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. Hence $\mu_P$ is a Smarandache fuzzy ideal of $X$. The converse follows from Proposition 3.6. □

Definition 3.8. Let $X$ be a Smarandache BCI-algebra. A map $\mu : X \to [0, 1]$ is called a Smarandache fuzzy clean ideal of $X$ if it satisfies $(SF_1)$ and

$(SF_3)$ $\mu(x) \geq \min\{\mu(x * (y * x)), \mu(z)\}$ for all $x, y, z \in P$. 

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where \( P \subseteq X \) and \( P \) is a BCK-algebra with \( |P| \geq 2 \). This Smarandache fuzzy clean ideal is denoted by \( \mu_P \), i.e., \( \mu_P : P \to [0, 1] \) is a Smarandache fuzzy clean ideal of \( X \).

**Example 3.9.** Let \( X := \{0, 1, 2, 3, 4, 5\} \) be a Smarandache BCI-algebra ([2]) with the following Cayley table:

\[
\begin{array}{cccccc}
* & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 \\
1 & 1 & 0 & 0 & 0 & 0 & 5 \\
2 & 2 & 1 & 0 & 1 & 0 & 5 \\
3 & 3 & 4 & 4 & 4 & 0 & 5 \\
4 & 4 & 4 & 4 & 4 & 0 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 0 \\
\end{array}
\]

Define a map \( \mu : X \to [0, 1] \) by

\[
\mu(x) := \begin{cases} 
0.4 & \text{if } x \in \{0, 1, 2, 3\} \\
0.8 & \text{otherwise}
\end{cases}
\]

Clearly \( \mu \) is a Samrandache fuzzy clean ideal of \( X \), but \( \mu \) is not a fuzzy clean ideal of \( X \), since \( \mu(3) = 0.4 \not\geq \min\{\mu((3 * (0 * 3)) * 5), \mu(5)\} = \min\{\mu(5), \mu(5)\} = \mu(5) = 0.8 \).

**Theorem 3.10.** Let \( X \) be a Smarandache BCI-algebra. Any Smarandache fuzzy clean ideal \( \mu_P \) of \( X \) must be a Smarandache fuzzy ideal of \( X \).

**Proof.** Let \( X \) be a BCK-algebra with \( P \subseteq X \) and \( |P| \geq 2 \). Let \( \mu_P : P \to [0, 1] \) be a Smarandache fuzzy clean ideal of \( X \). If we let \( y := x \) in \((SF_3)\), then \( \mu(x) \geq \min\{\mu((x * (x * x)) * z), \mu(z)\} = \min\{\mu((x * 0) * z), \mu(z)\} = \min\{\mu(x * z), \mu(z)\} \), for all \( x, y, z \in P \). This shows that \( \mu \) satisfies \((SF_2)\). Combining \((SF_1)\), \( \mu_P \) is a Smarandache fuzzy ideal of \( X \), proving the theorem. \( \square \)

**Corollary 3.11.** Every Smarandache fuzzy clean ideal \( \mu_P \) of a Smarandache BCI-algebra \( X \) must be a Smarandache fuzzy subalgebra of \( X \).

**Proof.** It follows from Theorem 3.5 and Theorem 3.10. \( \square \)

The converse of Theorem 3.10 may not be true as shown in the following example.

**Example 3.12.** Let \( X := \{0, 1, 2, 3, 4, 5\} \) be a Smarandache BCI-algebra with the following Cayley table:

\[
\begin{array}{cccccc}
* & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 \\
1 & 1 & 0 & 1 & 0 & 0 & 5 \\
2 & 2 & 2 & 0 & 0 & 0 & 5 \\
3 & 3 & 3 & 3 & 0 & 0 & 5 \\
4 & 4 & 3 & 4 & 1 & 0 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 0 \\
\end{array}
\]

Let \( \mu_P \) be a fuzzy set in \( P = \{0, 1, 2, 3, 4\} \) defined by \( \mu(0) = \mu(2) = 0.8 \) and \( \mu(1) = \mu(3) = \mu(4) = 0.3 \). It is easy to check that \( \mu_P \) is a fuzzy ideal of \( X \). Hence \( \mu : X \to [0, 1] \) is a Smarandache fuzzy ideal of \( X \). But it is not a Smarandache fuzzy clean ideal of \( X \) since \( \mu(1) = 0.3 \not\geq \min\{\mu((1 * (3 * 1)) * 2), \mu(2)\} = \min\{\mu(0), \mu(2)\} = 0.8 \).

**Theorem 3.13.** Let \( X \) be a Smarandache implicative BCI-algebra. Every Smarandache fuzzy ideal \( \mu_P \) of \( X \) is a Smarandache fuzzy clean ideal of \( X \).
Proof. Let $P$ be a $BCK$-algebra with $P \subseteq X$ and $|P| \geq 2$. Since $X$ is a Smarandache implicative $BCI$-algebra, we have $x = x * (y * x)$ for all $x, y \in P$. Let $\mu_P$ be a Smarandache fuzzy ideal of $X$. It follows from $(SF_2)$ that $\mu(x) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$, for all $x, y, z \in P$. Hence $\mu_P$ is a Smarandache clean ideal of $X$. The proof is complete. □

In what follows, we give characterizations of fuzzy implicative ideals.

Theorem 3.14. Let $X$ be a Smarandache $BCI$-algebra. Suppose that $\mu_P$ is a Smarandache fuzzy ideal of $X$. Then the following equivalent:

(i) $\mu_P$ is Smarandache fuzzy clean,
(ii) $\mu(x) \geq \mu((x * (y * x)))$ for all $x, y \in P$,
(iii) $\mu(x) = \mu((x * (y * x)))$ for all $x, y \in P$.

Proof. (i) $\Rightarrow$ (ii): Let $\mu_P$ be a Smarandache fuzzy clean ideal of $X$. It follows from $(SF_3)$ that $\mu(x) \geq \min\{\mu((x * (y * x)) * 0), \mu(0)\} = \min\{\mu((x * (y * x))), \mu(0)\} = \mu(x * (y * x))$, $\forall x, y \in P$. Hence the condition (ii) holds.

(ii) $\Rightarrow$ (iii): Since $X$ is a Smarandache $BCI$-algebra, we have $x * (y * x) \leq x$ for all $x, y \in P$. It follows from Lemma 3.4 that $\mu(x) \leq \mu((x * (y * x)))$. By (ii), $\mu(x) \geq \mu(x * (y * x))$. Thus the condition (iii) holds.

(iii) $\Rightarrow$ (i): Suppose that the condition (iii) holds. Since $\mu_P$ is a Smarandache fuzzy ideal, by $(SF_2)$, we have $\mu((x * (y * x))) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$. Combining (iii), we obtain $\mu(x) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$. Hence $\mu$ satisfies the condition $(SF_3)$. Obviously, $\mu$ satisfies $(SF_1)$. Therefore $\mu$ is a fuzzy clean ideal of $X$. Hence the condition (i) holds. The proof is complete. □

For any fuzzy sets $\mu$ and $\nu$ in $X$, we write $\mu \leq \nu$ if and only if $\mu(x) \leq \nu(x)$ for any $x \in X$.

Definition 3.15. Let $X$ be a Smarandache $BCI$-algebra and let $\mu_P : P \rightarrow [0, 1]$ be a Smarandache fuzzy $BCI$-algebra of $X$. For $t \leq \mu(0)$, the set $\mu_t := \{x \in P | \mu(x) \geq t\}$ is called a level subset of $\mu_P$.

Theorem 3.16. A fuzzy set $\mu$ in $P$ is a Smarandache fuzzy clean ideal of $X$ if and only if, for all $t \in [0, 1]$, $\mu_t$ is either empty or a Smarandache clean ideal of $X$.

Proof. Suppose that $\mu_P$ is a Smarandache fuzzy clean ideal of $X$ and $\mu_t \neq \emptyset$ for any $t \in [0, 1]$. It is clear that $0 \in \mu_t$ since $\mu(0) \geq t$. Let $\mu((x * (y * x)) * z) \geq t$ and $\mu(z) \geq t$. It follows from $(SF_3)$ that $\mu(x) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\} \geq t$, namely, $x \in \mu_t$. This shows that $\mu_t$ is a Smarandache clean ideal of $X$.

Conversely, assume that for each $t \in [0, 1]$, $\mu_t$ is either empty or a Smarandache clean ideal of $X$. For any $x \in P$, let $\mu(x) = t$. Then $x \in \mu_t$. Since $\mu_t(\neq \emptyset)$ is a Smarandache clean ideal of $X$, therefore $0 \in \mu_t$ and hence $\mu(0) \geq \mu(x) = t$. Thus $\mu(0) \geq \mu(x)$ for all $x \in P$. Now we show that $\mu$ satisfies $(SF_3)$. If not, then there exist $x', y', z' \in P$ such that $\mu(x') < \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}$. Taking $t_0 := \frac{1}{2}\{\mu(x') + \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}\}$, we have $\mu(x') < t_0 < \min\{\mu((x' * (y' * z')) * z'), \mu(z')\}$. Hence $x' \notin \mu_{t_0}$, $(x' * (y' * x')) * z \in \mu_{t_0}$, and $z' \in \mu_{t_0}$, i.e., $\mu_{t_0}$ is not a Smarandache clean ideal of $X$, which is a contradiction. Therefore, $\mu_P$ is a Smarandache fuzzy clean ideal, completing the proof. □

Theorem 3.17. ([2]) (Extension Property) Let $X$ be a Smarandache $BCI$-algebra. Let $I$ and $J$ be $Q$-Smarandache ideals of $X$ and $I \subseteq J \subseteq Q$. If $I$ is a $Q$-Smarandache clean ideal of $X$, then so is $J$.

Next we give the extension theorem of Smarandache fuzzy clean ideals.
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**Theorem 3.18.** Let $X$ be a Smarandache BCI-algebra. Let $\mu$ and $\nu$ be Smarandache fuzzy ideals of $X$ such that $\mu \leq \nu$ and $\mu(0) = \nu(0)$. If $\mu$ is a Smarandache fuzzy clean ideal of $X$, then so is $\nu$.

**Proof.** It suffices to show that for any $t \in [0,1]$, $\nu_t$ is either empty or a Smarandache clean ideal of $X$ if $\mu_t$ is a Smarandache fuzzy clean ideal of $X$. By the hypothesis, since $\mu$ is a Smarandache fuzzy clean ideal of $X$, $\mu_t$ is a Smarandache clean of $X$ by Theorem 3.16. It follows from Theorem 3.17 that $\nu_t$ is a Smarandache clean ideal of $X$. Hence $\nu$ is a Smarandache fuzzy clean of $X$. The proof is complete. \hfill \Box

**Definition 3.19.** Let $X$ be a Smarandache BCI-algebra. A map $\mu : X \rightarrow [0,1]$ is called a Smarandache fuzzy fresh ideal of $X$ if it satisfies $(SF_1)$ and

$$(SF_4) \quad \mu(x \ast z) \geq \min\{\mu((x \ast y) \ast z), \mu(y \ast z)\}$$

for all $x, y, z \in P$, where $P$ is a BCK-algebra with $P \subset X$ and $|P| \geq 2$. This Smarandache fuzzy ideal is denoted by $\mu_P$, i.e., $\mu_P : P \rightarrow [0,1]$ is a Smarandache fuzzy fresh ideal of $X$.

**Example 3.20.** Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra ([2]) with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</table>

Define a map $\mu : X \rightarrow [0,1]$ by

$$\mu(x) := \begin{cases} 0.5 & \text{if } x \in \{0, 1, 3\}, \\ 0.9 & \text{otherwise} \end{cases}$$

Clearly $\mu$ is a Smarandache fuzzy fresh ideal of $X$. But it is not a fuzzy fresh ideal of $X$, since $\mu(2 \ast 4) = \mu(0) = 0.5 \neq \min\{\mu((2 \ast 5) \ast 4), \mu(5 \ast 4)\} = \mu(5) = 0.9$.

**Theorem 3.21.** Any Smarandache fuzzy fresh ideal of a Smarandache BCI-algebra $X$ must be a Smarandache fuzzy ideal of $X$.

**Proof.** Taking $z := 0$ in $(SF_4)$ and $x \ast 0 = x$, we have $\mu(x \ast 0) \geq \min\{\mu((x \ast y) \ast 0), \mu(y \ast 0)\}$. Hence $\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\}$. Thus $(SF_2)$ holds. \hfill \Box

The converse of Theorem 3.21 may not be true as show in the following example.

**Example 3.22.** Let $X := \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI-algebra ([2]) with the following Cayley table:

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</table>
Proposition 3.23. Let \( X \) be a Smarandache BCI-algebra. A Smarandache fuzzy ideal \( \mu_P \) of \( X \) is a Smarandache fuzzy fresh ideal of \( X \) if and only if it satisfies the identity

\[
\mu((x \ast y) \ast y) = \mu(x \ast y) \ast y,
\]

for all \( x, y \in P \).

Proof. Assume that \( \mu_P \) is a Smarandache fuzzy fresh ideal of \( X \). Putting \( z := y \) in \( (SF_1) \), we have \( \mu(x \ast y) \geq \min\{\mu((x \ast y) \ast y), \mu(y \ast y)\} = \min\{\mu((x \ast y) \ast y), \mu(0)\} = \mu((x \ast y) \ast y), \forall x, y \in P \).

Conversely, let \( \mu_P \) be a Smarandache fuzzy ideal of \( X \) such that \( \mu(x \ast y) \geq \mu((x \ast y) \ast y) \). Since, for all \( x, y, z \in P \),

\[
((x \ast z) \ast z) \ast (y \ast z) \leq (x \ast z) \ast y = (x \ast y) \ast z,
\]

we have \( \mu((x \ast y) \ast z) \leq \mu(((x \ast z) \ast z) \ast (y \ast z)) \). Hence \( \mu(x \ast z) \geq \mu((x \ast y) \ast z) \geq \min\{\mu((x \ast z) \ast (y \ast z)), \mu(y \ast z)\} \geq \min\{\mu((x \ast y) \ast z), \mu(y \ast z)\} \). This completes the proof.

Since \( (x \ast y) \ast y \leq x \ast y \), it follows from Lemma 3.4 that \( \mu(x \ast y) \leq \mu((x \ast y) \ast y) \). Thus we have the following theorem.

Theorem 3.24. Let \( X \) be a Smarandache BCI-algebra. A Smarandache fuzzy ideal \( \mu_P \) of \( X \) is a Smarandache fuzzy fresh if and only if it satisfies the identity

\[
\mu(x \ast y) = \mu((x \ast y) \ast y), \text{ for all } x, y \in X.
\]

We give an equivalent condition for which a Smarandache fuzzy subalgebra of a Smarandache BCI-algebra to be a Smarandache fuzzy clean ideal of \( X \).

Theorem 3.25. A Smarandache fuzzy subalgebra \( \mu_P \) of \( X \) is a Smarandache fuzzy clean ideal of \( X \) if and only if it satisfies

\[
(x \ast (y \ast x)) \ast z \leq u \implies \mu(x) \geq \min\{\mu(z), \mu(u)\} \text{ for all } x, y, z, u \in P.
\]

Proof. Assume that \( \mu_P \) is a Smarandache fuzzy clean ideal of \( X \). Let \( x, y, z, u \in P \) be such that \( (x \ast (y \ast x)) \ast z \leq u \). Since \( \mu \) is a Smarandache fuzzy ideal of \( X \), we have \( \mu(x \ast (y \ast x)) \geq \min\{\mu(z), \mu(u)\} \) by Theorem 3.7. By Theorem 3.14-(iii), we obtain \( \mu(x) \geq \min\{\mu(z), \mu(u)\} \).

Conversely, suppose that \( \mu_P \) satisfies (\( \ast \)). Obviously, \( \mu_P \) satisfies \((SF_1)\), since \( (x \ast (y \ast x)) \ast ((x \ast (y \ast x)) \ast z) \leq z \), by (\( \ast \)), we obtain \( \mu(x) \geq \min\{\mu((x \ast (y \ast x)) \ast z), \mu(z)\} \), which shows that \( \mu_P \) satisfies \((SF_3)\). Hence \( \mu_P \) is a Smarandache fuzzy clean ideal of \( X \). The proof is complete.

REFERENCES

Smarandache fuzzy $BCI$-algebras