The $qq'$-Calculus

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Abstract

We present here a generalisation of the $q$-calculus, the $qq'$-calculus. The calculus is however limited.

1 The $\delta$-derivation

1.1 Definitions

The derivative of a function $f$ at the point $x$ is usually defined as:

$$d_h(f)(x) = f(x + h) - f(x)$$

$$\frac{df}{dx}(x) = f'(x) = \lim_{h \to 0} \frac{d_h(f)(x)}{h} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

if the limit exists.

Definition 1 Similarly, the $\delta$-derivative of a function is defined as:

$$\delta_{h,h'}(f)(x) = f(x + h) - f(x + h')$$

$$\frac{\delta f}{\delta x}(x) = \tilde{f}(x) = \lim_{h,h' \to 0} \frac{\delta_{h,h'}(f)}{\delta_{h,h'}(x)} = \lim_{h,h' \to 0} \frac{f(x + h) - f(x + h')}{h - h'} = \lim_{x_0,x_1 \to x} \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

If the $\delta$-derivative of a function exists, then the derivative of the function exists and we have $\frac{\delta f}{\delta x}(x) = \frac{df}{dx}(x)$.

1.2 A counter-example

The derivative can exist even if the $\delta$-derivative doesn’t. Indeed let be $f$ the function such that $f(x) = x^2$ if $x \in \mathbb{Q}$ and $f(x) = x^3$ if $x \notin \mathbb{Q}$. This function admits a derivative in zero which is zero, but has no $\delta$-derivative as one can verify:

$$\lim_{h,h' \to 0} \frac{h^2 - h'^3}{h - h'} = \lim_{h,h' \to 0} h + h' + \frac{h^2 - h'^3}{h - h'}$$

doesn’t exist because $h - h'$ can be as small as we want.
1.3 The Leibniz rule

The Leibniz rule can be verified:

\[
\frac{f(x+h)g(x+h) - f(x+h')g(x+h')}{h-h'} = \frac{f(x+h) - f(x+h')}{h-h'}g(x+h) + \frac{g(x+h) - g(x+h')}{h-h'}f(x+h')
\]

so that:

**Proposition 1**

\[
\frac{\delta (fg)}{\delta x} = (\frac{\delta f}{\delta x})g + f(\frac{\delta g}{\delta x})
\]

1.4 Some formulas

The following formulas can be easily verified:

\[
\frac{1}{f} = -\frac{1}{f^2} \frac{\delta f}{\delta x}
\]

\[
\frac{\frac{1}{g}}{g} = \frac{\delta g - g \delta \tilde{g}}{g^2}
\]

and also:

\[
\tilde{(f \circ g)} = (\tilde{f} \circ g) \times \tilde{g}
\]

1.5 \(\delta\)-derivative of a function of class \(C^1\)

**Theorem 1** If the function \(f\) is of class \(C^1\), then the \(\delta\)-derivative exists.

**Demonstration 1** By the Taylor’s formula

\[
\frac{f(x)}{x-x'} = f'(c), \quad c \in [x,x']
\]

So a smooth function is also infinitely \(\delta\)-derivable.

1.6 The \(qq'\)-limit

We have, if the limit exists, for \(x \neq 0\):

\[
\lim_{qq' \to 1} \frac{f(qx) - f(q'x)}{(q-q')x} = \tilde{f}(x)
\]

1.7 Integration and \(\delta\)-derivation

**Theorem 2** If \(f\) is continuous over the interval \([a,b]\), it is Riemann integrable and the primitive is \(\delta\)-derivable, so that we have:

\[
\frac{\delta}{\delta x} \int_a^x f(t)dt = f(x)
\]

**Demonstration 2**

\[
\frac{\delta}{\delta x} \int_a^x f(t)dt = \lim_{h,h' \to 0} \frac{\int_{h'}^h f(t+x)dt}{h-h'} = f(x)
\]

by the Taylor formula.

So, a function which is \(C^1\), is \(\delta\)-derivable.
2 \textbf{qq'-quantum derivation}

2.1 Definitions

\textbf{Definition 2} Let be two numbers \(q,q'\) and let be an arbitrary function \(f\), its \(qq'\)-differential is:

\[d_{qq'}(f)(x) = f(qx) - f(q'x)\]

In particular \(d_{qq'}x = (q - q')x\).

We have the following Leibniz rule:

\[d_{qq'}(fg)(x) = f(qx)g(qx) - f(q'x)g(q'x) = (f(qx) - f(q'x))g(qx) + f(q'x)g(qx) - g(q'x)\]

\textbf{Proposition 2}

\[d_{qq'}(fg)(x) = d_{qq'}(f)(x)g(qx) + f(q'x)d_{qq'}(g)(x)\]

\textbf{Definition 3} The following formula:

\[D_{qq'}f(x) = \frac{d_{qq'}(f(x))}{d_{qq'}(x)} = \frac{f(qx) - f(q'x)}{(q - q')x}\]

is called the \(qq'\)-derivative of the function \(f\)

2.2 The Leibniz rule

The Leibniz rule is:

\textbf{Proposition 3}

\[D_{qq'}(fg)(x) = D_{qq'}(f)(x)g(qx) + f(q'x)D_{qq'}(g)(x)\]

2.3 Some formulas

The \(qq'\)-derivative is a linear operator as we can verify:

\[D_{qq'}(af + bg) = aD_{qq'}(f) + bD_{qq'}(g)\]

for any scalars \(a, b\) and functions \(f, g\).

\textbf{Example 1}

\[D_{qq'}(x^n) = \left[n\right]_{qq'} x^{n-1}\]

with \(\left[n\right]_{qq'} = \frac{q^n - q'^n}{q - q'}\)

The number \(\left[n\right]_{qq'}\) is called the \(qq'\)-analog of \(n\) as \(\lim_{qq'\to 1}\left[n\right]_{qq'} = n\). We obtain also:

\textbf{Proposition 4}

\[D_{qq'} \left(\frac{f}{g}\right)(x) = \frac{g(q'x)D_{qq'}(f)(x) - f(q'x)D_{qq'}(g)(x)}{g(qx)g(q'x)} = \frac{g(qx)D_{qq'}(f)(x) - f(qx)D_{qq'}(g)(x)}{g(qx)g(q'x)}\]

For the composition we also have, if \(u = x^a\):

\textbf{Proposition 5}

\[D_{qq'}(f \circ u)(x) = (D_{qq'}^a(f) \circ u)(x) \times D_{qq'}(u)(x)\]
3 \( qq' \)-analogue of \((x - a)^n\)

3.1 Definition

Definition 4

\[ [0]_{qq'}! = 1 \]
\[ [n]_{qq'}! = [n]_{qq'} \times [n-1]_{qq'} \times \ldots \times [1]_{qq'} \quad \text{if } n \neq 0 \]

3.2 The exponential

Definition 5

\[ \exp_{qq'}(x) = \sum_{n \geq 0} \frac{x^n}{[n]_{qq'}!} \]

The derivative is:

\[ D_{qq'}(\exp_{qq'})(x) = \exp_{qq'}(x) \]

3.3 The \( qq' \)-analogue of \((x - a)^n\)

Definition 6

The \( qq' \)-analogue of \((x - a)^n\)

\[ (x - a)^{n_{qq'}} = \prod_{k,l, k+l = n-1} (x - q^k q'^l a) \]

We have the following theorem:

Theorem 3

\[ D_{qq'}(x - a)^{n_{qq'}} = [n]_{qq'} (x - a)^{n_{qq'}-1} \]

Demonstration 3

\[ (x - a)^{n_{qq'}} = (x - qa)^{n_{qq'}-1} (x - q^n a) \]

so that, by induction on \( n \), using Leibniz rule:

\[ D_{qq'}(x - a)^{n_{qq'}} = D_{qq'}(x - a)^{n_{qq'}-1} (q'x - q^{n-1} a) + (qx - qa)^{n_{qq'}-1} = \]
\[ = [n-1]_{qq'} q'(x - a)^{n_{qq'}-2} (x - q^{n-2} a) + q^{n-1} (x - a)^{n_{qq'}-1} = [n]_{qq'} (x - a)^{n_{qq'}-1} \]

We have also:

\[ (x - a)^{n_{qq'}+m} = (x - q^m a)^{n_{qq'}} (x - q^n a)^{m_{qq'}} \]

4 \( qq' \)-Taylor’s Formula for polynomials

4.1 The Taylor’s expansion

Theorem 4

For any polynomial \( P(X) \) of degree \( n \), and any number \( a \), we have the following \( qq' \)-Taylor expansion:

\[ P(x) = \sum_{j=0}^{n} (D_j^{'qq'} P)(a) \frac{(x - a)^j}{[j]_{qq'}!} \]
Demonstration 4 Due to the degree, we can write:

\[ P(x) = \sum_{j=0}^{n} c_j \frac{(x-a)^j}{[j]_{qq'}} \]

and now, by derivation, we have inductively on the degree of \( P \):

\[ c_k = (D^k_{qq'} P)(a) \]

4.2 A formula

The \( qq' \)-Taylor formula for \( x^n \) about \( x = 1 \) then gives:

\[ x^n = \sum_{j=0}^{n} [n]_{qq'} \ldots [n-j+1]_{qq'} \frac{(x-a)^j}{[j]_{qq'}} \]

Formula 1

\[ x^n = \sum_{j=0}^{n} \left[ \frac{n}{j} \right]_{qq'} (x-a)^j \]

with \( \left[ \frac{n}{j} \right]_{qq'} = \frac{[n]_{qq'}}{[j]_{qq'} [n-j]_{qq'}} \).

References
