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Bonferroni Mean Operators of Linguistic Neutrosophic Numbers and Their Multiple Attribute Group Decision-Making Methods

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Abstract: Linguistic neutrosophic numbers (LNN) is presented by Fang and Ye in 2017, which can describe the truth, falsity, and indeterminacy linguistic information independently. In this paper, the LNN and the Bonferroni mean operator are merged together to propose a LNN normalized weighted Bonferroni mean (LNNNWBM) operator and a LNN normalized weighted geometric Bonferroni mean (LNNNWGBM) operator and the properties of these two operators are proved. Further, multi-attribute group decision methods are introduced based on the proposed LNNNWBM and LNNNWGBM operators, and then an example is provided to demonstrate the application and validity of the proposed methods. In addition, in order to consider the effect of the parameters p and q on the decision results, different pairs of parameter values are employed to verify the decision results.

Keywords: linguistic neutrosophic numbers (LNN); LNN normalized weighted Bonferroni mean (LNNNWBM) operator; LNN normalized weighted geometric Bonferroni mean (LNNNWGBM) operator; group decision-making

1. Introduction

In dealing with the complex, unknown, and uncertain decision-making problems, a group of decision-makers are usually employed to analyze a set of alternatives and to get the optimal result in a certain way. Such a decision-making process is called multiple attribute group decision-making (MAGDM) problem. When making decisions, decision-makers tend to use words such as "excellent", "good", and "poor" to express their evaluations for objects. Zadeh proposed a linguistic variable set $S = \{S_0, S_1, S_2, S_3, \ldots S_g\}$ (g is an even number) to deal with the approximate reasoning problems [1,2]. The linguistic variable is an effective tool, it improves the reliability and flexibility of classical decision models [3,4]. In recent years, the linguistic variables have been frequently linked to other theories. Liu proposed the intuitionistic linguistic set (ILS) composed of linguistic variables and IFS, where the first component provides its qualitative evaluation value/linguistic value and the second component gives the credibility of its intuitionistic fuzzy value for the given linguistic value [5]. Then, Chen et al. proposed the linguistic intuitionistic fuzzy number (LIFN), which is composed of the intuitionistic fuzzy number (the basic element in IFS) and the linguistic variable [6]. On the other hand, some methods for multiple attribute group decision-making (MAGDM) were proposed based on two-dimension uncertain linguistic variable [7,8]. Some improved linguistic intuitionistic fuzzy aggregation operators and several corresponding applications were given in decision-making [9]. Although the IFS theory considers not only $T(x)$, but also $F(x)$, IFS is still not perfect enough because it ignores the indeterminate and inconsistent information. Thus, the intuitionistic fuzzy number can only be used for expressing incomplete information, but not for expressing indeterminate and inconsistent information. To make up for the insufficiency of the IFS theory, Smarandache put forward the neutrosophic set (NS) composed of three parts: truth $T(x)$, falsity $F(x)$, and indeterminacy $I(x)$ [10,11]. Wang et al. and Smarandache also proposed the concept of a single-valued neutrosophic set (SVNS) satisfying $T(x)$, $I(x)$, $F(x) \subseteq [0, 1]$, $0 \leq T(x) + F(x) + I(x) \leq 3$ [10–12]. Ye proposed an extended TOPSIS (technique for order preference by similarity to an ideal Solution) method for MAGDM based on single valued neutrosophic linguistic numbers (SVNLNs), which are basic elements in a single-valued neutrosophic linguistic set (SVNLS) [13]. Liu and Shi presented some neutrosophic uncertain linguistic number Heronian mean operators and their application to MAGDM [14]. Since the Bonferroni mean (BM) is a useful operator in decision-making [15], it was extended to hesitant fuzzy sets, IFSs, and interval-valued IFSs to propose their some Bonferroni mean operators for decision making [16–20]. Then, Fang and Ye proposed the linguistic neutrosophic numbers (LNN) and their basic operational laws [21]. LNN consists of the truth, indeterminacy, and falsity linguistic degrees, which can be expressed as the form $a = <l_T, l_I, l_F>$, but the LIFN and SVNLN cannot express such linguistic evaluation value. In [21], Fang and Ye also presented a LNN-weighted arithmetic averaging (LNNWAA) operator and a LNN-weighted geometric averaging (LNNWGA) operator for MAGDM. However, the Bonferroni mean operator is not extended to LNNs so far. Hence, this paper proposes a LNN normalized weighted Bonferroni mean (LNNNWBM) operator, a LNN normalized weighted geometric Bonferroni mean (LNNNWGBM) operator and their MAGDM methods. Compared with the aggregation operators in [14,21], the LNNNWBM and LNNNWGBM operators can calculate the final weights by the relation between attribute values, which can make the information aggregation more objective and reliable.

The rest organizations of this paper are as follows. Section 2 describes some basic concepts of LNN, the basic operational laws of LNNs, and the basic concepts of BM and the normalized weighted BM. Section 3 proposes the LNNNWBM and LNNNWGBM operators and investigates their properties. Section 4 establishes MAGDM methods by using the LNNNWBM operator and LNNNWGBM operator. Section 5 provides an illustrative example with different values of the parameters p and q to demonstrate the application of the proposed methods. Section 6 gives conclusions.

2. Some Concepts of LNNs and BM

2.1. Linguistic Neutrosophic Numbers and Their Operational Laws

Definition 1 [21]. *Set $L = \{l_0, l_1, l_2, \ldots, l_g\}$ as a language term set, in which g is an even number and $g + 1$ is the particle size of L. If $a = \langle l_T, l_I, l_F \rangle$ is defined for $l_T, l_I, l_F \in L$ and $T, I, F \in [0, g]$, where l_T expresses the truth degree, l_I expresses indeterminacy degree, and l_F expresses falsity degree by linguistic terms, then a is called an LNN.*

Definition 2 [21]. *Set $a = \langle l_T, l_I, l_F \rangle$, $a_1 = \langle l_{T_1}, l_{I_1}, l_{F_1} \rangle$, and $a_2 = \langle l_{T_2}, l_{I_2}, l_{F_2} \rangle$ as three LNNs in L, the number $\lambda \geq 0$, they have the follow operational laws:*

$$a_1 \oplus a_2 = \langle l_{T_1}, l_{I_1}, l_{F_1} \rangle \oplus \langle l_{T_2}, l_{I_2}, l_{F_2} \rangle = \langle l_{T_1+T_2-\frac{T_1T_2}{g}}, l_{\frac{I_1I_2}{g}}, l_{\frac{F_1F_2}{g}} \rangle; \quad (1)$$

$$a_1 \otimes a_2 = \langle l_{T_1}, l_{I_1}, l_{F_1} \rangle \otimes \langle l_{T_2}, l_{I_2}, l_{F_2} \rangle = \langle l_{\frac{T_1T_2}{g}}, l_{I_1+I_2-\frac{I_1I_2}{g}}, l_{F_1+F_2-\frac{F_1F_2}{g}} \rangle; \quad (2)$$

$$\lambda a = \lambda \langle l_T, l_I, l_F \rangle = \langle l_{g-g(1-\frac{T}{g})^\lambda}, l_{g(\frac{I}{g})^\lambda}, l_{g(\frac{F}{g})^\lambda} \rangle; \quad (3)$$

$$a^\lambda = \langle l_T, l_I, l_F \rangle^\lambda = \langle l_{g(\frac{T}{g})^\lambda}, l_{g-g(1-\frac{I}{g})^\lambda}, l_{g-g(1-\frac{F}{g})^\lambda} \rangle. \quad (4)$$

Definition 3 [21]. *Set $a = \langle l_T, l_I, l_F \rangle$ as an LNN in L, then the expectation $E(a)$ and the accuracy $H(a)$ can be defined as follows:*

$$E(a) = (2g + T - I - F)/3g \tag{5}$$

$$H(a) = (T - F)/g \tag{6}$$

Definition 4 [21]. *Set* $a_1 = \langle l_{T_1}, l_{I_1}, l_{F_1} \rangle$ *and* $a_2 = \langle l_{T_2}, l_{I_2}, l_{F_2} \rangle$ *as two LNNs, then:*

If $E(a_1) > E(a_2)$*, then* $a_1 \succ a_2$*;*
If $E(a_1) = E(a_2)$ *then*
If $H(a_1) > H(a_2)$*, then* $a_1 \succ a_2$*;*
If $H(a_1) = H(a_2)$*, then* $a_1 \sim a_2$*;*
If $H(a_1) < H(a_2)$*, then* $a_1 \prec a_2$*.*

2.2. Bonferroni Mean Operators

Definition 5 [15]. *Let* $(a_1, a_2, \dots a_n)$ *be a set of non-negative numbers, the function BM:* $R^n \to R$*. If* $p, q \geq 0$ *and BM satisfies:*

$$BM^{p,q}(a_1, a_2, \dots a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ j \neq i}}^{n} a_i{}^p \, a_j{}^q \right)^{\frac{1}{p+q}} \tag{7}$$

then $BM^{p,q}$ *is called a BM operator.*

Definition 6 [16]. *Let* $(a_1, a_2, \dots a_n)$ *be a set of non-negative numbers, the function NWBM:* $R^n \to R$*,* w_i $(i = 1,2,\dots,n)$ *be the relative weight of* a_i $(i = 1,2,\dots,n)$*,* $w_i \in [0,1]$*, and* $\sum_{i=1}^{n} w_i = 1$*. If* $p, q \geq 0$ *and NWBM satisfies:*

$$NWBM^{p,q}(a_1, a_2, \dots a_n) = \left(\sum_{\substack{i,j=1 \\ j \neq i}}^{n} \frac{w_i w_j}{1 - w_i} a_i{}^p \, a_j{}^q \right)^{\frac{1}{p+q}} \tag{8}$$

then $NWBM^{p,q}$ *is called a normalized weighted BM operator.*

Definition 7 [17]. *Let* $(a_1, a_2, \dots a_n)$ *be a set of non-negative numbers, the function GBM:* $R^n \to R$*. If* $p, q \geq 0$ *and GBM satisfies:*

$$GBM^{p,q}(a_1, a_2, \dots a_n) = \left(\frac{1}{n} \sum_{i}^{n} a_i{}^p \left(\prod_{j=1, j \neq i}^{n} a_j{}^q \right)^{\frac{1}{n-1}} \right)^{\frac{1}{p+q}} \tag{9}$$

then $GBM^{p,q}$ *is called a geometric BM operator.*

Definition 8 [18–20]. *Let* $(a_1, a_2, \dots a_n)$ *be a set of non-negative numbers, the function NWGBM:* $R^n \to R$*,* w_i $(i = 1,2,\dots,n)$ *be the relative weight of* a_i $(i = 1,2,\dots,n)$*,* $w_i \in [0,1]$*, and* $\sum_{i=1}^{n} w_i = 1$*. If* $p, q \geq 0$ *and NWGBM satisfies:*

$$NWGBM^{p,q}(a_1, a_2, \dots a_n) = \frac{1}{p+q} \otimes_{i,j=1, j \neq i}^{n} (pa_i \oplus qa_j)^{\frac{w_i w_j}{1 - w_i}} \tag{10}$$

then $NWGBM^{p,q}$ *is called a normalized weighted geometric BM (NWGBM) operator.*

3. Two BM Aggregation Operators of LNNs

3.1. Normalized Weighted BM Operators of LNNs

Definition 9. *Set* $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ $(i = 1,2,\dots,n)$ *as a collection of LNNs in L, then the LNNNWBM operator can be defined as follows:*

$$LNNNWBM^{p,q}(a_1, a_2, \dots a_n) = \left(\sum_{\substack{i,j=1 \\ j \neq i}}^{n} \frac{w_i w_j}{1 - w_i} a_i{}^p \, a_j{}^q \right)^{\frac{1}{p+q}} \tag{11}$$

where w_i *is the relative weight of* a_i, $w_i \in [0,1]$, *and* $\sum_{i=1}^{n} w_i = 1$, w_j *is the relative weight of* a_j, $w_j \in [0,1]$, *and* $\sum_{j=1}^{n} w_j = 1$.

According to Definitions 2 and 9, we can get the following theorem:

Theorem 1. *Set* $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ *(i = 1,2,…,n) as a collection of LNNs in L, then by the Equation (11), the aggregation result obtained is still an LNN, and we can get the following aggregation formula:*

$$LNNNWBM^{p,q}(a_1, a_2, \ldots a_n) = \left(\sum_{\substack{i,j=1 \\ j \neq i}}^{n} \frac{w_i w_j}{1-w_i} a_i^{\,p} a_j^{\,q} \right)^{\frac{1}{p+q}} =$$

$$\langle l_{g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(\frac{T_i}{g}\right)^p\left(\frac{T_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g-g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\frac{I_i}{g}\right)^p\left(1-\frac{I_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g-g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\frac{F_i}{g}\right)^p\left(1-\frac{F_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}} \rangle \quad (12)$$

where w_i *is the relative weight of* a_i, $w_i \in [0,1]$, *and* $\sum_{i=1}^{n} w_i = 1$, w_j *is the relative weight of* a_j, $w_j \in [0,1]$, *and* $\sum_{j=1}^{n} w_j = 1$.

Proof 1:

(1) $a_i^{\,p} = \langle l_{g(\frac{T_i}{g})^p}, l_{g-g(1-\frac{I_i}{g})^p}, l_{g-g(1-\frac{F_i}{g})^p} \rangle$;

(2) $a_j^{\,q} = \langle l_{g(\frac{T_j}{g})^q}, l_{g-g(1-\frac{I_j}{g})^q}, l_{g-g(1-\frac{F_j}{g})^q} \rangle$;

(3) $a_i^{\,p} \otimes a_j^{\,q} =$

$$\langle l_{\frac{g(\frac{T_i}{g})^p g(\frac{T_j}{g})^q}{g}}, l_{g-g\left(1-\frac{I_i}{g}\right)^p+g-g\left(1-\frac{I_j}{g}\right)^q-\frac{\left(g-g\left(1-\frac{I_i}{g}\right)^p\right)\left(g-g\left(1-\frac{I_j}{g}\right)^q\right)}{g}}, l_{g-g\left(1-\frac{F_i}{g}\right)^p+g-g\left(1-\frac{F_j}{g}\right)^q-\frac{\left(g-g\left(1-\frac{F_i}{g}\right)^p\right)\left(g-g\left(1-\frac{F_j}{g}\right)^q\right)}{g}} \rangle$$

$$=\langle l_{g(\frac{T_i}{g})^p(\frac{T_j}{g})^q}, l_{g-g\left(1-\frac{I_i}{g}\right)^p\left(1-\frac{I_j}{g}\right)^q}, l_{g-g\left(1-\frac{F_i}{g}\right)^p\left(1-\frac{F_j}{g}\right)^q} \rangle$$

(4) $\frac{w_iw_j}{1-w_i} a_i^{\,p} \otimes a_j^{\,q} = \langle l_{g-g\left(1-\frac{g\left(\frac{T_i}{g}\right)^p\left(\frac{T_j}{g}\right)^q}{g}\right)^{\frac{w_iw_j}{1-w_i}}}, l_{g\left(\frac{g-g\left(1-\frac{I_i}{g}\right)^p\left(1-\frac{I_j}{g}\right)^q}{g}\right)^{\frac{w_iw_j}{1-w_i}}}, l_{g\left(\frac{g-g\left(1-\frac{F_i}{g}\right)^p\left(1-\frac{F_j}{g}\right)^q}{g}\right)^{\frac{w_iw_j}{1-w_i}}} \rangle$

$$=\langle l_{g-g\left(1-\left(\frac{T_i}{g}\right)^p\left(\frac{T_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}}, l_{g\left(1-\left(1-\frac{I_i}{g}\right)^p\left(1-\frac{I_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}}, l_{g\left(1-\left(1-\frac{F_i}{g}\right)^p\left(1-\frac{F_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}} \rangle$$

(5) $\oplus_{i=1}^{n} \oplus_{j=1, j\neq i}^{n} \frac{w_iw_j}{1-w_i} a_i^{\,p} \otimes a_j^{\,q} =$

$$\langle l_{g-g\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(\frac{T_i}{g}\right)^p\left(\frac{T_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}}, l_{g\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\frac{I_i}{g}\right)^p\left(1-\frac{I_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}}, l_{g\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\frac{F_i}{g}\right)^p\left(1-\frac{F_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}} \rangle$$

(6) $(\oplus_{i=1}^{n} \oplus_{j=1, j\neq i}^{n} \frac{w_iw_j}{1-w_i} a_i^{\,p} \otimes a_j^{\,q})^{\frac{1}{p+q}} =$

$$\langle l_{g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(\frac{T_i}{g}\right)^p\left(\frac{T_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g-g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\frac{I_i}{g}\right)^p\left(1-\frac{I_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g-g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\frac{F_i}{g}\right)^p\left(1-\frac{F_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}} \rangle$$

So $\left(\sum_{\substack{i,j=1 \\ j \neq i}}^{n} \frac{w_i w_j}{1-w_i} a_i^{\,p} a_j^{\,q} \right)^{\frac{1}{p+q}} =$

$$\langle l_{g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(\frac{T_i}{g}\right)^p\left(\frac{T_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g-g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\frac{I_i}{g}\right)^p\left(1-\frac{I_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g-g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\left(1-\frac{F_i}{g}\right)^p\left(1-\frac{F_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}} \rangle$$

The proof of Theorem 1 is completed. □

Theorem 2. *(Idempotency). Set* $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ *(i = 1,2,…,n) as a collection of LNNs in L, if* $a_i = a$*, then*

$$LNNNWBM^{p,q}(a_1, a_2, \ldots, a_n) = LNNNWBM^{p,q}(a, a \ldots a) = a.$$

Proof 2:

Since $a_i = a$, i.e., $T_i = T$; $I_i = I$; $F_i = F$ for $i = 1,2, \ldots, n$, there are the following result:

$$LNNNWBM^{p,q}(a_1, a_2, \dots a_n) = LNNNWBM^{p,q}\ (a, a \dots a)\ = (\sum_{\substack{i,j=1 \\ j \neq i}}^{n} \frac{w_i w_j}{1-w_i} a^p\, a^q)^{\frac{1}{p+q}}$$

$$= \left\langle l_{g\left(1-\left(1-\left(\frac{T}{g}\right)^p\left(\frac{T}{g}\right)^q\right)^{\sum_{i=1}^n \sum_{j=1, j\neq i}^n \frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g-g\left(1-\left(1-\left(1-\frac{I}{g}\right)^p\left(1-\frac{I}{g}\right)^q\right)^{\sum_{i=1}^n \sum_{j=1, j\neq i}^n \frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g-g\left(1-\left(1-\left(1-\frac{F}{g}\right)^p\left(1-\frac{F}{g}\right)^q\right)^{\sum_{i=1}^n \sum_{j=1, j\neq i}^n \frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}} \right\rangle$$

$$= \left\langle l_{g\left(1-\left(1-\left(\frac{T}{g}\right)^{p+q}\right)\right)^{\frac{1}{p+q}}}, l_{g-g\left(1-\left(1-\left(1-\frac{I}{g}\right)^{p+q}\right)\right)^{\frac{1}{p+q}}}, l_{g-g\left(1-\left(1-\left(1-\frac{F}{g}\right)^{p+q}\right)\right)^{\frac{1}{p+q}}} \right\rangle$$

$$= \langle l_T, l_I, l_F \rangle\ = a.$$

The proof of Theorem 2 is completed. □

Theorem 3. *(Monotonicity). Set* $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ *and* $b_i = \langle l_{T_i'}, l_{I_i'}, l_{F_i'} \rangle$ *(i = 1,2,…, n) as two collections of LNNs in L, if* $T_i \leq T_i', I_i \geq I_i', and\ F_i \geq F_i'$ *then* $LNNNWBM^{p,q}(a_1, a_2, \dots a_n) \leq LNNNWBM^{p,q}\ (b_1, b_2, \dots b_n)$.

Proof 3:

Since $T_i \leq T_i', I_i \geq I_i'\ and\ F_i \geq F_i'$, we can easy obtain:

$$1 - \left(\frac{T_i}{g}\right)^p \left(\frac{T_j}{g}\right)^q \geq 1 - \left(\frac{T_i'}{g}\right)^p \left(\frac{T_j'}{g}\right)^q,$$

$$1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j\neq i}}^n \left(1 - \left(\frac{T_i}{g}\right)^p \left(\frac{T_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}} \leq 1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j\neq i}}^n \left(1 - \left(\frac{T_i'}{g}\right)^p \left(\frac{T_j'}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}},$$

$$g\left(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j\neq i}}^n \left(1 - \left(\frac{T_i}{g}\right)^p \left(\frac{T_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}} \leq g\left(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j\neq i}}^n \left(1 - \left(\frac{T_i'}{g}\right)^p \left(\frac{T_j'}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}.$$

Similarly

$$\left(1 - \frac{I_i}{g}\right)^p \left(1 - \frac{I_j}{g}\right)^q \leq \left(1 - \frac{I_i'}{g}\right)^p \left(1 - \frac{I_j'}{g}\right)^q,$$

$$1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j\neq i}}^n \left(1 - \left(1 - \frac{I_i}{g}\right)^p \left(1 - \frac{I_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}} \leq 1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j\neq i}}^n \left(1 - \left(1 - \frac{I_i'}{g}\right)^p \left(1 - \frac{I_j'}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}},$$

$$g - g\left(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j\neq i}}^n \left(1 - \left(1 - \frac{I_i}{g}\right)^p \left(1 - \frac{I_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}$$

$$\geq g - g\left(1 - \prod_{i=1}^n \prod_{\substack{j=1 \\ j\neq i}}^n \left(1 - \left(1 - \frac{I_i'}{g}\right)^p \left(1 - \frac{I_j'}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}$$

and

$$g-g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\left(1-\left(1-\frac{F_i}{g}\right)^p\left(1-\frac{F_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}$$

$$\geq g-g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\left(1-\left(1-\frac{F_i{}'}{g}\right)^p\left(1-\frac{F_j{}'}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}.$$

So, $LNNNWBM^{p,q}(a_1,a_2,\dots a_n) \leq LNNNWBM^{p,q}\ (b_1,b_2,\dots b_n)$ is true according to Theorem 3. Therefore, the proof of Theorem 3 is completed. □

Theorem 4. *(Boundedness). Set* $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i}\rangle$ *(i = 1,2,…,n) as a collections of LNNs in L, let* $a^- = \langle min(l_{T_i}), max(l_{I_i}), max(l_{F_i})\rangle$ *and* $a^+ = \langle max(l_{T_i}), min(l_{I_i}), min(l_{F_i})\rangle$*, then:*

$$a^- \leq LNNNWBM^{p,q}(a_1,a_2,\dots a_n) \leq a^+$$

Proof 4:

According Theorem 2, we can obtain:

$a^- = LNNNWB^{p,q}(a^-,a^- \dots a^-)$ and $a^+ = LNNNWBM^{p,q}(a^+,a^+ \dots a^+)$

According Theorem3, we can obtain:

$LNNNWBM^{p,q}\ (a^-,a^- \dots a^-) \leq LNNNWBM^{p,q}(a_1,a_2,\dots a_n) \leq LNNNWBM^{p,q}\ (a^+,a^+ \dots a^+)$.

Then $a^- \leq LNNNWBM^{p,q}(a_1,a_2,\dots a_n) \leq a^+$.

The proof of Theorem 4 is completed. □

3.2. Normalized Weighted Geometric BM Operators of LNNs

Definition 10. *Set* $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i}\rangle$ *(i = 1,2,…,n) as a collection of LNNs in L, then the LNNNWGBM operator can be defined as follows:*

$$LNNNWGBM^{p,q}\ (a_1,a_2,\dots a_n) = \frac{1}{p+q}\otimes_{i=1}^{n}\otimes_{j=1,j\neq i}^{n}(pa_i\oplus qa_j)^{\frac{w_iw_j}{1-w_i}} \tag{13}$$

where w_i *is the relative weight of* a_i, $w_i \in [0,1]$, *and* $\sum_{i=1}^{n} w_i = 1$, w_j *is the relative weighted of* a_j, $w_j \in [0,1]$, *and* $\sum_{j=1}^{n} w_j = 1$.

According to Definitions 2 and 10, we can get the following theorem:

Theorem 5. *Set* $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i}\rangle$ *(i = 1,2,…,n) as a collection of LNNs in L, then by the Equation (13) the aggregation result obtained is still an LNN, and we can get the following aggregation formula:*

$$LNNNWGBM^{p,q}\ (a_1,a_2,\dots a_n) = \frac{1}{p+q}\otimes_{i=1}^{n}\otimes_{j=1,j\neq i}^{n}(pa_i\oplus qa_j)^{\frac{w_iw_j}{1-w_i}} = \langle l_{g-g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\left(1-\left(1-\frac{T_i}{g}\right)^p\left(1-\frac{T_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\left(1-\left(\frac{I_i}{g}\right)^p\left(\frac{I_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\left(1-\left(\frac{F_i}{g}\right)^p\left(\frac{F_j}{g}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)} \tag{14}$$

where w_i *is the relative weight of* a_i, $w_i \in [0,1]$, *and* $\sum_{i=1}^{n} w_i = 1$, w_j *is the relative weighted of* a_j, $w_j \in [0,1]$, *and* $\sum_{j=1}^{n} w_j = 1$.

The proof of Theorem 5 is similar to that of Theorem 1, so we do not repeat it again.

Theorem 6. *(Idempotency). Set* $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i}\rangle$ *(i = 1,2,…,n) as a collection of LNNs in L, if* $a_i = a$*, then*

$$LNNNWGBM^{p,q}\ (a_1,a_2,\dots a_n) = LNNNWGBM^{p,q}\ (a,a \dots a) = a$$

The proof of Theorem 6 is similar to that of Theorem 2, so we don't repeat it again.

Theorem 7. *(Monotonicity). Set* $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ *and* $b_i = \langle l_{T_i'}, l_{I_i'}, l_{F_i'} \rangle$ *(i = 1,2,…,n) as two collections of LNNs in L, if* $T_i \le T_i', I_i \ge I_i'$ *and* $F_i \ge F_i'$ *then:*

$$LNNNWGBM^{p,q}(a_1, a_2, \ldots a_n) \le LNNNWGBM^{p,q}\,(b_1, b_2, \ldots b_n)$$

The proof of Theorem 7 is similar to that of Theorem 3, so we do not repeat it again.

Theorem 8. *(Boundedness). Set* $a_i = \langle l_{T_i}, l_{I_i}, l_{F_i} \rangle$ *(i = 1,2,…,n) as a collections of LNNs in L, let* $a^- = \langle min(l_{T_i}), max(l_{I_i}), max(l_{F_i}) \rangle$ *and* $a^+ = \langle max(l_{T_i}), min(l_{I_i}), min(l_{F_i}) \rangle$*, then:*

$$a^- \le LNNNWGBM^{p,q}(a_1, a_2, \ldots a_n) \le a^+$$

The proof of Theorem 8 is similar to that of Theorem 4, so we do not repeat it again.

4. MAGDM Methods Based on the LNNNWBM or LNNNWGBM Operator

In this section, we will use the LNNNWBM or LNNNWGBM operator to deal with the MAGDM problems with LNN information.

In a MAGDM problem, there is a set of several alternatives $A = \{A_1, A_2, \ldots, A_m\}$ with a set of some attributes $C = \{C_1, C_2, \ldots, C_n\}$. Then, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ with $\lambda_i \ge 0$ and $\sum_{i=1}^{n} \lambda_i = 1$ are the weights of $C_i (i = 1,2, \ldots, n)$. Now, there is a set of t experts $E = \{E_1, E_2, \ldots, E_t\}$ to evaluate the MAGDM problem. Assume that $w = (w_1, w_2, \ldots, w_t)^T$ with $w_j \ge 0$ and $\sum_{j=1}^{t} w_j = 1$ is the vector of the weights for $E_y (y = 1,2, \ldots, t)$ and $L = \{l_1, l_2, \ldots, l_g\}$ *is* the given linguistic term set. The assessed value of the expert E_y for A_i with attribute C_j is $a_{ij}^{(y)} = < l_{T_{ij}}^{y}, l_{I_{ij}}^{y}, l_{F_{ij}}^{y} > \in A (y = 1,2, \ldots, t;\ i = 1,2, \ldots, m; j = 1,2, \ldots, n)$, $l_{T_{ij}}^{y}, l_{I_{ij}}^{y}, l_{F_{ij}}^{y} \in L$. Then, we can get the neutrosophic linguistic decision evaluation matrix R^y, which is shown in Table 1.

Table 1. The neutrosophic linguistic decision matrix R^y of the expert E_y.

	C_1	…	C_n
A_1	$\langle l_{T_{11}}^{y}, l_{I_{11}}^{y}, l_{F_{11}}^{y} \rangle$	…	$\langle l_{T_{1n}}^{y}, l_{I_{1n}}^{y}, l_{F_{1n}}^{y} \rangle$
A_2	$\langle l_{T_{21}}^{y}, l_{I_{21}}^{y}, l_{F_{21}}^{y} \rangle$	…	$\langle l_{T_{2n}}^{y}, l_{I_{2n}}^{y}, l_{F_{2n}}^{y} \rangle$
…	…	…	…
A_m	$\langle l_{T_{m1}}^{y}, l_{I_{m1}}^{y}, l_{F_{m1}}^{y} \rangle$	…	$\langle l_{T_{mn}}^{y}, l_{I_{mn}}^{y}, l_{F_{mn}}^{y} \rangle$

Then, based on the *LNNNWBM* or *LNNNWGBM* operator, we propose two decision-making methods, which are described as the following decision steps:

Step 1: According to the weight vector $w = (w_1, w_2, \ldots, w_t)^T$ of experts and the *LNNNWBM* operator, we can obtain the integrated matrix $R = (a_{ij})_{m \times n}$, where the collective LNN a_{ij} can be obtained by the following formula:

$$\begin{aligned} a_{ij} &= LNNNWBM(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^t) = \left(\sum_{\substack{i,j=1 \\ j \ne i}}^{t} \frac{w_i w_j}{1 - w_i} a_i{}^p\, a_j{}^q \right)^{\frac{1}{p+q}} \\ &= \langle l_{g\left(1 - \prod_{i=1}^{t} \prod_{\substack{j=1 \\ j \ne i}}^{t} \left(1 - \left(\frac{T_i}{g}\right)^p \left(\frac{T_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g - g\left(1 - \prod_{i=1}^{t} \prod_{\substack{j=1 \\ j \ne i}}^{t} \left(1 - \left(1 - \frac{I_i}{g}\right)^p \left(1 - \frac{I_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g - g\left(1 - \prod_{i=1}^{t} \prod_{\substack{j=1 \\ j \ne i}}^{t} \left(1 - \left(1 - \frac{F_i}{g}\right)^p \left(1 - \frac{F_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}} \rangle \end{aligned} \quad (15)$$

Step 2: According to the weight vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ of attributes and the *LNNNWBM* operator or the *LNNNWGBM* operator, we can obtain the total collective LNN a_i for $A_i (i = 1,2, \ldots, m)$.

$$a_i = LNNNWBM(a_{i1}, a_{i2}, \dots, a_{in}) = \left(\sum_{\substack{i,j=1 \\ j \neq i}}^{n} \frac{w_i w_j}{1 - w_i} a_i{}^p\, a_j{}^q \right)^{\frac{1}{p+q}}$$
$$= \left\langle l_{g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1 \\ j\neq i}}^{n}\left(1-\left(\frac{T_i}{g}\right)^p\left(\frac{T_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g-g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1 \\ j\neq i}}^{n}\left(1-\left(1-\frac{I_i}{g}\right)^p\left(1-\frac{I_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g-g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1 \\ j\neq i}}^{n}\left(1-\left(1-\frac{F_i}{g}\right)^p\left(1-\frac{F_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}} \right\rangle \quad (16)$$

or:

$$a_i = LNNNWGBM(a_{i1}, a_{i2}, \dots, a_{in}) = \frac{1}{p+q} \otimes_{i=1}^{n} \otimes_{j=1, j\neq i}^{n} (pa_i \oplus qa_j)^{\frac{w_i w_j}{1-w_i}}$$
$$= \left\langle l_{g-g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1 \\ j\neq i}}^{n}\left(1-\left(1-\frac{T_i}{g}\right)^p\left(1-\frac{T_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1 \\ j\neq i}}^{n}\left(1-\left(\frac{I_i}{g}\right)^p\left(\frac{I_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, l_{g\left(1-\prod_{i=1}^{n}\prod_{\substack{j=1 \\ j\neq i}}^{n}\left(1-\left(\frac{F_i}{g}\right)^p\left(\frac{F_j}{g}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}} \right\rangle \quad (17)$$

Step 3: According to the Equation (5) (Equation (6) if necessary), we calculate the expected value $E(a_i)$ and the accuracy $H(a_i)$ of the LNN $A_i (i = 1,2, \dots, m)$.

Step 4: According to the value $E(a_i)$ ($H(a_i)$ if necessary), then we can rank the alternatives and choose the best one.

5. Illustrative Examples

The decision-making problem used in the literature [21] is considered here. There are four companies as a set of alternatives $A = \{A_1, A_2, A_3, A_4\}$, which are a car company (A_1), a food company (A_2), a computer company (A_3), and an arm company (A_4). An investment company needs to invest the best company, so they invite a set of three experts $E = \{E_1, E_2, E_3\}$ to evaluate these four companies. The evaluation of the alternatives must satisfy a set of three attributes $C = \{C_1, C_2, C_2\}$, which are the risk (C_1), the growth (C_2), and the environmental impact (C_3). The importance of three experts is given as a weight vector $w = (0.37, 0.33, 0.3)^T$ and the importance of three attributes is given as a weight vector $\lambda = (0.35, 0.25, 0.4)^T$. Then, the evaluation criteria are based on the linguistic term set $L = \{l_0 = \text{extremely bad}, l_1 = \text{very bad}, l_2 = \text{bad}, l_3 = \text{slightly bad}, l_4 = \text{medium}, l_5 = \text{slightly good}, l_6 = \text{good}, l_7 = \text{very good}, l_8 = \text{extremely good}\}$. Thus, we can establish the LNN decision matrix R^i (i = 1, 2, 3), which is listed in Tables 2–4.

Table 2. The LNN decision matrix R^1 of the expert E_1.

	C_1	C_2	C_3
A_1	$\langle l_6^1, l_1^1, l_2^1 \rangle$	$\langle l_7^1, l_2^1, l_1^1 \rangle$	$\langle l_6^1, l_2^1, l_2^1 \rangle$
A_2	$\langle l_7^1, l_1^1, l_1^1 \rangle$	$\langle l_7^1, l_3^1, l_2^1 \rangle$	$\langle l_7^1, l_2^1, l_1^1 \rangle$
A_3	$\langle l_6^1, l_2^1, l_2^1 \rangle$	$\langle l_7^1, l_1^1, l_1^1 \rangle$	$\langle l_6^1, l_2^1, l_2^1 \rangle$
A_4	$\langle l_7^1, l_1^1, l_2^1 \rangle$	$\langle l_7^1, l_2^1, l_3^1 \rangle$	$\langle l_7^1, l_2^1, l_1^1 \rangle$

Table 3. The LNN decision matrix R^2 of the expert E_2.

	C_1	C_2	C_3
A_1	$\langle l_6^2, l_1^2, l_2^2 \rangle$	$\langle l_6^2, l_1^2, l_1^2 \rangle$	$\langle l_4^2, l_2^2, l_3^2 \rangle$
A_2	$\langle l_7^2, l_2^2, l_3^2 \rangle$	$\langle l_6^2, l_1^2, l_1^2 \rangle$	$\langle l_4^2, l_2^2, l_3^2 \rangle$
A_3	$\langle l_5^2, l_1^2, l_2^2 \rangle$	$\langle l_5^2, l_1^2, l_2^2 \rangle$	$\langle l_5^2, l_4^2, l_2^2 \rangle$
A_4	$\langle l_6^2, l_1^2, l_1^2 \rangle$	$\langle l_5^2, l_1^2, l_1^2 \rangle$	$\langle l_5^2, l_2^2, l_3^2 \rangle$

Table 4. The LNN decision matrix R^3 of the expert E_3.

	C_1	C_2	C_3
A_1	$\langle l_7^3, l_3^3, l_4^3 \rangle$	$\langle l_7^3, l_3^3, l_3^3 \rangle$	$\langle l_5^3, l_2^3, l_5^3 \rangle$
A_2	$\langle l_6^3, l_3^3, l_4^3 \rangle$	$\langle l_5^3, l_1^3, l_2^3 \rangle$	$\langle l_6^3, l_2^3, l_3^3 \rangle$
A_3	$\langle l_7^3, l_2^3, l_4^3 \rangle$	$\langle l_6^3, l_1^3, l_2^3 \rangle$	$\langle l_7^3, l_2^3, l_4^3 \rangle$
A_4	$\langle l_7^3, l_2^3, l_3^3 \rangle$	$\langle l_5^3, l_2^3, l_1^3 \rangle$	$\langle l_6^3, l_1^3, l_1^3 \rangle$

5.1. The Decision-Making Process Based on the LNNNWBM Operator or LNNNWGBM Operator

Step 1: According to the weight vector $w = (0.37,0.33,0.3)^T$ of experts and the *LNNNWBM* operator (set p = 1 and q = 1), we can obtain the integrated matrix R = $(a_{ij})_{m\times n}$, which is listed in Table 5.

Table 5. The integrated matrix R.

	C_1	C_2	C_3
A_1	$\langle l_{6.3176}, l_{1.5682}, l_{2.6129}\rangle$	$\langle l_{6.6819}, l_{1.9641}, l_{1.5682}\rangle$	$\langle l_{5.0059}, l_{2.000}, l_{3.2898}\rangle$
A_2	$\langle l_{6.7045}, l_{1.9476}, l_{2.6308}\rangle$	$\langle l_{6.0524}, l_{1.6728}, l_{1.6636}\rangle$	$\langle l_{5.7033}, l_{2.000}, l_{2.3074}\rangle$
A_3	$\langle l_{5.9943}, l_{1.6636}, l_{2.6129}\rangle$	$\langle l_{6.0264}, l_{1.000}, l_{1.6430}\rangle$	$\langle l_{5.9943}, l_{2.6613}, l_{2.6129}$
A_4	$\langle l_{6.6819}, l_{1.2955}, l_{1.9641}\rangle$	$\langle l_{5.6926}, l_{1.6636}, l_{1.6728}\rangle$	$\langle l_{6.0264}, l_{1.6824}, l_{1.6170}$

Step 2: According to the weight vector $\lambda = (0.35,0.25,0.4)^T$ of attributes and the *LNNNWBM* operator (set p = 1 and q = 1), we can obtain the collective overall LNNs of a_i for $A_i (i = 1,2,3,4)$ as follows:

$$a_1 = \langle l_{5.9328}, l_{1.8388}, l_{2.5784}\rangle,\ a_2 = \langle l_{6.1489}, l_{1.8908}, l_{2.2399}\rangle,$$

$$a_3 = \langle l_{6.0032}, l_{1.8430}, l_{2.3427}\rangle,\ \text{and}\ a_4 = \langle l_{6.1675}, l_{1.5412}, l_{1.7536}\rangle.$$

Step 3: Calculating the expected values of $E(a_i)$ for $a_i (i = 1,2,3,4)$:

$$E(a_1) = 0.7298,\ E(a_2) = 0.7508,\ E(a_3) = 0.7424,\ \text{and}\ E(a_4) = 0.7864;$$

According to the results, we can rank $E(a_4) > E(a_2) > E(a_3) > E(a_1)$, so the company A_4 is the best choice among all the companies.

On the other hand, we also use the *LNNNWGBM* operator (set p = 1 and q = 1) to deal with this decision-making problem:

Step 1′: Just as step 1;

Step 2′: According to the weight vector $\lambda = (0.35,0.25,0.4)^T$ of attributes and the *LNNNWGBM* operator (set p = 1 and q = 1), we can obtain the collective overall LNNs of a_i for $A_i (i = 1,2,3,4)$ as follows:

$$a_1 = \langle l_{5.9970}, l_{1.8333}, l_{2.5434}\rangle,\ a_2 = \langle l_{6.1790}, l_{1.8897}, l_{2.2324}\rangle,\ a_3 = \langle l_{6.0032}, l_{1.7928}, l_{2.3332}\rangle, \text{and}\ a_4 = \langle l_{6.1824}, l_{1.5362}, l_{1.7500}\rangle.$$

Step 3′: Calculating the expected values of $E(a_i)$ for $a_i (i = 1,2,3,4)$:

$$E(a_1) = 0.7342,\ E(a_2) = 0.7524,\ E(a_3) = 0.7449,\ \text{and}\ E(a_4) = 0.7873.$$

According to the results, the ranking is $E\ (a_4) > E\ (a_2) > E\ (a_3) > E\ (a_1)$, so the company A_4 is the best choice among all the companies.

5.2. Analysis the Influence of the Parameters p and q on Decision Results

In order to analyze the effects of different parameters p and q on the decision results, in Steps 1 and 2, we take the different values of p and q, and all the results are shown in the Tables 6 and 7.

Table 6. The ranking based on the LNNNWBM operator with the different values of p and q.

p, q	*LNNNWBM* **Operator**	**Ranking**
p = 1, q = 0	$E(a_1)$ = 0.7528, $E(a_2)$ = 0.7777, $E(a_3)$ = 0.7613, $E(a_4)$ = 0.8060	$A_4 \succ A_2 \succ A_3 \succ A_1$
p = 1, q = 0.5	$E(a_1)$ = 0.7311, $E(a_2)$ = 0.7534, $E(a_3)$ = 0.7435, $E(a_4)$ = 0.7886	$A_4 \succ A_2 \succ A_3 \succ A_1$
p = 1, q = 2	$E(a_1)$ = 0.7329, $E(a_2)$ = 0.7545, $E(a_3)$ = 0.7453, $E(a_4)$ = 0.7897	$A_4 \succ A_2 \succ A_3 \succ A_1$
p = 0, q = 1	$E(a_1)$ = 0.7573, $E(a_2)$ = 0.7766, $E(a_3)$ = 0.7656, $E(a_4)$ = 0.8046	$A_4 \succ A_2 \succ A_3 \succ A_1$
p = 0.5, q = 1	$E(a_1)$ = 0.7326, $E(a_2)$ = 0.7530, $E(a_3)$ = 0.7449, $E(a_4)$ = 0.7879	$A_4 \succ A_2 \succ A_3 \succ A_1$
p = 2, q = 1	$E(a_1)$ = 0.7349, $E(a_2)$ = 0.7562, $E(a_3)$ = 0.7463, $E(a_4)$ = 0.7902	$A_4 \succ A_2 \succ A_3 \succ A_1$
p = 2, q = 2	$E(a_1)$ = 0.7343, $E(a_2)$ = 0.7537, $E(a_3)$ = 0.7458, $E(a_4)$ = 0.7884	$A_4 \succ A_2 \succ A_3 \succ A_1$

Table 7. The ranking based on the *LNNNWGBM* operator with the different values of p and q.

p, q	*LNNNWGBM* Operator	Ranking
$p = 1, q = 0$	$E(a_1) = 0.7397$, $E(a_2) = 0.7747$, $E(a_3) = 0.7531$, $E(a_4) = 0.8035$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 1, q = 0.5$	$E(a_1) = 0.7342$, $E(a_2) = 0.7545$, $E(a_3) = 0.7453$, $E(a_4) = 0.7891$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 1, q = 2$	$E(a_1) = 0.7343$, $E(a_2) = 0.7548$, $E(a_3) = 0.7457$, $E(a_4) = 0.7889$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 0, q = 1$	$E(a_1) = 0.7437$, $E(a_2) = 0.7730$, $E(a_3) = 0.7570$, $E(a_4) = 0.8019$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 0.5, q = 1$	$E(a_1) = 0.7356$, $E(a_2) = 0.7541$, $E(a_3) = 0.7467$, $E(a_4) = 0.7885$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 2, q = 1$	$E(a_1) = 0.7330$, $E(a_2) = 0.7553$, $E(a_3) = 0.7445$, $E(a_4) = 0.7895$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p = 2, q = 2$	$E(a_1) = 0.7334$, $E(a_2) = 0.7530$, $E(a_3) = 0.7441$, $E(a_4) = 0.7877$	$A_4 \succ A_2 \succ A_3 \succ A_1$

From above two tables, we can see that when the parameters p and q take different values, the sorting results are the same. Therefore, the influence of the two parameters is very little in this decision-making problem.

In the literature [21], the ranking is $\boldsymbol{A_4 \succ A_2 \succ A_3 \succ A_1}$, just according with the ranking result of this paper. Compared with the literature [21], the correlation between attributes is considered by the *LNNNWBM* operator and the *LNNNWGBM* operator for MAGDM, which make the information aggregation more objective and reliable. Hence, the proposed MAGDM methods with different p and q values are more flexible than the method in [21]. Compared to the literature [14], on the one hand, the literature [14] cannot express and deal with the decision-making problems with pure linguistic information like LNNs. However, in this paper, the proposed decision-making methods based on the *LNNNWBM* operator and the *LNNNWGBM* operator provide a new way for decision-makers under LNN environment.

6. Conclusions

In MADGM, how to tackle the problem of the interdependence between attributes is a challenging issue. Thus, MADGM methods based on the *LNNNWGBM* and *LNNNWGBM* operators for LNNs are proposed in this paper. First, a LNN normalized weight Bonferroni mean (*LNNNWBM*) operator and a LNN normalized weight geometric Bonferroni mean (*LNNNWGBM*) operator are proposed based on the BM operator, and the related properties of these operators are discussed. Second, based on the *LNNNWBM* operator and the *LNNNWGBM* operator, this paper puts forward two methods of MADGM in a LNN setting. Finally, an illustrative example was presented to show that these two methods were used for solving the MADGM problem with LNN information. In addition, the proposed decision-making methods may affect the decision results based on various parameters of p and q in some decision-making problems.

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References

1. Zadeh, L.A. The concept of a linguistic variable and its application to approximate reasoning-I. *Inf. Sci.* **1975**, *8*, 199–249.
2. Zadeh, L.A. A concept of a linguistic variable and its application to approximate reasoning-II. *Inf. Sci.* **1975**, *8*, 301–357.
3. Herrera, F.; Herrera-Viedma, E. Linguistic decision analysis: Steps for solving decision problems under linguistic information. *Fuzzy Sets Syst.* **2000**, *115*, 67–82.
4. Herrera, F.; Herrera-Viedma, E. A model of consensus in group decision making under linguistic assessments. *Fuzzy Sets Syst.* **1996**, *78*, 73–87.

5. Liu, P.D. Some generalized dependent aggregation operators with intuitionistic linguistic numbers and their application to group decision making. *J. Comput. Syst. Sci.* **2013**, *79*, 131–143.
6. Chen, Z.C.; Liu, P.D.; Pei, Z. An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers. *Int. J. Comput. Intell. Syst.* **2015**, *8*, 747–760.
7. Liu, P.D.; Teng, F. An extended TODIM method for multiple attribute group decision-making based on 2-dimension uncertain linguistic variable. *Complexity* **2016**, *21*, 20–30.
8. Liu, P.D.; Li, H.; Yu, X.C. Generalized hybrid aggregation operators based on the 2-dimension uncertain linguistic information for multiple attribute group decision making. *Group Decis. Negot.* **2016**, *25*, 103–126.
9. Liu, P.D.; Wang, P. Some improved linguistic intuitionistic fuzzy aggregation operators and their applications to multiple-attribute decision making. *Int. J. Inf. Technol. Decis. Mak.* **2017**, *16*, 817–850.
10. Smarandache, F. *Neutrosophy: Neutrosophic Probability, Set, and Logic*; American Research Press: Rehoboth, DE, USA, 1998.
11. Smarandache, F. A unifying field in logics. In *Neutrosophy: Neutrosophic Probability, Set and Logic*; American Research Press: Rehoboth, DE, USA, 1999.
12. Wang, H.; Smarandache, F.; Zhang, Y.Q.; Sunderraman, R. Single valued neutrosophic sets. *Multispace Multistruct.* **2010**, *4*, 410–413.
13. Ye, J. An extended TOPSIS method for multiple attribute group decision making based on single valued neutrosophic linguistic numbers. *J. Intell. Fuzzy Syst.* **2015**, *28*, 247–255.
14. Liu, P.D.; Shi, L.L. Some neutrosophic uncertain linguistic number Heronian mean operators and their application to multi-attribute group decision making. *Neural Comput. Appl.* **2017**, *28*, 1079–1093.
15. Bonferroni, C. Sulle medie multiple di potenze. *Bollettino dell'Unione Matematica Italiana* **1950**, *5*, 267–270. (in Italian)
16. Zhou, W.; He, J.M. Intuitionistic fuzzy normalized weighted Bonferroni mean and its application in multicriteria decision making. *J. Appl. Math.* **2012**, *2012*, 1–22.
17. Zhu, B.; Xu, Z.S.; Xia, M.M. Hesitant fuzzy geometric Bonferroni means. *Inf. Sci.* **2012**, *205*, 72–85.
18. Sun, M.; Liu, J. Normalized geometric Bonferroni operators of hesitant fuzzy sets and their application in multiple attribute decision making. *J. Inf. Comput. Sci.* **2013**, *10*, 2815–2822.
19. Liu, P.D.; Chen, S.M.; Liu, J.L. Some intuitionistic fuzzy interaction partitioned Bonferroni mean operators and their application to multi-attribute group decision making. *Inf. Sci.* **2017**, *411*, 98–121.
20. Liu, P.D.; Li, H.G. Interval-valued intuitionistic fuzzy power Bonferroni aggregation operators and their application to group decision making. *Cogn. Comput.* **2017**, *9*, 494–512.
21. Fang, Z.B.; Ye, J. Multiple Attribute Group Decision-Making Method Based on Linguistic Neutrosophic Numbers. *Symmetry* **2017**, *9*, 111.