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ORIGINAL ARTICLE

Multiple attribute group decision making based on interval neutrosophic uncertain linguistic variables

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Abstract To deal with decision-making problems with interval neutrosophic uncertain linguistic information, the paper proposes a multiple attribute group decision-making method under an interval neutrosophic uncertain linguistic environment. Firstly, the concept of an interval neutrosophic uncertain linguistic set and an interval neutrosophic uncertain linguistic variable (INULV) is presented by combining an uncertain linguistic variable with an interval neutrosophic set. Secondly, we introduce the operation rules of INULVs and the score function, accuracy function and certainty function of an INULV. Thirdly, we develop an interval neutrosophic uncertain linguistic weighted arithmetic averaging (INULWAA) operator and an interval neutrosophic uncertain linguistic weighted geometric averaging (INULWGA) operator and investigate their properties. Fourthly, a group decision-making method is established based on the INULWAA and INULWGA operators to solve multiple attribute group decision-making problems with interval neutrosophic uncertain linguistic information. Finally, an illustrative example is provided to demonstrate the application of the developed approach.

Keywords Interval neutrosophic uncertain linguistic set · Interval neutrosophic uncertain linguistic variable · Interval neutrosophic uncertain linguistic weighted arithmetic averaging (INLWAA) operator · Interval neutrosophic uncertain linguistic weighted geometric averaging (INLWGA) operator · Group decision making

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1 Introduction

Multiple attribute decision making is an important research topic in decision theory, and then it has been applied widely in many fields, such as engineering and economic management. Recently, new decision-making methods have been presented under various fuzzy environments [6–8]. In complex decision-making problems, however, there is a lot of qualitative information, where the evaluation results of decision makers may easily be expressed by linguistic variables or uncertain linguistic variables because of time pressure, lack of knowledge, and the decision maker's limited attention and information processing capabilities. Thus, Zadeh [19] originally proposed the concept of the linguistic variable and applied it to fuzzy reasoning. After that, Herrera et al. [2] put forward a model of consensus for group decision making under a linguistic assessment. Herrera and Herrera-Viedma [3] also proposed a linguistic decision analysis to solve decision-making problems with linguistic information. Then, Xu [15] presented a linguistic hybrid arithmetic averaging operator and applied it to multiple attribute group decision-making problems with linguistic information. Meanwhile, Xu [16] further proposed goal programming models for multiple attribute decision making under a linguistic environment. Furthermore, Zeng and Su [10] put forward linguistic induced generalized aggregation distance operators and applied them to multiple attribute decision making. Agarwal and Palpanas [1] further introduced a linguistic rough set (LRS) by integrating linguistic quantifiers in the rough set framework. On the other hand, Xu [14] proposed the uncertain linguistic ordered weighted averaging (ULOWA) operator and uncertain linguistic hybrid aggregation (ULHA) operator and applied them to multiple attribute group decision-making problems with uncertain linguistic information. Also, Xu [17] developed some induced uncertain linguistic ordered weighted averaging (IULOWA) operators and applied them to multiple attribute group decision making under an uncertain linguistic environment. Furthermore, by combining a linguistic variable with an intuitionistic fuzzy set (IFS), Wang and Li [12] introduced the intuitionistic linguistic set (ILS) that consists of a linguistic part and an intuitionistic part. Further, they [12] proposed an intuitionistic two-semantic, a Hamming distance between two intuitionistic two-semantics, and a ranking method for alternatives in accordance with the comprehensive membership degree to the ideal solution for each alternative. Wang and Li [13] also introduced the operation rules of intuitionistic linguistic variables (ILVs), the expected value, score and accuracy functions of an ILV and developed the intuitionistic linguistic weighted arithmetic averaging (ILWAA) operator and intuitionistic linguistic weighted geometric averaging (ILWGA) operator, and then they applied the ILWAA and ILWGA operators to multiple attribute decision-making problems with intuitionistic linguistic information. Furthermore, Liu and Jin [4] proposed the concept of an intuitionistic uncertain linguistic variable (IULV), which is an extension of the ILV concept, and developed an intuitionistic uncertain linguistic weighted geometric averaging (IULWGA) operator, an intuitionistic uncertain linguistic ordered weighted geometric (IULOWG) operator, and an intuitionistic uncertain linguistic hybrid geometric (IULHG) operator, which generalizes both the IULWGA operator and the IULOWG operator, and then they applied these operators to multiple attribute group decision-making problems with IULVs. Liu et al. [5] further developed some intuitionistic uncertain linguistic Heronian mean operators, including an intuitionistic uncertain linguistic arithmetic Heronian (IULAH) operator, an intuitionistic uncertain linguistic weighted arithmetic Heronian (IULWAH) operator, an intuitionistic uncertain linguistic geometric Heronian (IULGH) operator, and an intuitionistic uncertain linguistic weighted geometric Heronian (IULWGH) operator, and then applied them to group decision making.

In real decision-making problems, there is often incomplete, indeterminate and inconsistent information. Thus, the neutrosophic set proposed by Smarandache [9] can be better to express this kind of information. Therefore, Ye [18] proposed the concepts of interval neutrosophic linguistic set (INLS) and interval neutrosophic linguistic variable (INLV) by combining a linguistic variable with an interval neutrosophic set (INS). Since an INLV consists of a linguistic part and an interval neutrosophic part, we can also consider that the linguistic part of the INLV is the linguistic variable represented by decision maker's judgment to an evaluated object and the interval neutrosophic part of the INLV is the subjective evaluation value on the reliability of the given linguistic variable, which is expressed by a truth-membership degree interval and an indeterminacy-membership degree interval and a falsity-membership degree interval. However, an INLS is an extension of an ILS by replacing the intuitionistic part of the ILS with interval neutrosophic part, while the linguistic part in the INLS is still the linguistic variable rather then the uncertain linguistic variable that easily expresses the qualitative information. Hence, the INLS cannot represent and deal with interval neutrosophic uncertain linguistic information. To overcome the shortcoming, the INLS should be extended by expressing the linguistic part of an INLS with an uncertain linguistic variable to propose the concepts of an interval neutrosophic uncertain linguistic set (INULS) and an interval neutrosophic uncertain linguistic variable (INULV), which are composed of an uncertain linguistic part and an interval neutrosophic part. Therefore, INULSs can easily express and better handle the uncertain information and inconsistent information than ILSs, IULSs and INLSs. As the further extension of author's previous work [18], the purposes of this paper are: (1) to propose the concepts of an INULS and an INULV (2) to introduce basic operation rules of INULVs and the score, accuracy and certainty functions of an INULV (3) to develop an interval neutrosophic uncertain linguistic weighted arithmetic averaging (INULWAA) operator and an interval neutrosophic uncertain linguistic weighted geometric averaging (INULWGA) operator and to investigate their properties, and (4) to establish a group decision-making method based on the INULWAA and INULWGA operators for solving multiple attribute group decision-making problems with interval neutrosophic uncertain linguistic information.

To achieve the above purposes, the remainder of this paper is structured as follows. Section 2 briefly describes some concepts of linguistic variables, uncertain linguistic variables, INSs, INLSs and INLVs. In Sect. 3, we propose an INULS and an INULV, the operation rules of INULVs, and the score function, accuracy function and certainty function of an INULV. Section 4 introduces INULWAA and INULWGA operators of INULVs and investigates their properties. In Sect. 5, a multiple attribute group decision-making method based on the INULWAA and INULWGA operators is established in interval neutrosophic uncertain linguistic setting. In Sect. 6, an illustrative example is provided to demonstrate the application of the proposed method. Section 7 contains conclusions and future work.

2 Preliminaries

2.1 Linguistic variables and uncertain linguistic variables

Some concepts of linguistic variables, uncertain linguistic variables and their basic operation rules are introduced below.

Let $S = \{s_0, s_2, \ldots, s_{l\text{-}1}\}$ be a finite and totally ordered discrete linguistic term set with old cardinality, where s_i in the linguistic term set S represents a linguistic variable and l is an odd value. For example, Taking $l = 7$, we can give a linguistic term set S [3]:

$S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ = {extremely poor, very poor, poor, medium, good, very good, extremely good}.

For any two linguistic variables s_i and s_j in a linguistic term set S must satisfy the following characteristics [2, 3]:

1. Order of two variables: $s_i \geq s_j$ if $i \geq j$;
2. Negation operator: $\text{neg}(s_i) = s_{l\text{-}1\text{-}i}$;
3. Maximum operator: $\max(s_i, s_j) = s_i$ if $i > j$;
4. Minimum operator: $\min(s_i, s_j) = s_i$ if $j > i$.

Then, the discrete linguistic term set $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ can be extended to a continuous linguistic set $\bar{S} = \{s_\alpha | \alpha \in R\}$, which also satisfy the aforementioned characteristics, to minimize the linguistic information loss in the operation process. For any two linguistic variables s_i and s_j for $s_i, s_j \in \bar{S}$, the operation rules are defined as follows [15, 16]:

1. $s_i \oplus s_j = s_{i+j}$;
2. $s_i \otimes s_j = s_{i\times j}$;
3. $s_i / s_j = s_{i/j}$;
4. $\lambda s_i = s_{\lambda\times i}$ for $\lambda \geq 0$;
5. $(s_i)^\lambda = s_{i^\lambda}$ for $\lambda \geq 0$.

Definition 1 [14, 17] Suppose $\tilde{s} = [s_a, s_b]$, where $s_a, s_b \in \bar{S}$ with $a \leq b$ are the lower limit and the upper limit of $\tilde{s}$, respectively. Then $\tilde{s}$ is called an uncertain linguistic variable.

Let $\tilde{s}_1 = [s_{a1}, s_{b1}]$ and $\tilde{s}_2 = [s_{a2}, s_{b2}]$ be any two uncertain linguistic variables, then their operation rules are defined as follows [14, 17]:

1. $\tilde{s}_1 \oplus \tilde{s}_2 = [s_{a1}, s_{b1}] \oplus [s_{a2}, s_{b2}] = [s_{a1+a2}, s_{b1+b2}]$;
2. $\tilde{s}_1 \otimes \tilde{s}_2 = [s_{a1}, s_{b1}] \otimes [s_{a2}, s_{b2}] = [s_{a1\times a2}, s_{b1\times b2}]$;
3. $\tilde{s}_1 / \tilde{s}_2 = [s_{a1}, s_{b1}]/[s_{a2}, s_{b2}] = [s_{a1/b2}, s_{b1/a2}]$ if $a2 \neq 0$ and $b2 \neq 0$;
4. $\lambda\tilde{s}_1 = \lambda[s_{a1}, s_{b1}] = [s_{\lambda a1}, s_{\lambda b1}]$ for $\lambda \geq 0$.
5. $(\tilde{s}_1)^\lambda = [s_{(a1)^\lambda}, s_{(b1)^\lambda}]$ for $\lambda \geq 0$.

2.2 Some concepts of INSs

Smarandache [9] firstly proposed a neutrosophic set, which generalizes fuzzy sets, IFSs and interval valued intuitionistic fuzzy sets (IVIFSs), and gave the following definition of a neutrosophic set from philosophical point of view.

Definition 2 [9] Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]^-0, 1^+[$, i.e., $T_A(x)$: $X \to]^-0, 1^+[$, $I_A(x)$: $X \to]^-0, 1^+[$, and $F_A(x)$: $X \to]^-0, 1^+[$. Thus, the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ satisfies the condition $^-0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

For easy applications in real science and engineering areas, Wang et al. [11] introduced the concept of an INS, which is a subclass of a neutrosophic set, and gave the following definition of an INS.

Definition 3 [11] Let X be a space of points (objects) with generic elements in X denoted by x. An INS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. Then, there are $T_A(x) = [\inf T_A(x), \sup T_A(x)] \subseteq [0, 1]$, $I_A(x) = [\inf I_A(x), \sup I_A(x)] \subseteq [0, 1]$, and $F_A(x) = [\inf F_A(x), \sup F_A(x)] \subseteq [0, 1]$ for each point x in X. An INS A can be represented by

$$\begin{aligned} A &= \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\} \\ &= \{x, \langle[\inf T_A(x), \sup T_A(x)], [\inf I_A(x), \sup I_A(x)], \\ &\quad [\inf F_A(x), \sup F_A(x)]\rangle | x \in X\} \end{aligned}$$

Obviously, the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ satisfies the condition $0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3$.

Then, some relations of INSs are introduced as follows [11]:

1. The complement of an INS A is denoted by A^c and is defined as $T_A^c(x) = [\inf F_A(x), \sup F_A(x)]$ $I_A^c(x) = [1 - \sup I_A(x), 1 - \inf I_A(x)]$, $F_A^c(x) = [\inf T_A(x), \sup T_A(x)]$ for any x in X.
2. An INS A is contained in the other INS B, written as $A \subseteq B$, if and only if $\inf T_A(x) \leq \inf T_B(x)$, $\sup T_A(x) \leq \sup T_B(x)$, $\inf I_A(x) \geq \inf I_B(x)$, $\sup I_A(x) \geq \sup I_B(x)$, $\inf F_A(x) \geq \inf F_B(x)$, and $\sup F_A(x) \geq \sup F_B(x)$ for any x in X.
3. Two INSs A and B are equal, written as $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

2.3 Interval neutrosophic linguistic set

Based on combining INSs and linguistic variables, Ye [18] firstly proposed INLSs and gave the following definition:

Definition 4 [18] Let X be a finite universal set, then an INLS in X is defined as

$$A = \left\{ \left\langle x, \left[s_{\theta(x)}, (T_A(x), I_A(x), F_A(x)) \right] \right\rangle | x \in X \right\},$$

where $s_{\theta(x)} \in S$, $T_A(x) = [\inf T_A(x), \sup T_A(x)] \subseteq [0, 1]$, $I_A(x) = [\inf I_A(x), \sup I_A(x)] \subseteq [0, 1]$, $F_A(x) = [\inf F_A(x), \sup F_A(x)] \subseteq [0, 1]$, and $0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3$ for any $x \in X$. The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ express, respectively, the truth-membership degree interval, indeterminacy-membership degree interval, and falsity-membership degree interval of the element x in X to the linguistic variable $s_{\theta(x)}$.

Definition 5 [18] Let $A = \{\langle x, [s_{\theta(x)}, ([\inf T_A(x), \sup T_A(x)], [\inf I_A(x), \sup I_A(x),], [\inf F_A(x), \sup F_A(x)])]\rangle | x \in X\}$ be any INLS. Then the seven tuple $\langle s_{\theta(x)}, ([\inf T_A(x), \sup T_A(x)], [\inf I_A(x), \sup I_A(x),], [\inf F_A(x), \sup F_A(x)])\rangle$ is called an INLV and A can also be viewed as the collection of INLVs. Thus, the INLS can also be expressed by

$$A = \left\{ \left\langle s_{\theta(x)}, ([\inf T_A(x), \sup T_A(x)], [\inf I_A(x), \sup I_A(x)], [\inf F_A(x), \sup F_A(x)]) \right\rangle | x \in X \right\}$$

Let $a_1 = \langle s_{\theta(a_1)}, ([\inf T(a_1), \sup T(a_1)], [\inf I(a_1), \sup I(a_1)], [\inf F(a_1), \sup F(a_1)])\rangle$ and $a_2 = \langle s_{\theta(a_2)}, ([\inf T(a_2), \sup T(a_2)], [\inf I(a_2), \sup I(a_2)], [\inf F(a_2), \sup F(a_2)])\rangle$ be any two INLVs for a_1, $a_2 \in S$ and any real number $\lambda \geq 0$, then their operation rules are defined as follows [18]:

1. $a_1 \oplus a_2 = \langle s_{\theta(a_1)+\theta(a_2)}, ([\inf T(a_1) + \inf T(a_2) - \inf T(a_1) \inf T(a_2), \sup T(a_1) + \sup T(a_2) - \sup T(a_1) \sup T(a_2)], [\inf I(a_1) \inf I(a_2), \sup I(a_1) \sup I(a_2)], [\inf F(a_1) \inf F(a_2), \sup F(a_1) \sup F(a_2)])\rangle$;
2. $a_1 \otimes a_2 = \langle s_{\theta(a_1) \times \theta(a_2)}, ([\inf T(a_1) \inf T(a_2), \sup T(a_1) \sup T(a_2)], [\inf I(a_1) + \inf I(a_2) - \inf I(a_1) \inf I(a_2), \sup I(a_1) + \sup I(a_2) - \sup I(a_1) \sup I(a_2)], [\inf F(a_1) + \inf F(a_2) - \inf F(a_1) \inf F(a_2), \sup F(a_1) + \sup F(a_2) - \sup F(a_1) \sup F(a_2)])\rangle$;
3. $\lambda a_1 = \langle s_{\lambda\theta(a_1)}, \left([1 - (1 - \inf T(a_1))^{\lambda}, 1 - (1 - \sup T(a_1))^{\lambda}], ..[\inf I^{\lambda}(a_1), \sup I^{\lambda}(a_1)], [\inf F^{\lambda}(a_1), \sup F^{\lambda}(a_1)] \right)\rangle$;
4. $a_1^{\lambda} = \langle s_{\theta^{\lambda}(a_1)}, ([\inf T^{\lambda}(a_1), \sup T^{\lambda}(a_1)], [1 - (1 - \inf I(a_1))^{\lambda}, 1 - (1 - \sup I(a_1))^{\lambda}], [1 - (1 - \inf F(a_1))^{\lambda}, 1 - (1 - \sup F(a_1))^{\lambda}])\rangle$.

Let $a_1 = \langle s_{\theta(a_1)}, ([\inf T(a_1), \sup T(a_1)], [\inf I(a_1), \sup I(a_1)], [\inf F(a_1), \sup F(a_1)])\rangle$ and $a_2 = \langle s_{\theta(a_2)}, ([\inf T(a_2), \sup T(a_2)], [\inf I(a_2), \sup I(a_2)], [\inf F(a_2), \sup F(a_2)])\rangle$ be any two INLVs for a_1, $a_2 \in S$ and any real numbers λ, λ_1, $\lambda_2 \geq 0$, then they satisfy the following properties [18]:

1. $a_1 \oplus a_2 = a_2 \oplus a_1$;
2. $a_1 \otimes a_2 = a_2 \otimes a_1$;
3. $\lambda(a_1 \oplus a_2) = \lambda a_1 \oplus \lambda a_2$;
4. $\lambda_1 a_1 \oplus \lambda_2 a_1 = (\lambda_1 + \lambda_2) a_1$;
5. $a_1^{\lambda_1} \otimes a_1^{\lambda_2} = a_1^{\lambda_1 + \lambda_2}$;
6. $a_1^{\lambda_1} \otimes a_2^{\lambda_1} = (a_1 \otimes a_2)^{\lambda_1}$.

Then, Ye [18] defined the score function, accuracy function and certainty function of an INLV, which are important indexes for ranking alternatives in decision-making problems.

Definition 6 [18] Let $a = \langle s_{\theta(a)}, ([\inf T(a), \sup T(a)], [\inf I(a), \sup I(a)], [\inf F(a), \sup F(a)])\rangle$ be an INLV for $a \in S$. Then, the score function, accuracy function and certainty function for the INLV a are defined, respectively, as follows:

$$e(a) = \frac{1}{6}(4 + \inf T(a) - \inf I(a) - \inf F(a) + \sup T(a) - \sup I(a) - \sup F(a)) s_{\theta(a)} = s_{\frac{1}{6}(4 + \inf T(a) - \inf I(a) - \inf F(a) + \sup T(a) - \sup I(a) - \sup F(a))\theta(a)}, \tag{1}$$

$$h(a) = \frac{1}{2}(\inf T(a) - \inf F(a) + \sup T(a) - \sup F(a)) s_{\theta(a)} = s_{\frac{1}{2}(\inf T(a) - \inf F(a) + \sup T(a) - \sup F(a))\theta(a)}, \tag{2}$$

$$c(a) = \frac{1}{2}(\inf T(a) + \sup T(a)) s_{\theta(a)} = s_{\frac{1}{2}(\inf T(a) + \sup T(a))\theta(a)} \tag{3}$$

Definition 7 [18]. Let a_1 and a_2 be any two INLVs for a_1, $a_2 \in S$. Then, the ranking method can be defined as follows:

1. If $e(a_1) > e(a_2)$, then $a_1 > a_2$,
2. If $e(a_1) = e(a_2)$ and $h(a_1) > h(a_2)$, then $a_1 > a_2$,
3. If $e(a_1) = e(a_2)$, $h(a_1) = h(a_2)$, and $c(a_1) > c(a_2)$, then $a_1 > a_2$,
4. If $e(a_1) = e(a_2)$, $h(a_1) = h(a_2)$, and $c(a_1) = c(a_2)$, then $a_1 = a_2$.

Let a_j $(j = 1, 2, \ldots, n)$ be a collection of INLVs and $W = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of a_j $(j = 1, 2, \ldots, n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. Then Ye [18] proposed the INLWAA and INLWGA operators, respectively, as follows:

$$INLWAA(a_1,a_2,\cdots,a_n) = \left\langle s_{\sum_{j=1}^{n} w_j\theta(a_j)}, \left(\left[1-\prod_{j=1}^{n}(1-\inf T(a_j))^{w_j}, 1-\prod_{j=1}^{n}(1-\sup T(a_j))^{w_j} \right], \left[\prod_{j=1}^{n}\inf I^{w_j}(a_j), \prod_{j=1}^{n}\sup I^{w_j}(a_j) \right], \left[\prod_{j=1}^{n}\inf F^{w_j}(a_j), \prod_{j=1}^{n}\sup F^{w_j}(a_j) \right] \right) \right\rangle, \quad (4)$$

$$INLWGA(a_1,a_2,\cdots,a_n) = \left\langle s_{\prod_{j=1}^{n}\theta^{w_j}(a_j)}, \left(\left[\prod_{j=1}^{n}\inf T^{w_j}(a_j), \prod_{j=1}^{n}\sup T^{w_j}(a_j) \right], \left[1-\prod_{j=1}^{n}(1-\inf I(a_j))^{w_j}, 1-\prod_{j=1}^{n}(1-\sup I(a_j))^{w_j} \right], \left[1-\prod_{j=1}^{n}(1-\inf F(a_j))^{w_j}, 1-\prod_{j=1}^{n}(1-\sup F(a_j))^{w_j} \right] \right) \right\rangle. \quad (5)$$

3 Interval neutrosophic uncertain linguistic set

In real decision making, there is a lot of qualitative information, which is easily expressed by linguistic variables or uncertain linguistic variables by decision makers. An ILS consists of the intuitionistic part and the linguistic part. Then, an IULS composes of the intuitionistic part and the uncertain linguistic part. Hence, the IULS only extends the linguistic part of the ILS and can express the truth-membership degree and falsity-membership degree belonging to an uncertain linguistic variable, but cannot express the truth-membership degree, indeterminacy-membership degree, and falsity-membership degree belonging to an uncertain linguistic variable. Furthermore, an INLS consists of the interval neutrosophic part and the linguistic part and can handle the information expressed by the truth-membership degree, indeterminacy-membership degree and falsity-membership degree belonging to the linguistic variable, but cannot represent and deal with the information expressed by the truth-membership degree, indeterminacy-membership degree and falsity-membership degree belonging to the uncertain linguistic variable. To overcome the aforementioned shortcoming, this section proposes INULSs and INULVs by combining an uncertain linguistic variable with an INS, and then introduces the operation rules and ranking method of INULVs.

Definition 8 Let X be a finite universal set and $[s_{\theta(x)}, s_{\rho(x)}] \in S$ be an uncertain linguistic variable. An INULS in X is defined as

$$A = \left\{ \left\langle x, [s_{\theta(x)}, s_{\rho(x)}], (T_A(x), I_A(x), F_A(x)) \right\rangle | x \in X \right\},$$

where $s_{\theta(x)}, s_{\rho(x)} \in S$, $T_A(x) = [\inf T_A(x), \sup T_A(x)] \subseteq [0, 1]$, $I_A(x) = [\inf I_A(x), \sup I_A(x)] \subseteq [0, 1]$, and $F_A(x) = [\inf F_A(x), \sup F_A(x)] \subseteq [0, 1]$ with the condition $0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3$ for any $x \in X$. The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ express, respectively, the truth-membership degree interval, the indeterminacy-membership degree interval, and the falsity-membership degree interval of the element x in X belonging to the uncertain linguistic variable $[s_{\theta(x)}, s_{\rho(x)}] \in S$.

Definition 9 Let $A = \{\langle x, [s_{q(x)}, s_{r(x)}], ([\inf T_A(x), \sup T_A(x)], [\inf I_A(x), \sup I_A(x),], [\inf F_A(x), \sup F_A(x)])\rangle | x \in X\}$ be an INULS. Then the eight tuple $\langle [s_{q(x)}, s_{r(x)}], ([\inf T_A(x), \sup T_A(x)], [\inf I_A(x), \sup I_A(x),], [\inf F_A(x), \sup F_A(x)])\rangle$ is called an INULV and A can also be viewed as a collection of INULVs. Thus, the INULS can also be expressed as

$$A = \left\{ \left\langle [s_{\theta(x)}, s_{\rho(x)}], ([\inf\ T_A(x), \sup\ T_A(x)], [\inf\ I_A(x), \sup\ I_A(x)], [\inf\ F_A(x), \sup\ F_A(x)]) \right\rangle | x \in X \right\}.$$

Definition 10 Let $\tilde{a}_1 = \langle [s_{\theta(\tilde{a}_1)}, s_{\rho(\tilde{a}_1)}]$, ([inf $T(\tilde{a}_1)$, sup $T(\tilde{a}_1)$], [inf $I(\tilde{a}_1)$, sup $I(\tilde{a}_1)$], [inf $F(\tilde{a}_1)$, sup $F(\tilde{a}_1)$])$\rangle$ and $\tilde{a}_2 = \langle [s_{\theta(\tilde{a}_2)}, s_{\rho(\tilde{a}_2)}]$, ([inf $T(\tilde{a}_2)$, sup $T(\tilde{a}_2)$], [inf $I(\tilde{a}_2)$, sup $I(\tilde{a}_2)$], [inf $F(\tilde{a}_2)$, sup $F(\tilde{a}_2)$])$\rangle$ be two INULVs and $\lambda \ge 0$, then the operation rules of INULVs are defined as follows:

1. $\tilde{a}_1 \oplus \tilde{a}_2 = \langle [s_{\theta(\tilde{a}_1)+\theta(\tilde{a}_2)}, s_{\rho(\tilde{a}_1)+\rho(\tilde{a}_2)}], ([\inf\ T(\tilde{a}_1) + \inf\ T(\tilde{a}_2) - \inf\ T(\tilde{a}_1) \inf\ T(\tilde{a}_2), \sup\ T(\tilde{a}_1) + \sup\ T(\tilde{a}_2) - \sup\ T(\tilde{a}_1) \sup\ T(\tilde{a}_2)], [\inf I(\tilde{a}_1) \inf I(\tilde{a}_2), \sup I(\tilde{a}_1) \sup\ I(\tilde{a}_2)], [\inf F(\tilde{a}_1) \inf\ F(\tilde{a}_2), \sup F(\tilde{a}_1) \sup F(\tilde{a}_2)])\rangle$;
2. $\tilde{a}_1 \otimes \tilde{a}_2 = \langle [s_{\theta(\tilde{a}_1)\times\theta(\tilde{a}_2)}, s_{\rho(\tilde{a}_1)\times\rho(\tilde{a}_2)}], ([\inf T(\tilde{a}_1) \inf T(\tilde{a}_2), \sup T(\tilde{a}_1) \sup T(\tilde{a}_2)], [\inf I(\tilde{a}_1) + \inf I(\tilde{a}_2) - \inf I(\tilde{a}_1) \inf I(\tilde{a}_2), \sup I(\tilde{a}_1) + \sup I(\tilde{a}_2) - \sup I(\tilde{a}_1) \sup I(\tilde{a}_2)], [\inf F(\tilde{a}_1) + \inf F(\tilde{a}_2) - \inf F(\tilde{a}_1) \inf F(\tilde{a}_2), \sup F(\tilde{a}_1) + \sup F(\tilde{a}_2) - \sup F(\tilde{a}_1) \sup F(\tilde{a}_2)])\rangle$;
3. $\lambda\tilde{a}_1 = \langle [s_{\lambda\theta(\tilde{a}_1)}, s_{\lambda\rho(\tilde{a}_1)}], ([1 - (1 - \inf T(\tilde{a}_1))^{\lambda}, 1 - (1 - \sup T(\tilde{a}_1))^{\lambda}], [\inf I^{\lambda}(\tilde{a}_1), \sup I^{\lambda}(\tilde{a}_1)], [\inf F^{\lambda}(\tilde{a}_1), \sup F^{\lambda}(\tilde{a}_1)])\rangle$;
4. $\tilde{a}_1^{\lambda} = \langle [s_{\theta^{\lambda}}(\tilde{a}_1), s_{\rho^{\lambda}(\tilde{a}_1)}], ([\inf T^{\lambda}(\tilde{a}_1), \sup T^{\lambda}(\tilde{a}_1)], [1 - (1 - \inf I(\tilde{a}_1))^{\lambda}, 1 - (1 - \sup I(\tilde{a}_1))^{\lambda}], [1 - (1 - \inf F(\tilde{a}_1))^{\lambda}, 1 - (1 - \sup F(\tilde{a}_1))^{\lambda}])\rangle$.

Obviously, the above operation results are still INULVs.

Theorem 1 *For any two INULVs* $\tilde{a}_1 = \langle [s_{\theta(\tilde{a}_1)}, s_{\rho(\tilde{a}_1)}]$, ([inf $T(\tilde{a}_1)$, sup $T(\tilde{a}_1)$], [inf $I(\tilde{a}_1)$, sup $I(\tilde{a}_1)$], [inf $F(\tilde{a}_1)$, sup $F(\tilde{a}_1)$])$\rangle$ and $\tilde{a}_2 = \langle [s_{\theta(\tilde{a}_2)}, s_{\rho(\tilde{a}_2)}]$, ([inf $T(\tilde{a}_2)$, sup $T(\tilde{a}_2)$], [inf $I(\tilde{a}_2)$, sup $I(\tilde{a}_2)$], [inf $F(\tilde{a}_2)$, sup $F(\tilde{a}_2)$])$\rangle$, *and any real numbers* $\lambda, \lambda_1, \lambda_2 \ge 0$, *it is easy to prove that their operation rules have the following properties:*

1. $\tilde{a}_1 \oplus \tilde{a}_2 = \tilde{a}_2 \oplus \tilde{a}_1$;
2. $\tilde{a}_1 \otimes \tilde{a}_2 = \tilde{a}_2 \otimes \tilde{a}_1$;
3. $\lambda(\tilde{a}_1 \oplus \tilde{a}_2) = \lambda\tilde{a}_1 \oplus \lambda\tilde{a}_2$;
4. $\lambda_1\tilde{a}_1 \oplus \lambda_2\tilde{a}_1 = (\lambda_1 + \lambda_2)\tilde{a}_1$;

5. $\tilde{a}_1^{\lambda_1} \otimes \tilde{a}_1^{\lambda_2} = \tilde{a}_1^{\lambda_1+\lambda_2}$;
6. $\tilde{a}_1^{\lambda_1} \otimes \tilde{a}_2^{\lambda_1} = (\tilde{a}_1 \otimes \tilde{a}_2)^{\lambda_1}$.

To compare INULVs, we define the score function, accuracy function and certainty function of an INULV as the extension of author's previous work in [18].

Definition 11 Let $\tilde{a} = \langle [s_{\theta(\tilde{a})}, s_{\rho(\tilde{a})}]$, ([inf $T(\tilde{a})$, sup $T(\tilde{a})$], [inf $I(\tilde{a})$, sup $I(\tilde{a})$], [inf $F(\tilde{a})$, sup $F(\tilde{a})$])$\rangle$ be an INULV. Then, the score function, accuracy function, and certainty function for the INULV $\tilde{a}$ are defined, respectively, as follows:

$$E(\tilde{a}) = (4 + \inf T(\tilde{a}) - \inf I(\tilde{a}) - \inf F(\tilde{a}) + \sup T(\tilde{a}) - \sup I(\tilde{a}) - \sup F(\tilde{a}))/6 \times s_{(\theta(\tilde{a})+\rho(\tilde{a}))/2} = s_{(4+\inf T(\tilde{a})-\inf I(\tilde{a})-\inf F(\tilde{a})+\sup T(\tilde{a})-\sup I(\tilde{a})-\sup F(\tilde{a}))\times(\theta(\tilde{a})+\rho(\tilde{a}))/12}, \quad (6)$$

$$H(\tilde{a}) = (\inf T(\tilde{a}) - \inf F(\tilde{a}) + \sup T(\tilde{a}) - \sup F(\tilde{a}))/2 \times s_{(\theta(\tilde{a})+\rho(\tilde{a}))/2} = s_{(\inf T(\tilde{a})-\inf F(\tilde{a})+\sup T(\tilde{a})-\sup F(\tilde{a}))\times(\theta(\tilde{a})+\rho(\tilde{a}))/4}, \quad (7)$$

$$C(\tilde{a}) = (\inf T(\tilde{a}) + \sup T(\tilde{a}))/2 \times s_{(\theta(\tilde{a})+\rho(\tilde{a}))/2} = s_{(\inf T(\tilde{a})+\sup T(\tilde{a}))\times(\theta(\tilde{a})+\rho(\tilde{a}))/4}. \quad (8)$$

Based on Definitions 7 and 11, a ranking method between INULVs can be given as follows.

Definition 12 Let $\tilde{a}_1$ and $\tilde{a}_2$ be two INULVs. Then, their ranking method can be introduced as follows:

1. If $E(\tilde{a}_1) > E(\tilde{a}_2)$, then $\tilde{a}_1 > \tilde{a}_2$,
2. If $E(\tilde{a}_1) = E(\tilde{a}_2)$ and $H(\tilde{a}_1) > H(\tilde{a}_2)$, then $\tilde{a}_1 > \tilde{a}_2$,
3. If $E(\tilde{a}_1) = E(\tilde{a}_2)$, $H(\tilde{a}_1) = H(\tilde{a}_2)$, and $C(\tilde{a}_1) > C(\tilde{a}_2)$, then $\tilde{a}_1 > \tilde{a}_2$,
4. If $E(\tilde{a}_1) = E(\tilde{a}_2)$, $H(\tilde{a}_1) = H(\tilde{a}_2)$, and $C(\tilde{a}_1) = C(\tilde{a}_2)$, then $\tilde{a}_1 = \tilde{a}_2$.

4 Two weighted aggregation operators for INULVs

Based on the operation rules in Definition 10 and the extension of Eqs. (4) and (5), we can propose the weighted arithmetic aggregation operator and weighted geometric aggregation operator for INULVs to aggregate interval neutrosophic uncertain linguistic information.

4.1 Interval neutrosophic uncertain linguistic weighted arithmetic averaging operator

Definition 13 Let $\tilde{a}_j$ ($j = 1, 2, \ldots, n$) be a collection of INULVs. The INULWAA operator is defined by

$$INULWAA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \sum_{j=1}^{n} w_j \tilde{a}_j \quad (9)$$

where $W = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weight vector of $\tilde{a}_j$ ($j = 1, 2, \ldots, n$), $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Theorem 2 *Let $\tilde{a}_j$ ($j = 1, 2, \ldots, n$) be a collection of INULVs. Then by Eq. (9) and the operation rules in Definition 10, we have the following result:*

$$INULWAA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left\langle \left[s_{\sum_{j=1}^{n} w_j \theta(\tilde{a}_j)}, s_{\sum_{j=1}^{n} w_j \rho(\tilde{a}_j)} \right], \left(\left[1 - \prod_{j=1}^{n} (1 - \inf T(\tilde{a}_j))^{w_j}, 1 - \prod_{j=1}^{n} (1 - \sup T(\tilde{a}_j))^{w_j} \right], \left[\prod_{j=1}^{n} \inf I^{w_j}(\tilde{a}_j), \prod_{j=1}^{n} \sup I^{w_j}(\tilde{a}_j) \right], \left[\prod_{j=1}^{n} \inf F^{w_j}(\tilde{a}_j), \prod_{j=1}^{n} \sup F^{w_j}(\tilde{a}_j) \right] \right) \right\rangle \quad (10)$$

where W= (w_1, w_2, …, $w_n$$)^{\mathrm{T}}$ is the weight vector of $\tilde{a}_j$ (j= 1, 2, …, n), $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Proof The proof of Eq. (10) can be done by means of mathematical induction.

1. When $n = 2$, then

$$w_1 \tilde{a}_1 = \left\langle \left[s_{w_1 \theta(\tilde{a}_1)}, s_{w_1 \rho(\tilde{a}_1)} \right], \left([1 - (1 - \inf T(\tilde{a}_1))^{w_1}, 1 - (1 - \sup T(\tilde{a}_1))^{w_1}], [\inf I^{w_1}(\tilde{a}_1), \sup I^{w_1}(\tilde{a}_1)], [\inf F^{w_1}(\tilde{a}_1), \sup F^{w_1}(\tilde{a}_1)] \right) \right\rangle,$$

$$w_2 \tilde{a}_2 = \left\langle \left[s_{w_2 \theta(\tilde{a}_2)}, s_{w_2 \rho(\tilde{a}_2)} \right], \left([1 - (1 - \inf T(\tilde{a}_2))^{w_2}, 1 - (1 - \sup T(\tilde{a}_2))^{w_2}], [\inf I^{w_2}(a_2), \sup I^{w_2}(\tilde{a}_2)], [\inf F^{w_2}(\tilde{a}_2), \sup F^{w_2}(\tilde{a}_2)] \right) \right\rangle,$$

Thus,

$$
\begin{aligned}
&INULWAA(\tilde{a}_1,\tilde{a}_2)=w_1\tilde{a}_1\oplus w_2\tilde{a}_2\\
&=\left\langle\left[s_{\sum_{j=1}^{2}w_j\theta(\tilde{a}_j)},s_{\sum_{j=1}^{2}w_j\rho(\tilde{a}_j)}\right],\right.\\
&\quad([1-(1-\inf T(\tilde{a}_1))^{w_1}+1-(1-\inf T(\tilde{a}_2))^{w_2}\\
&\quad-(1-(1-\inf T(\tilde{a}_1))^{w_1})(1-(1-\inf T(\tilde{a}_2))^{w_2}),\\
&\quad 1-(1-\sup T(\tilde{a}_1))^{w_1}+1-(1-\sup T(\tilde{a}_2))^{w_2}\\
&\quad-(1-(1-\sup T(\tilde{a}_1))^{w_1})(1-(1-\sup T(\tilde{a}_2))^{w_2})],\\
&\quad[\inf I^{w_1}(\tilde{a}_1)\inf I^{w_2}(\tilde{a}_2),\sup I^{w_1}(\tilde{a}_1)\sup I^{w_2}(\tilde{a}_2)],\\
&\quad\quad\left.[\inf F^{w_1}(\tilde{a}_1)\inf F^{w_2}(\tilde{a}_2),\sup F^{w_1}(\tilde{a}_1)\sup F^{w_2}(\tilde{a}_2)])\right\rangle\\
&=\left\langle\left[s_{\sum_{j=1}^{2}w_j\theta(\tilde{a}_j)},s_{\sum_{j=1}^{2}w_j\rho(\tilde{a}_j)}\right],\right.\\
&\quad([1-(1-\inf T(\tilde{a}_1))^{w_1}(1-\inf T(\tilde{a}_2))^{w_2},\\
&\quad 1-(1-\sup T(\tilde{a}_1))^{w_1}(1-\sup T(\tilde{a}_2))^{w_2}],\\
&\quad\left[\prod_{j=1}^{2}\inf I^{w_j}(\tilde{a}_j),\prod_{j=1}^{2}\sup I^{w_j}(\tilde{a}_j)\right],\\
&\quad\left.\left[\prod_{j=1}^{2}\inf F^{w_j}(\tilde{a}_j),\prod_{j=1}^{2}\sup F^{w_j}(\tilde{a}_j)\right]\right)\right\rangle
\end{aligned}
\tag{11}
$$

2. When $n=k$, by using Eq. (10), we obtain

$$
\begin{aligned}
&INULWAA(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_k)\\
&=\left\langle\left[s_{\sum_{j=1}^{k}w_j\theta(\tilde{a}_j)},s_{\sum_{j=1}^{k}w_j\rho(\tilde{a}_j)}\right],\left(\left[1-\prod_{j=1}^{k}(1-\inf T(\tilde{a}_j))^{w_j},\right.\right.\right.\\
&1-\left.\prod_{j=1}^{k}(1-\sup T(\tilde{a}_j))^{w_j}\right],\left[\prod_{j=1}^{k}\inf I^{w_j}(\tilde{a}_j),\prod_{j=1}^{k}\sup I^{w_j}(\tilde{a}_j)\right],\\
&\left.\left.\left[\prod_{j=1}^{k}\inf F^{w_j}(\tilde{a}_j),\prod_{j=1}^{k}\sup F^{w_j}(\tilde{a}_j)\right]\right)\right\rangle
\end{aligned}
\tag{12}
$$

(3) When $n=k+1$, by applying Eqs. (11) and (12), we can get

$$
\begin{aligned}
&INULWAA(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_{k+1})=\left\langle\left[s_{\sum_{j=1}^{k+1}w_j\theta(\tilde{a}_j)},s_{\sum_{j=1}^{k+1}w_j\rho(\tilde{a}_j)}\right],\right.\\
&\left(\left[1-\prod_{j=1}^{k}(1-\inf T(\tilde{a}_j))^{w_j}+1-(1-\inf T(\tilde{a}_{k+1}))^{w_{k+1}}\right.\right.\\
&\quad-(1-\prod_{j=1}^{k}(1-\inf T(\tilde{a}_j))^{w_j})(1-(1-\inf T(\tilde{a}_{k+1}))^{w_{k+1}}),\\
&\quad 1-\prod_{j=1}^{k}(1-\sup T(\tilde{a}_j))^{w_j}+1-(1-\sup T(\tilde{a}_{k+1}))^{w_{k+1}}\\
&\quad\left.-(1-\prod_{j=1}^{k}(1-\sup T(\tilde{a}_j))^{w_j})(1-(1-\sup T(\tilde{a}_{k+1}))^{w_{k+1}})\right],\\
&\quad\left.\left.\left[\prod_{j=1}^{k+1}\inf I^{w_j}(\tilde{a}_j),\prod_{j=1}^{k+1}\sup I^{w_j}(\tilde{a}_j)\right],\left[\prod_{j=1}^{k+1}\inf F^{w_j}(\tilde{a}_j),\prod_{j=1}^{k+1}\sup F^{w_j}(\tilde{a}_j)\right]\right)\right\rangle\\
&\quad=\left\langle\left[s_{\sum_{j=1}^{k+1}w_j\theta(\tilde{a}_j)},s_{\sum_{j=1}^{k+1}w_j\rho(\tilde{a}_j)}\right],\left(\left[1-\prod_{j=1}^{k+1}(1-\inf T(\tilde{a}_j))^{w_j},\right.\right.\right.\\
&\quad\left.1-\prod_{j=1}^{k+1}(1-\sup T(\tilde{a}_j))^{w_j}\right],\\
&\quad\left.\left.\left[\prod_{j=1}^{k+1}\inf I^{w_j}(\tilde{a}_j),\prod_{j=1}^{k+1}\sup I^{w_j}(\tilde{a}_j)\right],\left[\prod_{j=1}^{k+1}\inf F^{w_j}(\tilde{a}_j),\prod_{j=1}^{k+1}\sup F^{w_j}(\tilde{a}_j)\right]\right)\right\rangle
\end{aligned}
$$

Therefore, according to the above results, we obtain Eq. (10) for any n. This completes the proof. □

Especially when $W=(1/n,\ 1/n,\ \ldots,\ 1/n)^{\mathrm{T}}$, then INULWAA operator reduces to an interval neutrosophic uncertain linguistic arithmetic averaging operator for INULVs.

It is obvious that the INULWAA operator satisfies the following properties:

1. Idempotency: Let $\tilde{a}_j$ $(j=1,2,\ldots,n)$ be a collection of INULVs. If $\tilde{a}_j$ $(j=1,2,\ldots,n)$ is equal, i.e., $\tilde{a}_j=\tilde{a}$ for $j=1,2,\ldots,n$, then $INULWAA(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n)=\tilde{a}$.
2. Boundedness: Let $\tilde{a}_j$ $(j=1,2,\ldots,n)$ be a collection of INULVs and $\tilde{a}_{\min}=\min(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n)$ and $\tilde{a}_{\max}=\max(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n)$ for $j=1,2,\ldots,n$, then $\tilde{a}_{\min}\le INULWAA(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n)\le\tilde{a}_{\max}$.
3. Monotonicity: Let $\tilde{a}_j$ $(j=1,2,\ldots,n)$ be a collection of INULVs. If $\tilde{a}_j\le\tilde{a}_j^*$ for $j=1,2,\ldots,n$, then $INULWAA(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n)\le INULWAA(\tilde{a}_1^*,\tilde{a}_2^*,\ldots,\tilde{a}_n^*)$.

Proof

1. Since $\tilde{a}_j=\tilde{a}$ for $j=1,2,\ldots,n$, we have

$$\begin{aligned} & INULWAA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \\ & = \left\langle \left[s_{\sum_{j=1}^{n} w_j \theta(\tilde{a}_j)}, s_{\sum_{j=1}^{n} w_j \rho(\tilde{a}_j)} \right], \left(\left[1 - \prod_{j=1}^{n} (1 - \inf T(\tilde{a}_j))^{w_j}, \right. \right. \right. \\ & \left. 1 - \prod_{j=1}^{n} (1 - \sup T(\tilde{a}_j))^{w_j} \right], \left[\prod_{j=1}^{n} \inf I^{w_j}(\tilde{a}_j), \prod_{j=1}^{n} \sup I^{w_j}(\tilde{a}_j) \right], \\ & \left. \left. \left[\prod_{j=1}^{n} \inf F^{w_j}(\tilde{a}_j), \prod_{j=1}^{n} \sup F^{w_j}(\tilde{a}_j) \right] \right) \right\rangle \\ & = \left\langle \left[s_{\sum_{j=1}^{n} w_j \theta(\tilde{a}_j)}, s_{\sum_{j=1}^{n} w_j \rho(\tilde{a}_j)} \right], \left(\left[1 - (1 - \inf T(\tilde{a}))^{\sum_{j=1}^{n} w_j}, \right. \right. \right. \\ & \left. 1 - (1 - \sup T(\tilde{a}))^{\sum_{j=1}^{n} w_j} \right], \left[\inf I^{\sum_{j=1}^{n} w_j}(\tilde{a}), \sup I^{\sum_{j=1}^{n} w_j}(\tilde{a}) \right], \\ & \left. \left. \left[\inf F^{\sum_{j=1}^{n} w_j}(\tilde{a}), \sup F^{\sum_{j=1}^{n} w_j}(\tilde{a}) \right] \right) \right\rangle \\ & = \langle [s_{\theta(\tilde{a})}, s_{\rho(\tilde{a})}], ([\inf T(\tilde{a})), \sup T(\tilde{a}))], \\ & [\inf I(\tilde{a}), \sup I(\tilde{a})], [\inf F(\tilde{a}), \sup F(\tilde{a})]) \rangle \\ & = \tilde{a} \end{aligned}$$

2. Since $\tilde{a}_{\min} = \min(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$ and $\tilde{a}_{\max} = \max(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$ for $j = 1, 2, \ldots, n$, there is $\tilde{a}_{\min} \le \tilde{a}_j \le \tilde{a}_{\max}$. Thus, there exists $\sum_{j=1}^{n} w_j \tilde{a}_{\min} \le \sum_{j=1}^{n} w_j \tilde{a}_j \le \sum_{j=1}^{n} w_j \tilde{a}_{\max}$. This is $\tilde{a}_{\min} \le \sum_{j=1}^{n} w_j \tilde{a}_j \le \tilde{a}_{\max}$, i.e., $\tilde{a}_{\min} \le INULWAA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \le \tilde{a}_{\max}$.
3. (3) Since $\tilde{a}_j \le \tilde{a}_j^*$ for $j = 1, 2, \ldots, n$, there is $\sum_{j=1}^{n} w_j \tilde{a}_j \le \sum_{j=1}^{n} w_j \tilde{a}_j^*$, i.e., $INULWAA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \le INULWAA(\tilde{a}_1^*, \tilde{a}_2^*, \ldots, \tilde{a}_n^*)$.

Thus, we complete the proofs of these properties. □

4.2 Interval neutrosophic uncertain linguistic weighted geometric averaging operator

Definition 14 Let $\tilde{a}_j$ ($j = 1, 2, \ldots, n$) be a collection of INULVs. Then the INULWGA operator is defined as

$$INULWGA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \prod_{j=1}^{n} \tilde{a}_j^{w_j}, \tag{13}$$

where $W = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\tilde{a}_j$ ($j = 1, 2, \ldots, n$), $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Theorem 3 *Let $\tilde{a}_j$ ($j = 1, 2, \ldots, n$) be a collection of INULVs. By Eq. (13) and the operation rules in Definition 10, we have the following result:*

$$\begin{aligned} & INULWGA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \\ & = \left\langle \left[s_{\prod_{j=1}^{n} \theta^{w_j}(\tilde{a}_j)}, s_{\prod_{j=1}^{n} \rho^{w_j}(\tilde{a}_j)} \right], \left(\left[\prod_{j=1}^{n} \inf T^{w_j}(\tilde{a}_j), \prod_{j=1}^{n} \sup T^{w_j}(\tilde{a}_j) \right], \right. \right. \\ & \left[1 - \prod_{j=1}^{n} (1 - \inf I(\tilde{a}_j))^{w_j}, 1 - \prod_{j=1}^{n} (1 - \sup I(\tilde{a}_j))^{w_j} \right], \\ & \left. \left. \left[1 - \prod_{j=1}^{n} (1 - \inf F(\tilde{a}_j))^{w_j}, 1 - \prod_{j=1}^{n} (1 - \sup F(\tilde{a}_j))^{w_j} \right] \right) \right\rangle, \end{aligned} \tag{14}$$

where $W = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\tilde{a}_j$ ($j = 1, 2, \ldots, n$), $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

By a similar proof manner of Theorem 2, we can also give the proof of Theorem 3 (omitted).

Especially when $W = (1/n, 1/n, \ldots, 1/n)^T$, the INULWGA operator reduces to an interval neutrosophic uncertain linguistic geometric averaging operator.

It is obvious that the INULWGA operator also satisfies the following properties:

1. Idempotency: Let $\tilde{a}_j$ ($j = 1, 2, \ldots, n$) be a collection of INULVs. If $\tilde{a}_j$ ($j = 1, 2, \ldots, n$) is equal, i.e., $\tilde{a}_j = \tilde{a}$ for $j = 1, 2, \ldots, n$, then $INULWGA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \tilde{a}$.
2. Boundedness: Let $\tilde{a}_j$ ($j = 1, 2, \ldots, n$) be a collection of INULVs and $\tilde{a}_{\min} = \min(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$ and $\tilde{a}_{\max} = \max(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$ for $j = 1, 2, \ldots, n$, then $\tilde{a}_{\min} \le INULWGA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \le \tilde{a}_{\max}$.
3. Monotonicity: Let $\tilde{a}_j$ ($j = 1, 2, \ldots, n$) be a collection of INULVs. If $\tilde{a}_j \le \tilde{a}_j^*$ for $j = 1, 2, \ldots, n$, then $INULWGA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \le INULWGA(\tilde{a}_1^*, \tilde{a}_2^*, \ldots, \tilde{a}_n^*)$.

Since the proof process of these properties is similar to the above proofs, we do not repeat it here.

5 Group decision-making method by the INULWAA and INULWGA operators

This section presents a method for multiple attribute group decision-making problems based on the INULWAA and INULWGA operators and the score, accuracy and certainty functions of INULVs under an interval neutrosophic uncertain linguistic environment.

In a multiple attribute group decision-making problem, assume that $A = \{A_1, A_2, \ldots, A_m\}$ is a set of m alternatives, $C = \{C_1, C_2, \ldots, C_n\}$ is a set of n attributes, and $E = \{E_1, E_2, \ldots, E_t\}$ is a set of t decision makers. Then, the weight vector of decision makers is $\boldsymbol{V} = (v_1, v_2, \ldots, v_t)^T$ for $v_k \ge 0$

and $\sum_{k=1}^{t} v_k = 1$. The weight vector of the attributes, entered by the decision maker, is $\boldsymbol{W} = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. In the group decision process, for each decision maker E_k ($k = 1, 2, \ldots, t$), the evaluation information of the alternative A_i ($i = 1, 2, \ldots, m$) with respect to the attribute C_j ($j = 1, 2, \ldots, n$) is represented by the form of an INULS:

$$A_i^k = \left\{ \left\langle \left[s_{\theta_i^k(C_j)}^k, s_{\rho_i^k(C_j)}^k \right], \left(T_{A_i^k}(C_j), I_{A_i^k}(C_j), F_{A_i^k}(C_j) \right) \right\rangle | C_j \in C \right\},$$

where, $T_{A_i^k}(C_j) = [\inf T_{A_i^k}(C_j), \sup T_{A_i^k}(C_j)] \subseteq [0,1]$, $I_{A_i^k}(C_j) = [\inf I_{A_i^k}(C_j), \sup I_{A_i^k}(C_j)] \subseteq [0,1]$, $F_{A_i^k}(C_j) = [\inf F_{A_i^k}(C_j), \sup F_{A_i^k}(C_j)] \subseteq [0,1]$, and $0 \le \sup T_{A_i^k}(C_j) + \sup I_{A_i^k}(C_j) + \sup F_{A_i^k}(C_j) \le 3$ for $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, t$. For convenience, an INULV in an INULS is denoted by $\tilde{d}_{ij}^k = \left\langle \left[s_{\theta_{ij}^k}^k, s_{\rho_{ij}^k}^k \right], \left([T_{ij}^{kL}, T_{ij}^{kU}], [I_{ij}^{kL}, I_{ij}^{kU}], [F_{ij}^{kL}, F_{ij}^{kU}] \right) \right\rangle$ ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, t$). Thus, one can establish the k-th interval neutrosophic uncertain linguistic decision matrix $D^k = (\tilde{d}_{ij}^k)_{m \times n}$ for $k = 1, 2, \ldots, t$.

The decision steps are described as follows:

Step 1: Obtain the integrated matrix $D = (\tilde{d}_{ij})_{m \times n}$ by the following aggregation formula:

$$\begin{aligned}\tilde{d}_{ij} &= \left\langle [s_{\theta_{ij}}, s_{\rho_{ij}}], \left([T_{ij}^L, T_{ij}^U], [I_{ij}^L, I_{ij}^U], [F_{ij}^L, F_{ij}^U] \right) \right\rangle \\ &= INULWAA(\tilde{d}_{ij}^1, \tilde{d}_{ij}^2, \ldots, \tilde{d}_{ij}^t) \\ &= \left\langle \left[s_{\sum_{k=1}^{t} v_k \theta_{ij}^k}, s_{\sum_{k=1}^{t} v_k \rho_{ij}^k} \right], \left(\left[1 - \prod_{k=1}^{t} (1 - T_{ij}^{kL})^{v_k}, 1 - \prod_{k=1}^{t} (1 - T_{ij}^{kU})^{v_k} \right], \right.\right. \\ &\quad \left.\left. \left[\prod_{k=1}^{t} \left(I_{ij}^{kL} \right)^{v_k}, \prod_{k=1}^{t} \left(I_{ij}^{kU} \right)^{v_k} \right], \left[\prod_{k=1}^{t} \left(F_{ij}^{kL} \right)^{v_k}, \prod_{k=1}^{t} \left(F_{ij}^{kU} \right)^{v_k} \right] \right) \right\rangle \end{aligned} \tag{15}$$

Step 2: Calculate the individual overall value of the INULV $\tilde{d}_i$ for A_i ($i = 1, 2, \ldots, m$) by the following aggregation formula:

$$\begin{aligned}\tilde{d}_i &= \left\langle [s_{\theta_i}, s_{\rho_i}], \left([T_i^L, T_i^U], [I_i^L, I_i^U], [F_i^L, F_i^U] \right) \right\rangle \\ &= INULWAA(\tilde{d}_{i1}, \tilde{d}_{i2}, \cdots, \tilde{d}_{in}) \\ &= \left\langle \left[s_{\sum_{j=1}^{n} w_j \theta_{ij}}, s_{\sum_{j=1}^{n} w_j \rho_{ij}} \right], \left(\left[1 - \prod_{j=1}^{n} (1 - T_{ij}^L)^{w_j}, 1 - \prod_{j=1}^{n} (1 - T_{ij}^U)^{w_j} \right], \right.\right. \\ &\quad \left.\left. \left[\prod_{j=1}^{n} \left(I_{ij}^L \right)^{w_j}, \prod_{j=1}^{n} \left(I_{ij}^U \right)^{w_j} \right], \left[\prod_{j=1}^{n} \left(F_{ij}^L \right)^{w_j}, \prod_{j=1}^{n} \left(F_{ij}^U \right)^{w_j} \right] \right) \right\rangle \end{aligned} \tag{16}$$

or

$$\begin{aligned}\tilde{d}_i &= \left\langle [s_{\theta_i}, s_{\rho_i}], \left([T_i^L, T_i^U], [I_i^L, I_i^U], [F_i^L, F_i^U] \right) \right\rangle \\ &= INULWGA(\tilde{d}_{i1}, \tilde{d}_{i2}, \ldots, \tilde{d}_{in}) \\ &= \left\langle \left[s_{\prod_{j=1}^{n} \theta_{ij}^{w_j}}, s_{\prod_{j=1}^{n} \rho_{ij}^{w_j}} \right], \left(\left[\prod_{j=1}^{n} \left(T_{ij}^L \right)^{w_j}, \prod_{j=1}^{n} \left(T_{ij}^U \right)^{w_j} \right], \right.\right. \\ &\quad \left[1 - \prod_{j=1}^{n} (1 - I_{ij}^L)^{w_j}, 1 - \prod_{j=1}^{n} (1 - I_{ij}^U)^{w_j} \right], \\ &\quad \left.\left. \left[1 - \prod_{j=1}^{n} (1 - F_{ij}^L)^{w_j}, 1 - \prod_{j=1}^{n} (1 - F_{ij}^U)^{w_j} \right] \right) \right\rangle \end{aligned} \tag{17}$$

Step 3: Calculate the score function $E(\tilde{d}_i)$ ($i = 1, 2, \ldots, m$) (accuracy function $H(\tilde{d}_i)$ and certainty function $C(\tilde{d}_i)$) by applying Eq. (6) (Eqs. (7) and (8)).

Step 4: Rank the alternatives according to the values of $E(\tilde{d}_i)$ ($H(\tilde{d}_i)$ and $C(\tilde{d}_i)$) ($i = 1, 2, \ldots, m$) by the ranking method in Definition 12, and then select the best one(s).

Step 5: End.

6 Illustrative example

An illustrative example about investment alternatives adapted from [18] is used to demonstrate the applications of the proposed decision-making method under an interval neutrosophic uncertain linguistic environment. There is an investment company, which wants to invest a sum of money in the best option. To invest the money, there is a panel with four possible alternatives: (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company. The investment company must take a decision according to the three attributes: (1) C_1 is the risk; (2) C_2 is the growth; (3) C_3 is the environmental impact. The weight vector of the attributes is $\boldsymbol{W} = (0.35, 0.25, 0.4)^{\mathrm{T}}$. A group of experts evaluate the four possible alternatives of A_i ($i = 1, 2, 3, 4$) with respect to the three attributes of C_j ($j = 1, 2, 3$), where the evaluation information is expressed by the form of INULV values under the linguistic term set $S = \{s_0$ = extremely poor, s_1 = very poor, s_2 = poor, s_3 = medium, s_4 = good, s_5 = very good, s_6 = extremely good$\}$.

Assume that three experts or decision makers are required in the evaluation process and their weight vector is given as $\boldsymbol{V} = (0.37, 0.33, 0.3)^{\mathrm{T}}$. Then, the evaluation information of an alternative A_i ($i = 1, 2, 3, 4$) with respect to an attribute C_j ($j = 1, 2, 3$) can be given by the three experts. For example, the INULV value of an alternative A_1 with respect to an attribute C_1 is given as $< \left[s_{4,}^1 s_5^1 \right]$ ([0.4, 0.5], [0.2, 0.3], [0.3, 0.4]) > by the first expert E_1, which indicates that the mark of the alternative A_1 with respect to the attribute C_1 is about the uncertain linguistic value $\left[s_4^1, s_5^1\right]$ with the satisfaction degree interval [0.4, 0.5], indeterminacy degree interval [0.2, 0.3], and dissatisfaction degree interval [0.3, 0.4]. Similarly, the four possible alternatives with respect to the above three attributes can be evaluated by the three experts, thus we can obtain the following three interval neutrosophic uncertain linguistic decision matrices:

$$D^1 = \left(\tilde{d}_{ij}^1\right)_{4\times 3} = \left[\begin{array}{l} \langle [s_4^1, s_5^1], ([0.4, 0.5], [0.2, 0.3], [0.3, 0.4])\rangle \\ \langle [s_5^1, s_6^1], ([0.5, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle \\ \langle [s_3^1, s_4^1], ([0.3, 0.4], [0.1, 0.2], [0.3, 0.4])\rangle \\ \langle [s_4^1, s_5^1], ([0.6, 0.7], [0.0, 0.1], [0.1, 0.2])\rangle \end{array}\right.$$

$$\begin{array}{l} \langle [s_4^1, s_5^1], ([0.4, 0.5], [0.1, 0.2], [0.3, 0.4])\rangle \\ \langle [s_4^1, s_5^1], ([0.6, 0.7], [0.1, 0.2], [0.1, 0.2])\rangle \\ \langle [s_3^1, s_4^1], ([0.4, 0.5], [0.1, 0.3], [0.3, 0.4])\rangle \\ \langle [s_3^1, s_4^1], ([0.5, 0.7], [0.0, 0.1], [0.1, 0.2])\rangle \end{array}$$

$$\left.\begin{array}{l} \langle [s_3^1, s_4^1], ([0.3, 0.4], [0.1, 0.2], [0.5, 0.6])\rangle \\ \langle [s_3^1, s_4^1], ([0.5, 0.6], [0.1, 0.2], [0.1, 0.2])\rangle \\ \langle [s_2^1, s_3^1], (0.5, 0.6], [0.2, 0.3], [0.0, 0.1])\rangle \\ \langle [s_2^1, s_3^1], ([0.3, 0.4], [0.1, 0.2], [0.1, 0.2])\rangle \end{array}\right],$$

$$D^2 = \left(\tilde{d}_{ij}^2\right)_{4\times 3} = \left[\begin{array}{l} \langle [s_4^2, s_5^2], ([0.4, 0.5], [0.2, 0.3], [0.3, 0.4])\rangle \\ \langle [s_5^2, s_6^2], ([0.6, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle \\ \langle [s_5^2, s_6^2], ([0.4, 0.5], [0.1, 0.2], [0.3, 0.4])\rangle \\ \langle [s_4^2, s_5^2], ([0.6, 0.8], [0.0, 0.1], [0.1, 0.2])\rangle \end{array}\right.$$

$$\begin{array}{l} \langle [s_5^2, s_6^2], ([0.4, 0.5], [0.0, 0.1], [0.2, 0.3])\rangle \\ \langle [s_4^2, s_5^2], ([0.6, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle \\ \langle [s_5^2, s_6^2], ([0.5, 0.6], [0.2, 0.3], [0.3, 0.4])\rangle \\ \langle [s_4^2, s_5^2], ([0.6, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle \end{array}$$

$$\left.\begin{array}{l} \langle [s_4^2, s_5^2], ([0.2, 0.3], [0.0, 0.1], [0.6, 0.7])\rangle \\ \langle [s_4^2, s_5^2], ([0.5, 0.7], [0.1, 0.2], [0.1, 0.2])\rangle \\ \langle [s_3^2, s_4^2], (0.5, 0.6], [0.1, 0.2], [0.2, 0.3])\rangle \\ \langle [s_3^2, s_4^2], ([0.3, 0.4], [0.1, 0.2], [0.1, 0.2])\rangle \end{array}\right],$$

$$D^3 = \left(\tilde{d}_{ij}^3\right)_{4\times 3} = \left[\begin{array}{l} \langle [s_5^3, s_6^3], ([0.4, 0.5], [0.1, 0.3], [0.2, 0.3])\rangle \\ \langle [s_4^3, s_5^3], ([0.5, 0.6], [0.1, 0.2], [0.2, 0.3])\rangle \\ \langle [s_5^3, s_6^3], ([0.4, 0.5], [0.0, 0.1], [0.2, 0.3])\rangle \\ \langle [s_5^3, s_6^3], ([0.6, 0.7], [0.0, 0.1], [0.1, 0.2])\rangle \end{array}\right.$$

$$\begin{array}{l} \langle [s_5^3, s_6^3], ([0.5, 0.6], [0.1, 0.2], [0.3, 0.4])\rangle \\ \langle [s_3^3, s_4^3], ([0.7, 0.8], [0.1, 0.2], [0.1, 0.2])\rangle \\ \langle [s_4^3, s_5^3], ([0.5, 0.6], [0.0, 0.1], [0.3, 0.4])\rangle \\ \langle [s_3^3, s_4^3], ([0.5, 0.6], [0.1, 0.2], [0.1, 0.2])\rangle \end{array}$$

$$\left.\begin{array}{l} \langle [s_4^3, s_5^3], ([0.1, 0.3], [0.0, 0.1], [0.5, 0.6])\rangle \\ \langle [s_4^3, s_5^3], ([0.7, 0.8], [0.1, 0.2], [0.0, 0.1])\rangle \\ \langle [s_3^3, s_4^3], (0.6, 0.7], [0.1, 0.2], [0.0, 0.1])\rangle \\ \langle [s_2^3, s_3^3], ([0.4, 0.5], [0.0, 0.1], [0.0, 0.1])\rangle \end{array}\right].$$

The proposed group decision-making method can handle this decision-making problem according to the following calculational steps:

Step 1: By using Eq. (15), we obtain the following integrated matrix:

$$D = \left(\tilde{d}_{ij}\right)_{4\times 3} = \left[\begin{array}{l} \langle [s_{4.30}, s_{5.30}], ([0.4000, 0.5000], [0.1625, 0.3000], [0.2656, 0.3669])\rangle \\ \langle [s_{4.70}, s_{5.70}], ([0.5355, 0.6730], [0.1000, 0.2000], [0.2000, 0.3000])\rangle \\ \langle [s_{4.26}, s_{5.26}], ([0.3648, 0.4651], [0.0000, 0.1625], [0.2656, 0.3669])\rangle \\ \langle [s_{4.30}, s_{5.30}], ([0.6000, 0.7000], [0.0000, 0.1000], [0.1000, 0.2000])\rangle \end{array}\right.$$

$$\begin{array}{l} \langle [s_{4.63}, s_{5.63}], ([0.4319, 0.5324], [0.0000, 0.1591], [0.2624, 0.3638))\rangle \\ \langle [s_{3.70}, s_{4.70}], ([0.6331, 0.7344], [0.1000, 0.2000], [0.1257, 0.2286])\rangle \\ \langle [s_{3.96}, s_{4.96}], ([0.4651, 0.5656], [0.0000, 0.2158], [0.3000, 0.4000))\rangle \\ \langle [s_{3.33}, s_{4.33}], ([0.5355, 0.6730], [0.0000, 0.1548], [0.1257, 0.2286])\rangle \end{array}$$

$$\left.\begin{array}{l} \langle [s_{3.63}, s_{4.63}], ([0.2112, 0.3388], [0.0000, 0.1292], [0.5310, 0.6313))\rangle \\ \langle [s_{3.63}, s_{4.63}], ([0.5710, 0.7045], [0.1000, 0.2000], [0.0000, 0.1625])\rangle \\ \langle [s_{2.63}, s_{3.63}], ([0.5324, 0.6331], [0.1292, 0.2324], [0.0000, 0.1437))\rangle \\ \langle [s_{2.33}, s_{3.33}], ([0.3316, 0.4319], [0.0000, 0.1625], [0.0000, 0.1625])\rangle \end{array}\right].$$

Step 2: By applying Eq. (16), we can obtain the individual overall values of the INULV $\tilde{d}_i$ for A_i ($i = 1, 2, 3, 4$):

$\tilde{d}_1$ = ⟨[$s_{4.1145}$, $s_{5.1145}$], ([0.3397, 0.4502], [0.0000, 0.1828], [0.3493, 0.4549])⟩,

$\tilde{d}_2$ = ⟨[$s_{4,0220}$, $s_{5.0220}$], ([0.5758, 0.7019], [0.1000, 0.2000], [0.0000, 0.2193])⟩,

$\tilde{d}_3$ = ⟨[$s_{3.5330}$, $s_{4.5330}$], ([0.4617, 0.5633], [0.0000, 0.2013], [0.0000, 0.2577])⟩,

$\tilde{d}_4$ = ⟨[$s_{3.7320}$, $s_{4.7320}$], ([0.4901, 0.6043], [0.0000, 0.1355], [0.0000, 0.1903])⟩.

Step 3: By applying Eq. (6), we can obtain the score values of $E(\tilde{d}_i)$ ($i = 1, 2, 3, 4$):

$E(\tilde{d}_1) = s_{2.9247}$, $E(\tilde{d}_2) = s_{3.5862}$, $E(\tilde{d}_3) = s_{3.0691}$, and $E(\tilde{d}_4) = s_{3.3635}$.

Step 4: Since $E(\tilde{d}_2) > E(\tilde{d}_4) > E(\tilde{d}_3) > E(\tilde{d}_1)$, the ranking order of four alternatives is $A_2 > A_4 > A_3 > A_1$. Therefore, we can see that the alternative A_2 is the best choice among all the alternatives.

On the other hand, we can also utilize the INULWGA operator as the following computational steps:

Step 1′: The same as Step 1.

Step 2′: By applying Eq. (17), we compute the individual overall values of the INULV $\tilde{d}_i$ for A_i ($i = 1, 2, 3, 4$):

$\tilde{d}_1 = \langle [s_{4.0933}, s_{5.0975}], ([0.3158, 0.4347], [0.0602, 0.2003], [0.3855, 0.4894]) \rangle$,,

$\tilde{d}_2 = \langle [s_{3.9925}, s_{4.9982}], ([0.5729, 0.7006], [0.1000, 0.2000], [0.1057, 0.2295]) \rangle$,

$\tilde{d}_3 = \langle [s_{3.4490}, s_{4.4687}], ([0.4509, 0.5525], [0.0538, 0.2044], [0.1790, 0.2951]) \rangle$,

$\tilde{d}_4 = \langle [s_{3.7040}, s_{4.7098}], ([0.4600, 0.5714], [0.0000, 0.1392], [0.0680, 0.1926]) \rangle$.

Step 3′: By using Eq. (6), we can get the score values of $E(\tilde{d}_i)$ $(i = 1, 2, 3, 4)$:

$E(\tilde{d}_1) = s_{2.7688}$, $E(\tilde{d}_2) = s_{3.4752}$, $E(\tilde{d}_3) = s_{2.8181}$, and $E(\tilde{d}_4) = s_{3.2475}$.

Step 4′: Since $E(\tilde{d}_2) > E(\tilde{d}_4) > E(\tilde{d}_3) > E(\tilde{d}_1)$, the ranking order of four alternatives is $A_2 \succ A_4 \succ A_3 \succ A_1$. Thus, we can see that the alternative A_2 is still the best choice among all the alternatives.

From the above decision results, we can see that the two kinds of ranking orders and the best alternative are identical, which are in agreement with the results of the method in [18].

Compared with the author's previous method in [18], although the decision results are in accordance with the ones in [18], the method proposed in this paper differs from the method in [18] for the multiple attribute decision-making problem not only due to the fact that the method proposed in this paper uses the INULV information and the weighted arithmetic aggregation operator and the weighted geometric aggregation operator for INULVs in the group decision-making problem, but also due to the consideration of the uncertain linguistic variable represented by decision makers' judgment to an evaluated object and the subjective evaluation value on the reliability of the given uncertain linguistic variable, which includes the indeterminacy information besides truth and falsity information belonging to the uncertain linguistic variable in the INULV. However, the method in [18] is a special case of the proposed method in this paper. Therefore, the group decision-making method proposed in this paper is more general and more feasible than the decision-making method in [18] since the former is a generalization of the later. The advantage is that the former easily reflects the ambiguous nature of a group of decision makers' judgment to an evaluated object because the INULV can provide the uncertain linguistic variable which easily expresses the qualitative information and the reliability of the given uncertain linguistic variable by a group of decision makers (experts) in the group decision-making problem, while the decision-making method in [18] can only provide the exact linguistic value, which difficultly expresses the qualitative information in some situations, and the reliability of the given linguistic value by unique decision maker (expert). Therefore, the group decision-making method in this paper is superior to the decision-making method in [18].

7 Conclusion

To deal with group decision making problems with INULVs, this paper proposed a group decision-making method based on the INULWAA and INULWGA operators to handle group decision-making problems with interval neutrosophic uncertain linguistic information. First, INULSs and the operation rules of INULVs were proposed as the generalization of the concepts of INLSs and INLVs. Then, the score function, accuracy function and certainty function of an INULV were defined to rank INULVs. Furthermore, we proposed the INULWAA and INULWGA operators and investigated their properties, and then applied them to group decision-making problems with interval neutrosophic uncertain linguistic information. Finally, an illustrative example was provided to demonstrate the application of the proposed method. The developed group decision-making method is the extension of existing decision-making method [18] and more suitable for expressing indeterminate and inconsistent information and handling group decision-making problems with interval neutrosophic uncertain linguistic information. The group decision-making method in this paper is superior to the one in [18]. In the future, we shall continue working on the extension and application of the developed operators to other domains.

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