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ORIGINAL ARTICLE

Probability multi-valued neutrosophic sets and its application in multi-criteria group decision-making problems

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Abstract This paper introduces probability multi-valued neutrosophic sets (PMVNSs) based on multi-valued neutrosophic sets and probability distribution. PMVNS can serve as a reliable tool to depict uncertain, incomplete, inconsistent and hesitant decision-making information and reflect the distribution characteristics of all provided evaluation values. This paper focuses on developing an innovative method to address multi-criteria group decision-making (MCGDM) problems in which the weight information is completely unknown and the evaluation values taking the form of probability multi-valued neutrosophic numbers (PMVNNs). First, the definition of PMVNSs is described. Second, an extended convex combination operation of PMVNNs is defined, and the probability multi-valued neutrosophic number weighted average operator is proposed. Moreover, two cross-entropy measures for PMVNNs are presented, and a novel qualitative flexible multiple criteria method (QUALIFLEX) is developed. Subsequently, an innovative MCGDM approach is established by incorporating the proposed aggregation operator and the developed QUALIFLEX method. Finally, an illustrative example concerning logistics outsourcing is provided to demonstrate the proposed method, and its feasibility and validity are further verified by comparison with other existing methods.

Keywords Multi-criteria group decision-making · Probability multi-valued neutrosophic sets · Cross-entropy · QUALIFLEX

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1 Introduction

To deal with fuzzy information, Zadeh [1] proposed fuzzy sets (FSs), which are now considered to be useful tools in the context of decision-making problems [2]. However, in some cases, the membership degree alone cannot precisely describe the information in decision-making problems. In order to address this issue, Atanassov [3] introduced intuitionistic fuzzy sets (IFSs), which measure both membership degree and non-membership degree. Since their introduction, IFSs have been researched in great detail, and some extensions of IFSs have been developed and applied to multi-criteria decision-making (MCDM) problems [4–6]. Torra and Narukawa [7] first introduced hesitant fuzzy sets (HFSs), an extension of traditional fuzzy sets that permit the membership degree of an element to be a set of several possible values in $[0, 1]$, and whose main purpose is to model the uncertainty produced by human doubt when eliciting information [8]. The information measures for HFSs have been studied in depth, including distance and similarity measures [9], correlation coefficients [10], entropy and cross-entropy [11].

Although the FSs theory has been developed and generalized, it cannot handle all types of uncertainties in real-life problems, especially those of inconsistent and incomplete information. For example, when several experts are asked for their comments about a given statement, some experts may have a consensus on the possibility that the statement is true is 0.6, some experts may have a consensus on the possibility that it is false is 0.4, and the others may have a consensus on the possibility that it is indeterminate is 0.3. Such kinds of issues cannot be appropriately dealt with using IFSs and HFSs. Thus, some new set theories are needed.

Smarandache [12, 13] proposed neutrosophic logic and neutrosophic sets (NSs). An NS employs the functions of truth, indeterminacy and falsity to depict decision-making information and considers the truth-membership degree, indeterminacy-membership degree and falsity-membership degree simultaneously. NSs can effectively deal with incomplete, imprecise and inconsistent information, and they are more flexible and applicable than IFSs and HFSs in decision-making problems. For the aforesaid example, the experts' comment can be presented as $x(0.6, 0.4, 0.3)$ resort to NS. Nevertheless, NSs are hard to apply in practical problems since the values of the functions with respect to truth, indeterminacy and falsity lie in $]0^-, 1^+[$. In view of this, a series of particular cases of NSs were introduced, including single-valued neutrosophic sets (SVNSs) [12, 14, 46, 49], interval neutrosophic sets (INSs) [15–18, 36, 39, 43, 47, 48], neutrosophic soft sets [19–22], neutrosophic refined sets [23–25, 45], bipolar neutrosophic sets (BNSs) [26, 27, 42], neutrosophic linguistic sets [28, 29, 64] and neutrosophic graphs [30–35]. Subsequently, studies of these particular sets have focused on defining operations and aggregation operators [36–38], distance measures [39, 40], similarity measures [41, 42], correlation coefficients [43–45], entropy and cross-entropy measures [46–48] and subsethood measures [49] to address decision-making problems. In addition, Ye [50] introduced simplified neutrosophic sets (SNSs), which can be represented using three discrete real numbers in $[0, 1]$, and the corresponding aggregation operators of SNSs were proposed. The similarity measures of SNSs were proposed by Ye [51]. The operations and comparison method of SNSs were improved by Peng et al. [38], and the cross-entropy measurements of SNNs were proposed by Wu et al. [52].

In some real-life situations, the truth-membership degree, indeterminacy-membership degree and falsity-membership degree in SNSs may be represented by several possible values due to the hesitance of experts. For the preceding example, some experts may estimate the possibility that the statement is true is 0.5, 0.6 or 0.7, some experts may estimate the possibility that it is false is 0.4 or 0.5, and the others may estimate the possibility that it is indeterminate is 0.2 or 0.3. Such issues cannot be properly solved using SNNs. Under these circumstances, Wang and Li [53] and Ye [54] introduced multi-valued neutrosophic sets (MVNSs) and single-valued neutrosophic hesitant fuzzy sets (SVNHFSs), respectively, based on NSs and HFSs. Actually, both MVNSs and SVNHFSs are characterized by truth-membership, indeterminacy-membership and falsity-membership functions that have a set of crisp values in $[0, 1]$, and there is no distinction between MVNSs and SVNHFSs. In the above example, the experts' comment can be described as $\langle\{0.5, 0.6, 0.7\}, \{0.4, 0.5\}, \{0.2, 0.3\}\rangle$ by means of MVNSs. In recent years, MVNSs have been studied deeply, and the corresponding MCDM methods have been developed and applied in various fields. Peng et al. [55] defined the Einstein operations for MVNSs and proposed the multi-valued neutrosophic power weighted average (MVNPWA) and multi-valued neutrosophic power weighted geometric (MVNPWG) aggregation operators. Liu et al. [56] proposed a series of Bonferroni mean (BM) aggregation operators for MVNSs. Ji et al. [57] defined the normalized projection measurement for MVNSs, and a projection-based TODIM method was developed to deal with personnel selection problems. Peng et al. [58] proposed an extended Elimination and Choice Translating Reality (ELECTRE) method to address MCDM problems under MVNSs environment. Peng et al. [59] proposed a multi-valued neutrosophic qualitative flexible multiple criteria method (QUALIFLEX) based on the likelihood of MVNSs to solve MCDM problems.

Nevertheless, in the current studies in regard to MVNSs, all possible values in specific part in a MVNN have equal weight and importance. It is quite apparent that it is not in conformity to the reality. In practice, some experts may prefer some of the possible values in practical decision-making problems, and different possible values in specific part in a MVNN may have inconsistent weights or importance degrees. In the preceding example, more experts may prefer 0.6 in the truth-membership degree and 0.5 in the indeterminacy-membership degree, as well as 0.2 in the falsity-membership degree. To deal with such kinds of issues, the characteristics of all possible values must be reflected and the difference among them need to be identified, creating an opportunity to utilize the probability distribution. Therefore, this paper proposes the concept of probability multi-valued neutrosophic sets (PMVNSs) by integrating MVNSs and probability distribution. PMVNS includes not only several possible values in the truth-membership degree, indeterminacy-membership degree and falsity-membership degree, but also the associated probabilistic information, which can be interpreted as importance degree, probability, weight, belief degree and so on. In the aforementioned example, experts have different preferences for distinct possible values in specific part in $\langle\{0.5, 0.6, 0.7\}, \{0.4, 0.5\}, \{0.2, 0.3\}\rangle$, and the corresponding probabilities or weights for these possible values can be obtained according to their distribution. Assume that the weights for 0.5, 0.6 and 0.7 in the truth-membership degree are 0.2, 0.5 and 0.3, respectively; the weights for 0.4 and 0.5 in the indeterminacy-membership degree are 0.3 and 0.6, respectively; the weights for 0.2 and 0.3 in the falsity-membership degree are 0.5 and 0.2, respectively. Therefore, the evaluation value can be described as $\langle\{0.5(0.2), 0.6(0.5), 0.7(0.3)\}, \{0.4(0.3), 0.5(0.6)\}, \{0.2(0.5), 0.3(0.2)\}\rangle$ using PMVNN.

Based on the analyses above, the primary motivations for this paper can be summarized as follows:

1. In view of the existing drawback of MVNSs that the different characteristics of all possible values in MVNSs cannot be distinguished and identified, this paper introduces probability distribution to characterize these possible values and proposes PMVNSs to facilitate the description of information.
2. In order to establish applicable and valid models to address decision-making problems with probability multi-valued neutrosophic information, this paper defines the aggregation operator and cross-entropy measures for PMVNNs and proposes a novel TOPSIS-based QUALIFLEX method, as well as integrates them to develop a comprehensive MCGDM method based on PMVNNs.

To do this, the rest part of this paper is organized as follows. In Sect. 2, some concepts, such as NSs, SNSs and MVNSs, are reviewed briefly. In Sect. 3, the concept of PMVNSs is defined, and the operation and aggregation operator for PMVNNs are proposed. In Sect. 4, two cross-entropy measures of PMVNNs are proposed, and a novel QUALIFLEX method is developed. In Sect. 5, the proposed aggregation operator and developed QUALIFLEX method are combined to establish an innovative MCGDM method. In Sect. 6, an illustrative example is used to verify the validity of the proposed approach, and the comparison analysis is conducted. Finally, the conclusion is drawn in Sect. 7.

2 Preliminaries

In this section, the definitions of NSs, SNSs and MVNSs will be introduced, all of which are necessary to the subsequent analysis.

2.1 NSs and SNSs

Definition 1 [13] Let X be a space of points (objects) with a generic element in X, denoted by x. Then an NS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. For each point x in X, $T_A(x) \in]0^-, 1^+[$, $I_A(x) \in]0^-, 1^+[$ and $F_A(x) \in]0^-, 1^+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, then $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

In fact, NSs are difficult to be applied in practical problems. To remove this shortcoming, the NSs of non-standard intervals are reduced into the SNSs of standard intervals [12, 50].

Definition 2 [50] Let X be a space of points (objects) with a generic element in X, denoted by x. Then an NS A in X is characterized by $T_A(x)$, $I_A(x)$ and $F_A(x)$, which are single subintervals/subsets in the real standard $[0, 1]$; that is, $T_A(x) : X \to [0, 1]$, $I_A(x) : X \to [0, 1]$ and $F_A(x) : X \to [0, 1]$. Thus, a simplification of A can be denoted by $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}$, which is an SNS and is a subclass of NSs. And the complement set of A is denoted by A^c and defined as $A^c = \{\langle x, F_A(x), I_A(x), T_A(x)\rangle | x \in X\}$. For convenience, a simplified neutrosophic number (SNN) can be described as $a = \langle T_A(x), I_A(x), F_A(x)\rangle$, and the set of all SNNs is presented as SNS.

2.2 Multi-valued neutrosophic sets

Definition 3 [53] Let X be a space of points (objects) with a generic element in X, denoted by x. Then a MVNS A in X is characterized by $\tilde{T}_A(x)$, $\tilde{I}_A(x)$ and $\tilde{F}_A(x)$ in the form of subset of $[0, 1]$ and can be expressed as:

$$A = \left\{\left\langle x, \tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x)\right\rangle | x \in X\right\}, \quad (1)$$

where $\tilde{T}_A(x)$, $\tilde{I}_A(x)$ and $\tilde{F}_A(x)$ are three sets of discrete real numbers in $[0, 1]$, showing the truth-membership degree, indeterminacy-membership degree and falsity-membership degree, respectively, satisfying $0 \leq \eta, \xi, \tau \leq 1$ and $0 \leq \eta^+ + \xi^+ + \tau^+ \leq 3$ where $\eta \in \tilde{T}_A(x)$, $\xi \in \tilde{I}_A(x)$, $\tau \in \tilde{F}_A(x)$ and $\eta^+ = \sup \tilde{T}_A(x)$, $\xi^+ = \sup \tilde{I}_A(x)$, $\tau^+ = \sup \tilde{F}_A(x)$.

In addition, $\left\langle \tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x)\right\rangle$, which is an element in A, is a multi-valued neutrosophic number (MVNN). For convenience, a MVNN is denoted as $a = \left\langle \tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x)\right\rangle$, and the set of all MVNNs is expressed as MVNS.

3 Probability multi-valued neutrosophic sets and its operation

In this section, the concept of probability multi-valued neutrosophic sets (PMVNSs) is proposed based on the elicitation of MVNSs and probability distribution. Subsequently, an extended convex combination operation and an aggregation operator for PMVNNs are developed.

All the existing studies with respect to MVNNs in decision-making problems assume that DMs have a consensus on several possible values with equal weight and importance for an evaluation. For example, to evaluate a given object using MVNNs, suppose one DM provides $\langle\{0.3, 0.5\}, \{0.4, 0.6\}, \{0.5\}\rangle$, and the other provides $\langle\{0.5, 0.6\}, \{0.4, 0.5\}, \{0.3\}\rangle$, then the overall evaluation values can be identified as $\langle\{0.3, 0.5, 0.6\}, \{0.4, 0.5, 0.6\}, \{0.3, 0.5\}\rangle$ according to the existing assumption. Obviously, the value 0.5 in truth-membership degree and the value 0.4 in indeterminacy-membership degree provided by one of the two DMs are lost, and the difference among these possible values in $\{0.3, 0.5, 0.6\}/\{0.4, 0.5, 0.6\}/\{0.3, 0.5\}$ cannot be ascertained. There is no doubt that this does not correspond to the reality and is lack of applicability. In order to effectively apply MVNNs in decision-making problems with multiple DMs, it is necessary to explore more feasible and practical presentation for DMs' preferences. Therefore, we suggest utilizing probability distribution to characterize all possible values in MVNNs and developing probability multi-valued neutrosophic sets (PMVNSs) to facilitate the description of information.

3.1 Probability multi-valued neutrosophic sets

Definition 4 Let X be a space of points (objects), with a generic element in X denoted by x. A probability multi-valued neutrosophic set (PMVNS) A in X can be defined as

$$A = \left\{ \left\langle x, T_A(P_t(x)), I_A(P_i(x)), F_A\left(P_f(x)\right) \right\rangle | x \in X \right\}, \tag{2}$$

where $T_A(P_t(x)) = \cup_{t_A^j \in T_A, p_t^j \in P_t(x)} \{t_A^j(p_t^j)\}$ is a set consisting of all possible truth-membership degrees $t_A^j \in T_A$ associated with the probability $p_t^j \in P_t(x)$, $I_A(P_i(x)) = \cup_{i_A^k \in I_A, p_i^k \in P_i(x)} \{i_A^k(p_i^k)\}$ is a set consisting of all possible indeterminacy-membership degrees $i_A^k \in I_A$ associated with the probability $p_i^k \in P_i(x)$, and $F_A\left(P_f(x)\right) = \cup_{f_A^l \in F_A, p_f^l \in P_f(x)} \{f_A^l(p_f^l)\}$ is a set consisting of all possible falsity-membership degrees $f_A^l \in F_A$ associated with the probability $p_f^l \in P_f(x)$, satisfying $0 \le t_A^j, i_A^k, f_A^l \le 1$, $0 \le t_A^{j+} + i_A^{k+} + f_A^{l+} \le 3$, $0 \le p_t^j, p_i^k, p_f^l \le 1$, $\sum_{j=1}^{\#T_A} p_t^j \le 1$, $\sum_{k=1}^{\#I_A} p_i^k \le 1$, $\sum_{l=1}^{\#F_A} p_f^l \le 1$, where $t_A^{j+} = \sup\,(T_A)$, $i_A^{k+} = \sup(I_A)$ and $f_A^{l+} = \sup\,(F_A)$, $\#T_A$, $\#I_A$ and $\#F_A$ are the number of all elements in $T_A(P_t(x))$, $I_A(P_i(x))$ and $F_A\left(P_f(x)\right)$, respectively. The complement set of A is denoted by A^c and defined as $A^c = \left\{ \left\langle x, F_A\left(P_f(x)\right), I_A(P_i(x)), T_A(P_t(x)) \right\rangle | x \in X \right\}$.

When X includes only one element, then the PMVNS A is reduced to a probability multi-valued neutrosophic number (PMVNN), denoting by $\langle T_A(P_t(x)), I_A(P_i(x)), F_A\left(P_f(x)\right) \rangle$. For convenience, a PMVNN can be described as $a = \left\langle T_A(P_t), I_A(P_i), F_A\left(P_f\right) \right\rangle$, and the set of all PMVNNs is presented as PMVNS. Moreover, when $p_t^1 = p_t^2 = \cdots = p_t^{\#T_A}$, $p_i^1 = p_i^2 = \cdots = p_i^{\#I_A}$ and $p_f^1 = p_f^2 = \cdots = p_f^{\#F_A}$, then the PMVNS A is reduced to the MVNS given in Definition 3. In particular, if $\#T_A = 1$, $\#I_A = 1$, $\#F_A = 1$ and $p_t^1 = 1$, $p_i^1 = 1$, $p_f^1 = 1$, then the PMVNS A is degenerated to the SNS presented in Definition 2. Therefore, SNS and MVNS are all special cases of PMVNN.

The probabilistic information associated with the possible values in PMVNNs can be interpreted as probability degree, importance, weight, belief degree and so on. And the probability multi-valued neutrosophic information can be collected in various real-life decision-making problems.

Example 1 In a personnel selection problem, four vice managers of human resources evaluate a candidate. They need to express three kinds of degrees to which she/he supports the candidate is capable for the position, and she/he is not sure whether the candidate is qualified for the position, as well as she/he deems the candidate is not suitable for the position in the form of a real number, or more than one due to hesitance. Therefore, MVNNs can be employed to depict such evaluation information. Assume that one provides $\langle \{0.5, 0.6\}, \{0.3\}, \{0.4\} \rangle$, one provides $\langle \{0.6, 0.7\}, \{0.2, 0.3\}, \{0.2\} \rangle$, one provides $\langle \{0.6, 0.7\}, \{0.1, 0.2\}, \{0.3\} \rangle$, another provides $\langle \{0.8\}, \{0.1\}, \{0.2, 0.3\} \rangle$. Then, the overall evaluation information is collected as $\langle \{0.5, 0.6, 0.6, 0.6, 0.7, 0.7, 0.8\}, \{0.1, 0.1, 0.2, 0.2, 0.3, 0.3\}, \{0.2, 0.2, 0.3, 0.3, 0.4\} \rangle$ without loss of any original information. Therefore, the final evaluation information taking the form of PMVNN can be identified as $\langle \{0.5(0.14), 0.6(0.43), 0.7(0.29), 0.8(0.14)\}, \{0.1(0.33), 0.2(0.33), 0.3(0.33)\}, \{0.2(0.4), 0.3(0.4), 0.4(0.2)\} \rangle$ according to the probability distribution.

For a PMVNN $a = \left\langle T_A(P_t), I_A(P_i), F_A\left(P_f\right) \right\rangle$, if $\sum_{j=1}^{\#T_A} p_t^j = 1$, then the probability distribution of all possible truth-membership degrees is complete; if $\sum_{j=1}^{\#T_A} p_t^j < 1$, then it is incomplete, and a normalized process is needed so that $p_t^j \in P_t(x)$ can be regarded as a complete probability distribution. The above analysis for indeterminacy-membership degree and falsity-membership degree is identical.

Definition 5 Given a PMVNN $a = \langle T_A(P_t), I_A(P_i), F_A\left(P_f\right) \rangle$ with $\sum p_\eta < 1, (\eta = t, i, f)$, then the normalized PMVNN $\tilde{a}$ of a is defined as follows:

$$\tilde{a} = \left\langle T_A\left(\tilde{P}_t\right), I_A\left(\tilde{P}_i\right), F_A\left(\tilde{P}_f\right) \right\rangle, \tag{3}$$

where $\tilde{p}_t^j = \frac{p_t^j}{\sum_{j=1}^{\#T_A} p_t^j} \in \tilde{P}_t$, $\tilde{p}_i^k = \frac{p_i^k}{\sum_{k=1}^{\#I_A} p_i^k} \in \tilde{P}_i$ and $\tilde{p}_f^l = \frac{p_f^l}{\sum_{l=1}^{\#F_A} p_f^l} \in \tilde{P}_f$.

3.2 Convex combination operation and aggregation operator of PMVNNs

Delgado et al. [60] firstly defined the convex combination of two linguistic terms. Subsequently, Wei et al. [61]

applied the convex combination operation to hesitant fuzzy linguistic term sets. Based on these extant studies and the operations of MVNNs [53], we define an extended convex combination operation for PMVNNs in the following.

Definition 6 Let $\phi_1 = \langle T_1(P_{t1}), I_1(P_{i1}), F_1(P_{f1}) \rangle$ and $\phi_2 = \langle T_2(P_{t2}), I_2(P_{i2}), F_2(P_{f2}) \rangle$ be two arbitrary PMVNNs, an extended convex combination operation of ϕ_1, and ϕ_2 is defined as

$$\begin{aligned}
&\mathrm{CC}_2(\omega_1, \phi_1, \omega_2, \phi_2) = \omega_1 \otimes \phi_1 \oplus \omega_2 \otimes \phi_2 \\
&= (\omega_1 + \omega_2) \otimes \left(\frac{\omega_1}{\omega_1 + \omega_2} \otimes \phi_1 \oplus \frac{\omega_2}{\omega_1 + \omega_2} \otimes \phi_2 \right) \\
&= \left\langle \cup_{t_1(p_{t1}) \in T_1(P_{t1}), t_2(p_{t2}) \in T_2(P_{t2})} \left\{ 1 - (1 - t_1)^{\omega_1} (1 - t_2)^{\omega_2} \left(\frac{\frac{(1-(1-t_1)^{\omega_1})\omega_1 p_{t1} + (1-(1-t_2)^{\omega_2})\omega_2 p_{t2}}{1-(1-t_1)^{\omega_1} + 1 - (1-t_2)^{\omega_2}}}{\sum_{t_1(p_{t1}) \in T_1(P_{t1})} \sum_{t_2(p_{t2}) \in T_2(P_{t2})} \frac{(1-(1-t_1)^{\omega_1})\omega_1 p_{t1} + (1-(1-t_2)^{\omega_2})\omega_2 p_{t2}}{1-(1-t_1)^{\omega_1} + 1 - (1-t_2)^{\omega_2}}} \right) \right\}, \right. \\
&\quad \cup_{i_1(p_{i1}) \in I_1(P_{i1}), i_2(p_{i2}) \in I_2(P_{i2})} \left\{ i_1^{\omega_1} i_2^{\omega_2} \left(\frac{\frac{i_1^{\omega_1} \omega_1 p_{i1} + i_2^{\omega_2} \omega_2 p_{i2}}{i_1^{\omega_1} + i_2^{\omega_2}}}{\sum_{i_1(p_{i1}) \in I_1(P_{i1})} \sum_{i_2(p_{i2}) \in I_2(P_{i2})} \frac{i_1^{\omega_1} \omega_1 p_{i1} + i_2^{\omega_2} \omega_2 p_{i2}}{i_1^{\omega_1} + i_2^{\omega_2}}} \right) \right\}, \\
&\quad \left. \cup_{f_1(p_{f1}) \in F_1(P_{f1}), f_2(p_{f2}) \in F_1(P_{f1})} \left\{ f_1^{\omega_1} f_2^{\omega_2} \left(\frac{\frac{f_1^{\omega_1} \omega_1 p_{f1} + f_2^{\omega_2} \omega_2 p_{f2}}{f_1^{\omega_1} + f_2^{\omega_2}}}{\sum_{f_1(p_{f1}) \in F_1(P_{f1})} \sum_{f_2(p_{f2}) \in F_2(P_{f2})} \frac{f_1^{\omega_1} \omega_1 p_{f1} + f_2^{\omega_2} \omega_2 p_{f2}}{f_1^{\omega_1} + f_2^{\omega_2}}} \right) \right\} \right\rangle,
\end{aligned} \tag{4}$$

where $0 \le \omega_1, \omega_2 \le 1$, and $\omega_1 + \omega_2 = 1$. It is quite apparent that the result obtained by the above operation is also a PMVNN.

3.3 The weighted average aggregation operator with PMVNNs

Definition 7 Let $\phi_k = \langle T_1(P_{t1}), I_1(P_{i1}), F_1(P_{f1}) \rangle (k = 1, 2, \ldots, n)$ be a collection of PMVNNs, denoted by Ω, and $w = (w_1, w_2, \ldots, w_n)$ be the weight vector of $\phi_k (k = 1, 2, \ldots, n)$, with $w_k \in [0, 1]$ and $\sum_{k=1}^{n} w_k = 1$. Then, the probability multi-valued neutrosophic number weighted average (PMVNNWA) operator is the mapping $\mathrm{PMVNNWA} : \Omega^n \to \Omega$ and is defined as follows:

$$\begin{aligned}
&\mathrm{PMVNNWA}(\phi_1, \phi_2, \ldots, \phi_n) \\
&= \mathrm{CC}_n(\omega_k, \phi_k, k = 1, 2, \ldots, n) \\
&= \left(\sum_{k_n=1}^{n} \omega_{k_n} \right) \otimes \left(\frac{\sum_{k_{n-1}=1}^{n-1} \omega_{k_{n-1}}}{\sum_{k_{n-1}=1}^{n-1} \omega_{k_{n-1}}} \otimes \left(\frac{\sum_{k_{n-2}=1}^{n-2} \omega_{k_{n-2}}}{\sum_{k_{n-1}=1}^{n-1} \omega_{k_{n-1}}} \right. \right. \\
&\times \left(\cdots \left(\frac{\sum_{k_2=1}^{2} \omega_{k_2}}{\sum_{k_3=1}^{3} \omega_{k_3}} \otimes \left(\frac{\omega_1}{\sum_{k_2=1}^{2} \omega_{k_2}} \otimes \phi_1 \oplus \frac{\omega_2}{\sum_{k_2=1}^{2} \omega_{k_2}} \otimes \phi_2 \right) \right. \right. \\
&\times \left. \left. \left. \left. \oplus \frac{\omega_3}{\sum_{k_3=1}^{3} \omega_{k_3}} \otimes \phi_3 \right) \right) \right) \oplus \frac{\omega_n}{\sum_{k_n=1}^{n} \omega_{k_n}} \otimes \phi_n \right).
\end{aligned} \tag{5}$$

Based on the extended convex combination operation of PMVNNs given in Definition 6, the following result can be obtained.

Theorem 1 *Let* $\phi_k = \langle T_1(P_{t1}), I_1(P_{i1}), F_1(P_{f1}) \rangle (k = 1, 2, \ldots, n)$ *be a collection of PMVNNs, denoted by* Ω, *and* $w = (w_1, w_2, \ldots, w_n)$ *be the weight vector of* $\phi_k (k = 1, 2, \ldots, n)$, with $w_k \in [0, 1]$ and $\sum_{k=1}^{n} w_k = 1$. Then, the aggregated value calculated by the PMVNNWA operator is also a PMVNN, and

$$
\begin{aligned}
&\mathrm{PMVNNWA}(\phi_1,\phi_2,\ldots,\phi_n)=\mathrm{CC}_n(\omega_k,\phi_k,k=1,2,\ldots,n)\\
&=\left\langle \cup_{t_1(p_{t1})\in T_1(P_{t1}),t_2(p_{t2})\in T_2(P_{t2}),\ldots,t_n(p_{tn})\in T_n(P_{tn})}\left\{1-\prod_{j=1}^{n}\left(1-t_j\right)^{\omega_j}\left(\frac{\frac{\sum_{j=1}^{n}\left(1-\left(1-t_j\right)^{\omega_j}\right)\omega_j p_{tj}}{\sum_{j=1}^{n}\left(1-\left(1-t_j\right)^{\omega_j}\right)}}{\sum_{t_1(p_{t1})\in T_1(P_{t1})}\sum_{t_2(p_{t2})\in T_2(P_{t2})}\cdots\sum_{t_n(p_{tn})\in T_n(P_{tn})}\frac{\sum_{j=1}^{n}\left(1-\left(1-t_j\right)^{\omega_j}\right)\omega_j p_{tj}}{\sum_{j=1}^{n}\left(1-\left(1-t_j\right)^{\omega_j}\right)}}\right)\right\},\right.\\
&\quad \cup_{i_1(p_{i1})\in I_1(P_{i1}),i_2(p_{i2})\in I_2(P_{i2}),\ldots,i_n(p_{in})\in I_n(P_{in})}\left\{\prod_{h=1}^{n}i_h^{\omega_h}\left(\frac{\frac{\sum_{h=1}^{n}i_h^{\omega_h}\omega_h p_{ih}}{\sum_{h=1}^{n}i_h^{\omega_h}}}{\sum_{i_1(p_{i1})\in I_1(P_{i1})}\sum_{i_2(p_{i2})\in I_2(P_{i2})}\cdots\sum_{i_n(p_{in})\in I_n(P_{in})}\frac{\sum_{h=1}^{n}i_h^{\omega_h}\omega_h p_{ih}}{\sum_{h=1}^{n}i_h^{\omega_h}}}\right)\right\},\\
&\quad \left.\cup_{f_1(p_{f1})\in F_1(P_{f1}),f_2(p_{f2})\in F_1(P_{f1}),\ldots,f_n(p_{fn})\in F_n(P_{fn})}\left\{\prod_{l=1}^{n}f_l^{\omega_l}\left(\frac{\frac{\sum_{l=1}^{n}f_l^{\omega_l}\omega_l p_{fl}}{\sum_{l=1}^{n}f_l^{\omega_l}}}{\sum_{f_1(p_{f1})\in F_1(P_{f1})}\sum_{f_2(p_{f2})\in F_2(P_{f2})}\cdots\sum_{f_n(p_{fn})\in F_n(P_{fn})}\frac{\sum_{l=1}^{n}f_l^{\omega_l}\omega_l p_{fl}}{\sum_{l=1}^{n}f_l^{\omega_l}}}\right)\right\}\right\rangle.
\end{aligned}
\tag{6}
$$

In the following, Theorem 1 will be proved by the mathematical induction of n.

Proof

(1) When $n=2$, the following equation can be obtained according to Definition 6,

$$
\begin{aligned}
&\mathrm{PMVNNWA}(\phi_1,\phi_2)=\mathrm{CC}_2(\omega_1,\phi_1,\omega_2,\phi_2)=(\omega_1+\omega_2)\otimes\left(\frac{\omega_1}{\omega_1+\omega_2}\otimes\phi_1\oplus\frac{\omega_2}{\omega_1+\omega_2}\otimes\phi_2\right)\\
&=\left\langle \cup_{t_1(p_{t1})\in T_1(P_{t1}),t_2(p_{t2})\in T_2(P_{t2})}\left\{1-(1-t_1)^{\omega_1}(1-t_2)^{\omega_2}\left(\frac{\frac{(1-(1-t_1)^{\omega_1})\omega_1 p_{t1}+(1-(1-t_2)^{\omega_2})\omega_2 p_{t2}}{1-(1-t_1)^{\omega_1}+1-(1-t_2)^{\omega_2}}}{\sum_{t_1(p_{t1})\in T_1(P_{t1})}\sum_{t_2(p_{t2})\in T_2(P_{t2})}\frac{(1-(1-t_1)^{\omega_1})\omega_1 p_{t1}+(1-(1-t_2)^{\omega_2})\omega_2 p_{t2}}{1-(1-t_1)^{\omega_1}+1-(1-t_2)^{\omega_2}}}\right)\right\},\right.\\
&\quad \cup_{i_1(p_{i1})\in I_1(P_{i1}),i_2(p_{i2})\in I_2(P_{i2})}\left\{i_1^{\omega_1}i_2^{\omega_2}\left(\frac{\frac{i_1^{\omega_1}\omega_1 p_{i1}+i_2^{\omega_2}\omega_2 p_{i2}}{i_1^{\omega_1}+i_2^{\omega_2}}}{\sum_{i_1(p_{i1})\in I_1(P_{i1})}\sum_{i_2(p_{i2})\in I_2(P_{i2})}\frac{i_1^{\omega_1}\omega_1 p_{i1}+i_2^{\omega_2}\omega_2 p_{i2}}{i_1^{\omega_1}+i_2^{\omega_2}}}\right)\right\},\\
&\quad \left.\cup_{f_1(p_{f1})\in F_1(P_{f1}),f_2(p_{f2})\in F_1(P_{f1})}\left\{f_1^{\omega_1}f_2^{\omega_2}\left(\frac{\frac{f_1^{\omega_1}\omega_1 p_{f1}+f_2^{\omega_2}\omega_2 p_{f2}}{f_1^{\omega_1}+f_2^{\omega_2}}}{\sum_{f_1(p_{f1})\in F_1(P_{f1})}\sum_{f_2(p_{f2})\in F_2(P_{f2})}\frac{f_1^{\omega_1}\omega_1 p_{f1}+f_2^{\omega_2}\omega_2 p_{f2}}{f_1^{\omega_1}+f_2^{\omega_2}}}\right)\right\}\right\rangle\\
&=\left\langle \cup_{t_1(p_{t1})\in T_1(P_{t1}),t_2(p_{t2})\in T_2(P_{t2})}\left\{1-\prod_{j=1}^{2}\left(1-t_j\right)^{\omega_j}\left(\frac{\frac{\sum_{j=1}^{2}\left(1-\left(1-t_j\right)^{\omega_j}\right)\omega_j p_{tj}}{\sum_{j=1}^{2}\left(1-\left(1-t_j\right)^{\omega_j}\right)}}{\sum_{t_1(p_{t1})\in T_1(P_{t1})}\sum_{t_2(p_{t2})\in T_2(P_{t2})}\frac{\sum_{j=1}^{2}\left(1-\left(1-t_j\right)^{\omega_j}\right)\omega_j p_{tj}}{\sum_{j=1}^{2}\left(1-\left(1-t_j\right)^{\omega_j}\right)}}\right)\right\},\right.\\
&\quad \cup_{i_1(p_{i1})\in I_1(P_{i1}),i_2(p_{i2})\in I_2(P_{i2})}\left\{\prod_{h=1}^{2}i_h^{\omega_h}\left(\frac{\frac{\sum_{h=1}^{2}i_h^{\omega_h}\omega_h p_{ih}}{\sum_{h=1}^{2}i_h^{\omega_h}}}{\sum_{i_1(p_{i1})\in I_1(P_{i1})}\sum_{i_2(p_{i2})\in I_2(P_{i2})}\frac{\sum_{h=1}^{2}i_h^{\omega_h}\omega_h p_{ih}}{\sum_{h=1}^{2}i_h^{\omega_h}}}\right)\right\},\\
&\quad \left.\cup_{f_1(p_{f1})\in F_1(P_{f1}),f_2(p_{f2})\in F_1(P_{f1})}\left\{\prod_{l=1}^{2}f_l^{\omega_l}\left(\frac{\frac{\sum_{l=1}^{2}f_l^{\omega_l}\omega_l p_{fl}}{\sum_{l=1}^{2}f_l^{\omega_l}}}{\sum_{f_1(p_{f1})\in F_1(P_{f1})}\sum_{f_2(p_{f2})\in F_2(P_{f2})}\frac{\sum_{l=1}^{2}f_l^{\omega_l}\omega_l p_{fl}}{\sum_{l=1}^{2}f_l^{\omega_l}}}\right)\right\}\right\rangle,
\end{aligned}
$$

That is, when $n = 2$, Eq. (2) is true.

(2) If Eq. (6) holds for $n = g$, then

$$\mathrm{PMVNNWA}(\phi_1, \phi_2, \ldots, \phi_g) = \mathrm{CC}_g(\omega_k, \phi_k, k = 1, 2, \ldots, g)$$
$$= \left\langle \cup_{t_1(p_{t1})\in T_1(P_{t1}), t_2(p_{t2})\in T_2(P_{t2}), \ldots, t_g(p_{tg})\in T_g(P_{tg})} \left\{ 1 - \prod_{j=1}^{g} (1 - t_j)^{\omega_j} \left(\frac{\frac{\sum_{j=1}^{g} (1-(1-t_j)^{\omega_j})\omega_j p_{tj}}{\sum_{j=1}^{g} (1-(1-t_j)^{\omega_j})}}{\sum_{t_1(p_{t1})\in T_1(P_{t1})} \sum_{t_2(p_{t2})\in T_2(P_{t2})} \cdots \sum_{t_g(p_{tg})\in T_g(P_{tg})} \frac{\sum_{j=1}^{g} (1-(1-t_j)^{\omega_j})\omega_j p_{tj}}{\sum_{j=1}^{g} (1-(1-t_j)^{\omega_j})}} \right) \right\}, \right.$$
$$\cup_{i_1(p_{i1})\in I_1(P_{i1}), i_2(p_{i2})\in I_2(P_{i2}), \ldots, i_g(p_{ig})\in I_g(P_{ig})} \left\{ \prod_{h=1}^{g} i_h^{\omega_h} \left(\frac{\frac{\sum_{h=1}^{g} i_h^{\omega_h} \omega_h p_{ih}}{\sum_{h=1}^{g} i_h^{\omega_h}}}{\sum_{t_1(p_{t1})\in T_1(P_{t1})} \sum_{t_2(p_{t2})\in T_2(P_{t2})} \cdots \sum_{t_g(p_{tg})\in T_g(P_{tg})} \frac{\sum_{j=1}^{g} (1-(1-t_j)^{\omega_j})\omega_j p_{tj}}{\sum_{j=1}^{g} (1-(1-t_j)^{\omega_j})}} \right) \right\},$$
$$\left. \cup_{f_1(p_{f1})\in F_1(P_{f1}), f_2(p_{f2})\in F_1(P_{f1}), \ldots, f_g(p_{fg})\in F_g(P_{fg})} \left\{ \prod_{l=1}^{g} f_l^{\omega_l} \left(\frac{\frac{\sum_{l=1}^{g} f_l^{\omega_l} \omega_l p_{fl}}{\sum_{l=1}^{g} f_l^{\omega_l}}}{\sum_{f_1(p_{f1})\in F_1(P_{f1})} \sum_{f_2(p_{f2})\in F_2(P_{f2})} \cdots \sum_{f_g(p_{fg})\in F_g(P_{fg})} \frac{\sum_{l=1}^{g} f_l^{\omega_l} \omega_l p_{fl}}{\sum_{l=1}^{g} f_l^{\omega_l}}} \right) \right\} \right\rangle.$$

Thus, when $n = g + 1$, the following result can be computed.

$$\mathrm{PMVNNWA}(\phi_1, \phi_2, \ldots, \phi_g, \phi_{g+1}) = \mathrm{CC}_{g+1}(\omega_k, \phi_k, k = 1, 2, \ldots, g+1)$$
$$= \left(\sum_{k_{g+1}=1}^{g+1} \omega_{k_{g+1}} \right) \otimes \left(\frac{\sum_{k_g=1}^{g} \omega_{k_g}}{\sum_{k_{g+1}=1}^{g+1} \omega_{k_{g+1}}} \otimes (CC_g(\omega_k, \phi_k, k = 1, 2, \ldots, g)) \oplus \frac{\omega_{k_{g+1}}}{\sum_{k_{g+1}=1}^{g+1} \omega_{k_{g+1}}} \otimes \phi_{g+1} \right)$$
$$= \left\langle \cup_{t_1(p_{t1})\in T_1(P_{t1}), \ldots, t_{g+1}(p_{tg+1})\in T_{g+1}(P_{tg+1})} \left\{ 1 - \prod_{j=1}^{g+1} (1 - t_j)^{\omega_j} \left(\frac{\frac{\sum_{j=1}^{g+1} (1-(1-t_j)^{\omega_j})\omega_j p_{tj}}{\sum_{j=1}^{g+1} (1-(1-t_j)^{\omega_j})}}{\sum_{t_1(p_{t1})\in T_1(P_{t1})} \cdots \sum_{t_{g+1}(p_{tg+1})\in T_{g+1}(P_{tg+1})} \frac{\sum_{j=1}^{g+1} (1-(1-t_j)^{\omega_j})\omega_j p_{tj}}{\sum_{j=1}^{g+1} (1-(1-t_j)^{\omega_j})}} \right) \right\}, \right.$$
$$\cup_{i_1(p_{i1})\in I_1(P_{i1}), \ldots, i_g(p_{ig+1})\in I_g(P_{ig+1})} \left\{ \prod_{h=1}^{g+1} i_h^{\omega_h} \left(\frac{\frac{\sum_{h=1}^{g+1} i_h^{\omega_h} \omega_h p_{ih}}{\sum_{h=1}^{g+1} i_h^{\omega_h}}}{\sum_{i_1(p_{i1})\in I_1(P_{i1})} \cdots \sum_{i_g(p_{ig+1})\in I_g(P_{ig+1})} \frac{\sum_{h=1}^{g+1} i_h^{\omega_h} \omega_h p_{ih}}{\sum_{h=1}^{g+1} i_h^{\omega_h}}} \right) \right\},$$
$$\left. \cup_{f_1(p_{f1})\in F_1(P_{f1}), \ldots, f_{g+1}(p_{fg+1})\in F_{g+1}(P_{fg+1})} \left\{ \prod_{l=1}^{g+1} f_l^{\omega_l} \left(\frac{\frac{\sum_{l=1}^{g+1} f_l^{\omega_l} \omega_l p_{fl}}{\sum_{l=1}^{g+1} f_l^{\omega_l}}}{\sum_{f_1(p_{f1})\in F_1(P_{f1})} \cdots \sum_{f_{g+1}(p_{fg+1})\in F_{g+1}(P_{fg+1})} \frac{\sum_{l=1}^{g+1} f_l^{\omega_l} \omega_l p_{fl}}{\sum_{l=1}^{g+1} f_l^{\omega_l}}} \right) \right\} \right\rangle.$$

That is, Eq. (6) also holds for $n = g + 1$. Therefore, Eq. (6) is true for all n.

4 TOPSIS-based QUALIFLEX method with PMVNNs

In this section, the comparison method and cross-entropy measures of PMVNNs are proposed. Subsequently, a novel QUALIFLEX method is developed by incorporating the proposed cross-entropy measures and the closeness coefficient of TOPSIS.

4.1 The comparison method of PMVNNs

Definition 8 Let $\phi_1 = \langle T_1(P_{t1}), I_1(P_{i1}), F_1(P_{f1})\rangle$ and $\phi_2 = \langle T_2(P_{t2}), I_2(P_{i2}), F_2(P_{f2})\rangle$ be two arbitrary PMVNNs, all elements in $T_\varepsilon(P_{t\varepsilon})$, $I_\varepsilon(P_{i\varepsilon})$ and $F_\varepsilon(P_{f\varepsilon})(\varepsilon = 1, 2)$ be arranged in ascending order according to the values of $t_\varepsilon \times p_{t\varepsilon}$, $i_\varepsilon \times p_{i\varepsilon}$ and $f_\varepsilon \times p_{f\varepsilon}(\varepsilon = 1, 2)$, respectively. And $t_\varepsilon^{(j)}(p_{t\varepsilon})$, $i_\varepsilon^{(j)}(p_{i\varepsilon})$ and $f_\varepsilon^{(j)}(p_{f\varepsilon})(\varepsilon = 1, 2)$ be referred to as the j *th* value in $T_\varepsilon(P_{t\varepsilon})$, $I_\varepsilon(P_{i\varepsilon})$ and $F_\varepsilon(P_{f\varepsilon})(\varepsilon = 1, 2)$. Then, the comparison method of PMVNNs is provided as follows:

$\phi_1 \le \phi_2$, if $t_1^{(j)}(p_{t1}) \le t_2^{(j)}(p_{t2})$ and $t_1^{(\#T_1)}(p_{t1}) \le t_2^{(\#T_2)}(p_{t2})$, $i_1^{(k)}(p_{i1}) \ge i_2^{(k)}(p_{i2})$ and $i_1^{(\#I_1)}(p_{i1}) \ge i_2^{(\#I_2)}(p_{i2})$, and $f_1^{(l)}(p_{f1}) \ge f_2^{(l)}(p_{f2})$ and $f_1^{(\#F_1)}(p_{f1}) \ge f_2^{(\#F_2)}(p_{f2})$, where $j = 1, 2, \ldots, \alpha_T$, $k = 1, 2, \ldots, \alpha_I$, $l = 1, 2, \ldots, \alpha_F$ and $\alpha_T = \min(\#T_1, \#T_2)$, $\alpha_I = \min(\#I_1, \#I_2)$, $\alpha_F = \min(\#F_1, \#F_2)$.

Although a complete ordering of all PMVNNs cannot be determined based on Definition 8, it is sufficient to demonstrate the following property of the proposed cross-entropy of PMVNNs.

4.2 Cross-entropy of PMVNNs

Ye [46] firstly defined the cross-entropy of SNSs, but it is questionable in some specific situations. To overcome this drawback, Wu et al. [52] further proposed two effective cross-entropy measures for SNSs. Based on the two existing simplified neutrosophic cross-entropy measures, the cross-entropy for PMVNNs is defined in the following.

Definition 9 Let $\phi_1 = \langle T_1(P_{t1}), I_1(P_{i1}), F_1(P_{f1})\rangle$ and $\phi_2 = \langle T_2(P_{t2}), I_2(P_{i2}), F_2(P_{f2})\rangle$ be two arbitrary PMVNNs, $\mathrm{CE}: \mathrm{PMVNN} \times \mathrm{PMVNN} \to R^+$, then two cross-entropy measures $\mathrm{CE}(\phi_1, \phi_2)$ between ϕ_1 and ϕ_2 are defined as follows:

$$\begin{aligned}\mathrm{CE}_1(\phi_1,\phi_2) = {} & \frac{1}{\#T_1 \times \#T_2} \sum_{t_1(p_{t1}) \in T_1(P_{t1})} \sum_{t_2(p_{t2}) \in T_2(P_{t2})} \sin(t_1 p_{t1}) \times \sin(t_1 p_{t1} - t_2 p_{t2}) \\ & + \frac{1}{\#I_1 \times \#I_2} \sum_{i_1(p_{i1}) \in I_1(P_{i1})} \sum_{i_2(p_{i2}) \in I_2(P_{i2})} \sin(i_1 p_{i1}) \times \sin(i_1 p_{i1} - i_2 p_{i2}) \\ & + \frac{1}{\#F_1 \times \#F_2} \sum_{f_1(p_{f1}) \in F_1(P_{f1})} \sum_{f_2(p_{f2}) \in F_1(P_{f1})} \sin(f_1 p_{f1}) \times \sin(f_1 p_{f1} - f_2 p_{f2}),\end{aligned} \tag{7}$$

$$\begin{aligned}\mathrm{CE}_2(\phi_1,\phi_2) = {} & \frac{1}{\#T_1 \times \#T_2} \sum_{t_1(p_{t1}) \in T_1(P_{t1})} \sum_{t_2(p_{t2}) \in T_2(P_{t2})} \tan(t_1 p_{t1}) \times \tan(t_1 p_{t1} - t_2 p_{t2}) \\ & + \frac{1}{\#I_1 \times \#I_2} \sum_{i_1(p_{i1}) \in I_1(P_{i1})} \sum_{i_2(p_{i2}) \in I_2(P_{i2})} \tan(i_1 p_{i1}) \times \tan(i_1 p_{i1} - i_2 p_{i2}) \\ & + \frac{1}{\#F_1 \times \#F_2} \sum_{f_1(p_{f1}) \in F_1(P_{f1})} \sum_{f_2(p_{f2}) \in F_1(P_{f1})} \tan(f_1 p_{f1}) \times \tan(f_1 p_{f1} - f_2 p_{f2}),\end{aligned} \tag{8}$$

which can signify the discrimination degree between ϕ_1 and ϕ_2. It is worth noting that $\mathrm{CE}_1(\phi_1, \phi_2)$ and $\mathrm{CE}_2(\phi_1, \phi_2)$ are not symmetric in regard to their arguments. Therefore, two symmetric discrimination information measures for PMVNNs are defined as

$$\mathrm{CE}_1^*(\phi_1, \phi_2) = \mathrm{CE}_1(\phi_1, \phi_2) + \mathrm{CE}_1(\phi_2, \phi_1); \tag{9}$$

$$\mathrm{CE}_2^*(\phi_1, \phi_2) = \mathrm{CE}_2(\phi_1, \phi_2) + \mathrm{CE}_2(\phi_2, \phi_1). \tag{10}$$

Property 1 *Let* $\phi_1 = \langle T_1(P_{t1}), I_1(P_{i1}), F_1(P_{f1})\rangle$ *and* $\phi_2 = \langle T_2(P_{t2}), I_2(P_{i2}), F_2(P_{f2})\rangle$ *be two arbitrary PMVNNs, all elements in* $T_\varepsilon(P_{t\varepsilon})$, $I_\varepsilon(P_{i\varepsilon})$ *and* $F_\varepsilon(P_{f\varepsilon})(\varepsilon = 1, 2)$ *be arranged in ascending order according to the* *values of $t_\varepsilon \times p_{t\varepsilon}$, $i_\varepsilon \times p_{i\varepsilon}$ and $f_\varepsilon \times p_{f\varepsilon}$, respectively. And $t_\varepsilon^{(j)}(p_{t\varepsilon})$, $i_\varepsilon^{(j)}(p_{i\varepsilon})$ and $f_\varepsilon^{(j)}(p_{f\varepsilon})$ be referred to as the jth value in $T_\varepsilon(P_{t\varepsilon})$, $I_\varepsilon(P_{i\varepsilon})$ and $F_\varepsilon(P_{f\varepsilon})$. Then, the above cross-entropy satisfy the following properties*:

(1) $\mathrm{CE}_1^*(\phi_1,\phi_2) = \mathrm{CE}_1^*(\phi_2,\phi_1)$ and $\mathrm{CE}_2^*(\phi_1,\phi_2) = \mathrm{CE}_2^*(\phi_2,\phi_1)$.

(2) $\mathrm{CE}_1^*(\phi_1,\phi_2) = \mathrm{CE}_1^*(\phi_1^c,\phi_2^c)$ and $\mathrm{CE}_2^*(\phi_1,\phi_2) = \mathrm{CE}_2^*(\phi_1^c,\phi_2^c)$, where ϕ_1^c and ϕ_2^c are the complement set of ϕ_1 and ϕ_2, respectively.

(3) $\mathrm{CE}_1^*(\phi_1,\phi_2) \geq 0$ and $\mathrm{CE}_2^*(\phi_1,\phi_2) \geq 0$.

(4) The greater the difference between ϕ_1 and ϕ_2 is, the greater $\mathrm{CE}_1^*(\phi_1,\phi_2)$ and $\mathrm{CE}_2^*(\phi_1,\phi_2)$ will be.

Proof Obviously, the cross-entropy measures described in Definition 9 satisfy (1) and (2) of Property 1, and the proofs of (3) and (4) of Property 1 are as follows.

(3) Firstly, it is necessary to analysis the following functions:

$$f_1(a,b) = \sin a \times \sin(a-b) + \sin b \times \sin(b-a) = (\sin a - \sin b) \times \sin(a-b),$$

and $f_2(a,b) = \tan a \times \tan(a-b) + \tan b \times \tan(b-a) = (\tan a - \tan b) \times \tan(a-b)$, where $a,b \in [0,1]$. It is worth noting that whether $a \geq b$ or $a \leq b$, the two inequalities $f_1(a,b) \geq 0$ and $f_2(a,b) \geq 0$ always hold.

Moreover, the following equations can be obtained:

$$\mathrm{CE}_1^*(\phi_1,\phi_2) = \frac{1}{\#T_1 \times \#T_2}\sum_{t_1(p_{t1})\in T_1(P_{t1})}\sum_{t_2(p_{t2})\in T_2(P_{t2})} \sin(t_1p_{t1} - t_2p_{t2}) \times (\sin(t_1p_{t1}) - \sin(t_2p_{t2}))$$
$$+ \frac{1}{\#I_1 \times \#I_2}\sum_{i_1(p_{i1})\in I_1(P_{i1})}\sum_{i_2(p_{i2})\in I_2(P_{i2})} \sin(i_1p_{i1} - i_2p_{i2}) \times (\sin(i_1p_{i1}) - \sin(i_2p_{i2}))$$
$$+ \frac{1}{\#F_1 \times \#F_2}\sum_{f_1(p_{f1})\in F_1(P_{f1})}\sum_{f_2(p_{f2})\in F_2(P_{f2})} \sin(f_1p_{f1} - f_2p_{f2}) \times (\sin(f_1p_{f1}) - \sin(f_2p_{f2})),$$

and

$$\mathrm{CE}_2^*(\phi_1,\phi_2) = \frac{1}{\#T_1 \times \#T_2}\sum_{t_1(p_{t1})\in T_1(P_{t1})}\sum_{t_2(p_{t2})\in T_2(P_{t2})} \tan(t_1p_{t1} - t_2p_{t2}) \times (\tan(t_1p_{t1}) - \tan(t_2p_{t2}))$$
$$+ \frac{1}{\#I_1 \times \#I_2}\sum_{i_1(p_{i1})\in I_1(P_{i1})}\sum_{i_2(p_{i2})\in I_2(P_{i2})} \tan(i_1p_{i1} - i_2p_{i2}) \times (\tan(i_1p_{i1}) - \tan(i_2p_{i2}))$$
$$+ \frac{1}{\#F_1 \times \#F_2}\sum_{f_1(p_{f1})\in F_1(P_{f1})}\sum_{f_2(p_{f2})\in F_2(P_{f2})} \tan(f_1p_{f1} - f_2p_{f2}) \times (\tan(f_1p_{f1}) - \tan(f_2p_{f2})).$$

Since $\forall(t_1p_{t1}, i_1p_{i1}, f_1p_{f1}, t_2p_{t2}, i_2p_{i2}, f_2p_{f2}) \in [0,1]$, $\mathrm{CE}_1^*(\phi_1,\phi_2) \geq 0$ and $\mathrm{CE}_2^*(\phi_1,\phi_2) \geq 0$ can be easily obtained according to the property of $f_1(a,b)$ and $f_2(a,b)$.

(4) For three PMVNNs $\phi_1 = \langle T_1(P_{t1}), I_1(P_{i1}), F_1(P_{f1})\rangle$, $\phi_2 = \langle T_2(P_{t2}), I_2(P_{i2}), F_2(P_{f2})\rangle$ and $\phi_3 = \langle T_3(P_{t3}), I_3(P_{i3}), F_3(P_{f3})\rangle$, if $\phi_1 \leq \phi_2 \leq \phi_3$, then $t_1^{(j)}(p_{t1}) \leq t_2^{(j)}(p_{t2}) \leq t_3^{(j)}(p_{t3})$ and $t_1^{(\#T_1)}(p_{t1}) \leq t_2^{(\#T_2)}(p_{t2}) \leq t_3^{(\#T_3)}(p_{t3})$, $i_1^{(k)}(p_{i1}) \geq i_2^{(k)}(p_{i2}) \geq i_3^{(k)}(p_{i3})$ and $i_1^{(\#I_1)}(p_{i1}) \geq i_2^{(\#I_2)}(p_{i2}) \geq i_3^{(\#I_3)}(p_{i3})$, $f_1^{(l)}(p_{f1}) \geq f_2^{(l)}(p_{f2}) \geq f_3^{(l)}(p_{f3})$ and $f_1^{(\#F_1)}(p_{f1}) \geq f_2^{(\#F_2)}(p_{f2}) \geq f_3^{(\#F_3)}(p_{f3})$ can be obtained according to Definition 8. Thus, the following inequalities are true,

$$\sin(t_1^{(j)}(p_{t1})) \leq \sin(t_2^{(j)}(p_{t2})) \leq \sin(t_3^{(j)}(p_{t3})) \Rightarrow 0 \leq \sin(t_2^{(j)}(p_{t2})) - \sin(t_1^{(j)}(p_{t1})) \leq \sin(t_3^{(j)}(p_{t3})) - \sin(t_1^{(j)}(p_{t1})),$$

$$0 \leq t_2^{(j)}(p_{t2}) - t_1^{(j)}(p_{t1}) \leq t_3^{(j)}(p_{t3}) - t_1^{(j)}(p_{t1}) \leq 1 \Rightarrow 0 \leq \sin(t_2^{(j)}(p_{t2}) - t_1^{(j)}(p_{t1})) \leq \sin(t_3^{(j)}(p_{t3}) - (t_1^{(j)}(p_{t1})).$$

Then,

$$\sin(t_1^{(j)}(p_{t1}) - t_2^{(j)}(p_{t2}))(\sin(t_1^{(j)}(p_{t1})) - \sin(t_2^{(j)}(p_{t2}))) \leq \sin((t_1^{(j)}(p_{t1}) - t_3^{(j)}(p_{t3}))(\sin(t_1^{(j)}(p_{t1})) - \sin(t_3^{(j)}(p_{t3}))).$$

And then,

$$\frac{1}{\#T_2}\sum_{t_2(p_{t2})\in T_2(P_{t2})}\sin(t_1p_{t1}-t_2p_{t2})\times(\sin(t_1p_{t1})-\sin(t_2p_{t2}))$$
$$\leq\frac{1}{\#T_3}\sum_{t_3(p_{t3})\in T_3(P_{t3})}\sin(t_1p_{t1}-t_3p_{t3})\times(\sin(t_1p_{t1})-\sin(t_3p_{t3})).$$

Thus,

$$\frac{1}{\#T_1\times\#T_2}\sum_{t_1(p_{t1})\in T_1(P_{t1})}\sum_{t_2(p_{t2})\in T_2(P_{t2})}\sin(t_1p_{t1}-t_2p_{t2})\times(\sin(t_1p_{t1})-\sin(t_2p_{t2}))$$
$$\leq\frac{1}{\#T_1\times\#T_3}\sum_{t_1(p_{t1})\in T_1(P_{t1})}\sum_{t_3(p_{t3})\in T_3(P_{t3})}\sin(t_1p_{t1}-t_3p_{t3})\times(\sin(t_1p_{t1})-\sin(t_3p_{t3})).$$

In the same way,

$$\frac{1}{\#I_1\times\#I_2}\sum_{i_1(p_{i1})\in I_1(P_{i1})}\sum_{i_2(p_{i2})\in I_2(P_{i2})}\sin(i_1p_{i1}-i_2p_{i2})\times(\sin(i_1p_{i1})-\sin(i_2p_{i2}))$$
$$\leq\frac{1}{\#I_1\times\#I_3}\sum_{i_1(p_{i1})\in I_1(P_{i1})}\sum_{i_3(p_{i3})\in I_3(P_{i3})}\sin(i_1p_{i1}-i_3p_{i3})\times(\sin(i_1p_{i1})-\sin(i_3p_{i3})),$$

and

$$\frac{1}{\#F_1\times\#F_2}\sum_{f_1(p_{f1})\in F_1(P_{f1})}\sum_{f_2(p_{f2})\in F_2(P_{f2})}\sin(f_1p_{f1}-f_2p_{f2})\times\left(\sin(f_1p_{f1})-\sin(f_2p_{f2})\right)$$
$$\leq\frac{1}{\#F_1\times\#F_3}\sum_{f_1(p_{f1})\in F_1(P_{f1})}\sum_{f_3(p_{f3})\in F_3(P_{f3})}\sin(f_1p_{f1}-f_3p_{f3})\times\left(\sin(f_1p_{f1})-\sin(f_3p_{f3})\right),$$

can also be obtained. Therefore, $\mathrm{CE}_1^*(\phi_1,\phi_2)\leq CE_1^*(\phi_1,\phi_3)$. The inequality $\mathrm{CE}_1^*(\phi_2,\phi_3)\leq CE_1^*(\phi_1,\phi_3)$ can be proved in a similar way. The same proof can also be provided for $\mathrm{CE}_2^*(\phi_1,\phi_2)$.

4.3 TOPSIS-based QUALIFLEX method with PMVNNs

In this subsection, a novel TOPSIS-based QUALIFLEX method is developed based on the cross-entropy measures of PMVNNs described in Definition 9.

The QUALIFLEX method, introduced by Paelinck [62], is a useful outranking method for multi-criteria decision analysis because of its flexibility with respect to ordinal and cardinal information. The QUALIFLEX method compares each pair of alternatives for all possible alternative permutations under each criterion and determines the optimal permutation by searching for the maximal value of permutation' concordance/discordance index.

Taking into account a decision-making problem that refers to n alternatives under m criteria. And the evaluation value of alternative $a_j(j=1,2,\ldots,n)$ associated with criterion $c_k(k=1,2,\ldots,m)$ is in the form of PMVNN, denoting as $r_{jk}=\left\langle T_{jk}(P_t),I_{jk}(P_i),F_{jk}\left(P_f\right)\right\rangle$.

In order to ranking all alternatives, the positive ideal solution (PIS) and negative ideal solution (NIS) are identified as $a^+=\langle\{1(1)\},\{0(0)\},\{0(0)\}\rangle$ and $a^-=\langle\{0(0)\},\{1(1)\},\{1(1)\}\rangle$, respectively. The cross-entropy between alternative $a_j(j=1,2,\ldots,n)$ and the PIS a^+, expressed as $\mathrm{CE}^*\left(a_j,a^+\right)$, and the cross-entropy between alternative $a_j(j=1,2,\ldots,n)$ and the NIS a^-, expressed as $\mathrm{CE}^*\left(a_j,a^-\right)$; both of them can be computed utilizing Eqs. (9) and (10). Therefore, the extended closeness coefficient of TOPSIS for a given alternative a_j under criterion c_k is defined as

$$\mathrm{CC}^k(a_j)=\frac{\mathrm{CE}^*\left(a_j,a^-\right)}{\mathrm{CE}^*\left(a_j,a^-\right)+\mathrm{CE}^*\left(a_j,a^+\right)}.\tag{11}$$

It is apparent that $\mathrm{CC}^k(a_j)\in[0,1]$, and the bigger value of $\mathrm{CC}^k(a_j)$ indicates alternative a_j is better.

In the following, the above extended closeness coefficient is employed to identify the concordance/discordance index. For an alternative set A with n alternatives, then $n!$ permutations of the ranking for all alternatives exist. Let P_l denotes the lth permutation, then

$$P_l=(\ldots,a_\alpha,\ldots,a_\beta,\ldots),\quad\text{for } l=1,2,\ldots,n!,$$

where $a_\alpha, a_\beta \in A$, and the alternative a_α is ranked better than or equal to a_β. If a_α and a_β are ranked in the same order within two preorders, then concordance exists. If a_α and a_β have the same ranking, then ex aequo exists. If a_α and a_β are counter-ranked in two preorders, then discordance exists.

Therefore, the concordance/discordance index $\varphi_l^k(a_\alpha, a_\beta)$ for each pair of alternatives (a_α, a_β) $(a_\alpha, a_\beta \in A)$ in regard to the criterion c_k and the permutation P_l is defined as follows:

$$\varphi_l^k(a_\alpha, a_\beta) = \mathrm{CC}(a_\alpha) - \mathrm{CC}(a_\beta) = \frac{\mathrm{CE}^*(a_\alpha, a^-)}{\mathrm{CE}^*(a_\alpha, a^-) + \mathrm{CE}^*(a_\alpha, a^+)} - \frac{\mathrm{CE}^*\left(a_\beta, a^-\right)}{\mathrm{CE}^*\left(a_\beta, a^-\right) + \mathrm{CE}^*\left(a_\beta, a^+\right)}, \tag{12}$$

where $\varphi_l^k(a_\alpha, a_\beta) \in [-1, 1]$.

Based on the extended closeness coefficient comparison method and the above concordance/discordance index, the following three situations can be obtained.

1. If $\varphi_l^k(a_\alpha, a_\beta) > 0$, that is $\mathrm{CC}(a_\alpha) > \mathrm{CC}(a_\beta)$, then a_α is ranked better than a_β under the criterion c_k. Thus, concordance exists between the extended closeness coefficient-based ranking and the preorder of a_α and a_β under the lth permutation P_l.
2. If $\varphi_l^k(a_\alpha, a_\beta) = 0$, that is $\mathrm{CC}(a_\alpha) = \mathrm{CC}(a_\beta)$, then a_α and a_β have the same ranking under the criterion c_k. Thus, ex aequo exists between the extended closeness coefficient-based ranking and the preorder of a_α and a_β under the lth permutation P_l.
3. If $\varphi_l^k(a_\alpha, a_\beta) < 0$, that is $\mathrm{CC}(a_\alpha) < \mathrm{CC}(a_\beta)$, then a_β is ranked better than a_α under the criterion c_k. Thus, discordance exists between the extended closeness coefficient-based ranking and the preorder of a_α and a_β under the lth permutation P_l.

Suppose that the weight of criterion $c_k (k = 1, 2, \ldots, m)$ is $w_k (k = 1, 2, \ldots, m)$, satisfying $w_k \in [0, 1]$ and $\sum_{k=1}^m w_k = 1$, then, the weighted concordance/discordance index $\varphi_l^k(a_\alpha, a_\beta)$ for each pair of alternatives (a_α, a_β) $(a_\alpha, a_\beta \in A)$ with respect to the criterion c_k and the permutation P_l can be obtained based on Eq. (12) as follows:

$$\varphi_l(a_\alpha, a_\beta) = \sum_{k=1}^m w_k \varphi_l^k(a_\alpha, a_\beta) = \sum_{k=1}^m w_k \left(\mathrm{CC}(a_\alpha) - \mathrm{CC}(a_\beta)\right). \tag{13}$$

Moreover, the comprehensive concordance/discordance index φ_l associated with the permutation P_l can be calculated as follows:

$$\varphi_l = \sum_{a_\alpha, a_\beta \in A} \sum_{k=1}^m w_k \varphi_l^k(a_\alpha, a_\beta) = \sum_{a_\alpha, a_\beta \in A} \sum_{k=1}^m w_k \left(\mathrm{CC}(a_\alpha) - \mathrm{CC}(a_\beta)\right). \tag{14}$$

According to the extended closeness coefficient-based comparison method, we can conclude that the bigger φ_l is, the more reliable the permutation P_l is. Therefore, the optimal ranking P^* of all alternatives can be identified as

$$\varphi^* = \max_{l=1}^{n!} \{\varphi_l\}. \tag{15}$$

5 A MCGDM method under PMVNNs circumstance

In this section, two objective weight determination methods are established based on the cross-entropy measurement of PMVNNs. Furthermore, a novel MCGDM method is developed by combining the proposed aggregation operator and TOPSIS-based QUALIFLEX method.

MCGDM problems with PMVNNs information consist of a group of alternatives, denoted by $A = \{a_1, a_2, \ldots, a_n\}$. Suppose $C = \{c_1, c_2, \ldots, c_m\}$ to be the set of criteria, whose weight vector is $w = (w_1, w_2, \ldots, w_m)$, satisfying $w_k \in [0, 1]$ and $\sum_{k=1}^m w_k = 1$; let $D = \{d_1, d_2, \ldots, d_q\}$ be a finite set of decision-makers (DMs), whose weight vector is $v = (v_1, v_2, \ldots, v_q)$, satisfying $v_l \in [0, 1]$ and $\sum_{l=1}^q v_l = 1$. Then, for a DM $d_l (l = 1, 2, \ldots, q)$, the evaluation information of $a_i (i = 1, 2, \ldots, n)$ with respect to $c_j (j = 1, 2, \ldots, m)$ is presented in the form of PMVNNs, denoted by $z_{jk}^l = \left\langle T\left(P_{jkl}^t\right), I\left(P_{jkl}^i\right), F\left(P_{jkl}^f\right)\right\rangle$, where $T\left(P_{jkl}^t\right) = \cup_{t_{jkl} \in T, p_{jkl}^t \in P_{jkl}^t} \{t_{jkl}(p_{jkl}^t)\}$, and t_{jkl} indicates the truth-membership degree that the alternative a_i satisfies the criterion c_j, and p_{jkl}^t indicates the importance of the provided truth-membership degree t_{jkl}; $I\left(P_{jkl}^i\right) = \cup_{i_{jkl} \in I, p_{jkl}^i \in P_{jkl}^i} \{i_{jkl}(p_{jkl}^i)\}$, and i_{jkl} indicates the indeterminacy-membership degree that the alternative a_i satisfies the criterion c_j, and p_{jkl}^i indicates the importance of the provided indeterminacy-membership degree i_{jkl}; $F\left(P_{jkl}^f\right) = \cup_{f_{jkl} \in F, p_{jkl}^f \in P_{jkl}^f} \{f_{jkl}(p_{jkl}^f)\}$, and f_{jkl} indicates the falsity-membership degree that the alternative a_i satisfies the criterion c_j, and p_{jkl}^f indicates the importance of the provided falsity-membership degree f_{jkl}. Finally, the decision matrix $R^l = \left(z_{jk}^l\right)_{n \times m}$ can be constructed.

5.1 To determine the weights of criteria and DMs

The weights of DMs and criteria are important parameters in MCGDM problems because they directly influence the accuracy of the final results. In practical MCGDM problems, weight information is usually uncertain. Furthermore, because DMs are selected from distinct backgrounds with different degrees of expertise, they cannot be directly endowed with arbitrary weights or equal weights. Moreover, because of the time pressure, problem complexity and lack of knowledge, criteria weights also should not be determined according to empirical values or subjectively assigned in advance. As a result, for a practical MCGDM problem, weights for DMs and criteria should be regarded as unknown and to be determined.

According to the above discussion, two reliable models are established, based on the proposed cross-entropy measurement of PMVNNs, to objectively calculate the unknown weights of DMs and criteria in the following.

In the evaluation process, different DMs may have different opinions on each criterion, and then the criteria in different decision matrices should have different importance. In this way, the criteria weights $w^l = \left(w_1^l, w_2^l, \ldots, w_m^l\right)$ in each decision matrix $R^l = \left(z_{jk}^l\right)_{n\times m}(l = 1, 2, \ldots, q)$ should be determined, respectively. If a MCDM problem features marked differences between any two distinct alternatives' evaluation values under a given criterion c_j, then c_j plays a relatively important role in the decision-making process; therefore, a higher weight should be assigned to c_j. In contrast, if a criterion makes the evaluation values of all alternatives appear to be similar, then this criterion plays a less important role in the decision-making process and should be assigned a low weight. Thus, taking into account the viewpoint of sorting the alternatives, the proposed cross-entropy measurement in Definition 9 can be employed to distinguish different evaluation values under a given criterion.

Therefore, for a decision matrix $R^l = \left(z_{jk}^l\right)_{n\times m}(l = 1, 2, \ldots, q)$, an extended maximizing deviation model can be established to derive the weights of criteria as follows:

$$\max F^l\left(w_k^l\right) = \sum_{k=1}^{m} w_k^l \sum_{j=1}^{n} \sum_{h=1,h\neq j}^{n} \mathrm{CE}^*\left(z_{jk}^l, z_{hk}^l\right)$$
$$\text{s.t}\begin{cases} \sum_{k=1}^{m} w_k^{l2} = 1 \\ w_k^l \geq 0, k = 1, 2, \ldots, m; l = 1, 2, \ldots, q. \end{cases} \tag{M-1}$$

where $\mathrm{CE}^*\left(z_{jk}^l, z_{hk}^l\right)$ signify the cross-entropy measurement between z_{jk}^l and z_{hk}^l.

To deal with this model, the Lagrange function is constructed as

$$F^l\left(w_k^l, \lambda\right) = \sum_{k=1}^{m} w_k^l \sum_{j=1}^{n} \sum_{h=1,h\neq j}^{n} \mathrm{CE}^*\left(z_{jk}^l, z_{hk}^l\right) + \frac{\lambda}{2}\left(\sum_{k=1}^{m} w_k^{l2} - 1\right), \tag{16}$$

where λ is the Lagrange multiplier. Then, the partial derivatives of $F^l\left(w^l, \lambda\right)$ can be calculated as follows:

$$\begin{cases} \dfrac{\partial F^l\left(w_k^l, \lambda\right)}{\partial w_k^l} = \sum_{j=1}^{n} \sum_{h=1,h\neq j}^{n} \mathrm{CE}^*\left(z_{jk}^l, z_{hk}^l\right) + \lambda w_k^l = 0, \\ \dfrac{\partial F^l\left(w_k^l, \lambda\right)}{\partial \lambda} = \sum_{k=1}^{m} w_k^{l2} - 1 = 0. \end{cases} \tag{17}$$

By solving Eq. (17), the optimal weights of criteria can be identified as

$$w_k^{l*} = \frac{\sum_{j=1}^{n} \sum_{h=1,h\neq j}^{n} \mathrm{CE}^*\left(z_{jk}^l, z_{hk}^l\right)}{\sqrt{\sum_{k=1}^{m}\left(\sum_{j=1}^{n} \sum_{h=1,h\neq j}^{n} \mathrm{CE}^*\left(z_{jk}^l, z_{hk}^l\right)\right)^2}}. \tag{18}$$

To normalize $w_k^{l*}(l = 1, 2, \ldots, q)$ be a unit, the final weights of criteria can be calculated as

$$w_k^l = \frac{w_k^{l*}}{\sum_{k=1}^{m} w_k^{l*}} = \frac{\sum_{j=1}^{n} \sum_{h=1,h\neq j}^{n} \mathrm{CE}^*\left(z_{jk}^l, z_{hk}^l\right)}{\sum_{k=1}^{m} \sum_{j=1}^{n} \sum_{h=1,h\neq j}^{n} \mathrm{CE}^*\left(z_{jk}^l, z_{hk}^l\right)}. \tag{19}$$

Motivated by the closeness coefficient of TOPSIS [63] and the variation coefficient approach [64, 65], an objective weight determination method for DMs can be developed based on the proposed cross-entropy measurement in the following.

1. Determine the PIS and NIS.
   The PIS and NIS can be identified as $z^+ = \langle\{1(1)\}, \{0(0)\}, \{0(0)\}\rangle$ and $z^- = \langle\{0(0)\}, \{1(1)\}, \{1(1)\}\rangle$, respectively.
2. Calculate the overall weighted cross-entropy measurement between z_{jk}^l and z^+/z^- in each decision matrix $R^l = \left(z_{jk}^l\right)_{n\times m}(l = 1, 2, \ldots, q)$.
   According to the cross-entropy of PMVNNs given in Definition 9, the overall weighted cross-entropy measurement between z_{jk}^l and z^+/z^- can be obtained as follows:

$$C^{l+} = \sum_{k=1}^{m}\left(w_k^l \sum_{j=1}^{n} \mathrm{CE}^*\left(z_{jk}^l, z^+\right)\right) \text{ and }$$
$$C^{l-} = \sum_{k=1}^{m}\left(w_k^l \sum_{j=1}^{n} \mathrm{CE}^*\left(z_{jk}^l, z^-\right)\right). \tag{20}$$

3. Determine the closeness degree of each DM.
According to the closeness coefficient of TOPSIS method, the closeness degree of DM $d_l(l = 1, 2, \ldots, q)$ can be calculated as follows:

$$\varphi^l = \frac{C^{l-}}{C^{l+} + C^{l-}}. \tag{21}$$

The larger the value of φ^l, the larger the closeness degree of criteria values involved in decision matrix $R^l = \left(z_{jk}^l\right)_{n\times m}(l = 1, 2, \ldots, q)$ will be, and accordingly the more precise the evaluation provided by DM d_l, which suggests DM d_l provides less conflicting and controversial information and acts a relatively important role in decision-making procedure, and thus a greater weight should be endowed with DM d_l. The above discussion is consistent with the main principle of DMs weight determination.

4. Calculate the final weights of DMs.
The final weight of DM $d_l(l = 1, 2, \ldots, q)$ can be obtained as

$$v_l = \frac{\varphi^l}{\sum_{l=1}^{q} \varphi^l}. \tag{22}$$

5.2 A MCGDM method based on PMVNNWA operator and QUALIFLEX with PMVNNs

According to the above analysis, the proposed probability multi-valued neutrosophic aggregation operator and TOPSIS-based QUALIFLEX method are integrated to develop a MCGDM method, and its main procedure can be described as follows:

Step 1 Normalize the evaluation information.

It is necessary to normalize all evaluation values to the same magnitude grade to eliminate the influence of different dimensions in the operation process. The normalization of decision matrix needs to consider the criteria type and probability distribution simultaneously, and the complement set and Eq. (3) can be employed as follows:

$$z'_{jkl} = \begin{cases} \left\langle T\left(\tilde{P}_{jkl}^t\right), I\left(\tilde{P}_{jkl}^i\right), F\left(\tilde{P}_{jkl}^f\right)\right\rangle, & \text{for benefit criterion } c_k \\ \left\langle F\left(\tilde{P}_{jkl}^f\right), I\left(\tilde{P}_{jkl}^i\right), T\left(\tilde{P}_{jkl}^t\right)\right\rangle, & \text{for cost criterion } c_k \end{cases} \tag{23}$$

And the normalized decision matrix is expressed as $R^{l\prime} = \left(z'_{jkl}\right)_{n\times m}(l = 1, 2, \ldots, q)$.

Step 2 Determine the criteria weights in each decision matrix $R^{l'} = \left(z'_{jkl}\right)_{n\times m}(l = 1, 2, \ldots, q)$.

The criteria weights $w_k^l(k = 1, 2, \ldots, m; l = 1, 2, \ldots, q)$ in each decision matrix $R^{l\prime} = \left(z'_{jkl}\right)_{n\times m}(l = 1, 2, \ldots, q)$ can be obtained using model (M-1).

Step 3 Determine the weights associated to DMs.

The DMs weights $v_l(l = 1, 2, \ldots, q)$ can be obtained using Eq. (22).

Step 4 Calculate the collective weights of criteria.

After obtaining the criteria weights $w_k^l(k = 1, 2, \ldots, m; l = 1, 2, \ldots, q)$ in individual decision matrix and the DM weights $v_l(l = 1, 2, \ldots, q)$, the collective weight of criteria $w_k(k = 1, 2, \ldots, m)$ can be calculated as

$$w_k = \sum_{l=1}^{q} w_k^l v_l. \tag{24}$$

Step 5 Obtain the collective evaluation information.

The individual evaluation values z'_{jkl} can be aggregated by utilizing the PMVNNWA operator, and the collective evaluation matrix $R = \left(z_{jk}\right)_{n\times m}$ can be acquired.

Step 6 List all of the possible permutations for the alternatives.

List $n!$ permutations for the n alternatives, and the qth permutation can be denoted by P_q.

Step 7 Obtain all of the concordance/discordance indices for the pairwise alternatives under each criterion.

The concordance/discordance index $\varphi_q^k(a_\alpha, a_\beta)$ for the pairwise alternatives (a_α, a_β) $(a_\alpha, a_\beta \in A)$ under each criterion $c_k(k = 1, 2, \ldots, m)$ can be obtained by using Eq. (12), and the total concordance/discordance index matrix can be constructed.

Step 8 Compute the comprehensive concordance/discordance index.

The comprehensive concordance/discordance index φ_q for each permutation P_q can be calculated using Eq. (14).

Step 9 Determine the final ranking of all alternatives.

The optimal ranking can be obtained using Eq. (15), and then the final ranking of all alternatives can be determined.

6 Illustrative example

In this section, an illustrative example in the form of logistics outsourcing problem is provided in order to highlight the applicability of the proposed method. Furthermore, the method's availability and strengths are confirmed through the comparative analysis with other existing methods.

The following background is adapted from Wang et al. [66]. Logistics has been taken seriously by lots of companies due to its great influence on business operations. More and more companies choose to outsource logistics to third party logistics providers because of the numerous advantages of logistics outsourcing, such as increased competitiveness, professional services, improved performance and reductions in cost. The selection of a third party logistics provider is one of a business' most important decision-making projects. Numerous companies plan to implement logistics outsourcing to improve operational efficiency and lower costs.

ABC Machinery Manufacturing Co., Ltd. is a medium-sized automotive component manufacturer in China that is mainly involved in the exploitation, manufacture and sale of automotive components and mechanical products. The company's management team has decided to select a third party logistics provider for logistics outsourcing. A professional team was formed to assist in decision-making, consisting of a general manager, a logistics manager and a production manager, denoted by $\{d_1, d_2, d_3\}$, whose weight vector is $v = (v_1, v_2, v_3)$. Initially, information on some logistics providers was collected. After preliminary filtrating, many unqualified alternatives were weeded out, and four potential logistics providers remained, denoted by $\{a_1, a_2, a_3, a_4\}$. The professional team chose the following four criteria to evaluate these alternatives: c_1, information and equipment systems; c_2, service; c_3, quality; c_4, relationship. The weight vector of the four criteria is $w = (w_1, w_2, w_3, w_4)$. The weights of criteria and DMs are completely unknown. Moreover, the evaluation information can be collected in the form of PMVNNs in Tables 1, 2 and 3.

6.1 An illustration of the proposed method

The main procedures for evaluating the four logistics providers can be summarized in the following steps. Let $\mathrm{CE}^*(\phi_1, \phi_2) = \mathrm{CE}_1^*(\phi_1, \phi_2)$, here.

Table 1 Decision matrix of d_1

	c_1	c_2
a_1	$\langle\{0.4(0.6), 0.6(0.2)\}, \{0.4(0.6)\}, \{0.3(0.4), 0.4(0.5)\}\rangle$	$\langle\{0.3(0.4), 0.6(0.4)\}, \{0.5(0.5), 0.6(0.4)\}, \{0.3(0.4)\}\rangle$
a_2	$\langle\{0.5(0.4), 0.6(0.3)\}, \{0.4(0.2), 0.6(0.5)\}, \{0.3(0.4)\}\rangle$	$\langle\{0.6(0.5)\}, \{0.4(0.3), 0.6(0.5)\}, \{0.4(0.6), 0.6(0.3)\}\rangle$
a_3	$\langle\{0.5(0.7)\}, \{0.4(0.3), 0.5(0.4)\}, \{0.4(0.3), 0.6(0.5)\}\rangle$	$\langle\{0.4(0.5), 0.6(0.5)\}, \{0.5(0.6)\}, \{0.4(0.4), 0.5(0.4)\}\rangle$
a_4	$\langle\{0.5(0.3)\}, \{0.2(0.1), 0.4(0.5), 0.6(0.2)\}, \{0.5(0.7)\}\rangle$	$\langle\{0.6(0.5)\}, \{0.4(0.5), 0.6(0.5)\}, \{0.5(0.3), 0.6(0.5)\}\rangle$
	c_3	c_4
a_1	$\langle\{0.7(0.5), 0.8(0.5)\}, \{0.3(0.5), 0.4(0.4)\}, \{0.5(0.6)\}\rangle$	$\langle\{0.5(0.4), 0.7(0.6)\}, \{0.3(0.5), 0.5(0.4)\}, \{0.5(0.4)\}\rangle$
a_2	$\langle\{0.7(0.3), 0.8(0.5)\}, \{0.4(0.6)\}, \{0.4(0.5), 0.6(0.4)\}\rangle$	$\langle\{0.6(0.4), 0.8(0.4)\}, \{0.4(0.2), 0.6(0.5)\}, \{0.5(0.3)\}\rangle$
a_3	$\langle\{0.6(0.5)\}, \{0.4(0.5), 0.5(0.3)\}, \{0.4(0.5), 0.6(0.4)\}\rangle$	$\langle\{0.6(0.5)\}, \{0.5(0.4), 0.6(0.4)\}, \{0.5(0.6), 0.6(0.4)\}\rangle$
a_4	$\langle\{0.6(0.3), 0.8(0.5)\}, \{0.4(0.6)\}, \{0.5(0.3), 0.6(0.5)\}\rangle$	$\langle\{0.6(0.5), 0.8(0.4)\}, \{0.4(0.6)\}, \{0.4(0.5), 0.5(0.4)\}\rangle$

Table 2 Decision matrix of d_2

	c_1	c_2
a_1	$\langle\{0.6(0.5)\}, \{0.4(0.2), 0.6(0.6)\}, \{0.4(0.6), 0.6(0.2)\}\rangle$	$\langle\{0.5(0.4), 0.7(0.4)\}, \{0.6(0.4)\}, \{0.4(0.6), 0.5(0.4)\}\rangle$
a_2	$\langle\{0.3(0.4)\}, \{0.5(0.4)\}, \{0.2(0.2), 0.4(0.5), 0.6(0.3)\}\rangle$	$\langle\{0.5(0.6)\}, \{0.6(0.4)\}, \{0.5(0.3), 0.6(0.4), 0.7(0.2)\}\rangle$
a_3	$\langle\{0.4(0.6), 0.6(0.2)\}, \{0.6(0.3)\}, \{0.5(0.4), 0.6(0.5)\}\rangle$	$\langle\{0.6(0.4), 0.8(0.4)\}, \{0.5(0.3), 0.7(0.5)\}, \{0.5(0.4)\}\rangle$
a_4	$\langle\{0.5(0.4), 0.6(0.4)\}, \{0.5(0.3)\}, \{0.3(0.4), 0.6(0.5)\}\rangle$	$\langle\{0.7(0.5)\}, \{0.5(0.6), 0.6(0.3)\}, \{0.5(0.6)\}\rangle$
	c_3	c_4
a_1	$\langle\{0.5(0.3), 0.6(0.5)\}, \{0.4(0.4), 0.6(0.6)\}, \{0.3(0.6)\}\rangle$	$\langle\{0.6(0.6)\}, \{0.3(0.5)\}, \{0.4(0.4), 0.5(0.3), 0.6(0.3)\}\rangle$
a_2	$\langle\{0.5(0.4), 0.6(0.3)\}, \{0.5(0.6), 0.6(0.3)\}, \{0.5(0.5)\}\rangle$	$\langle\{0.5(0.6), 0.6(0.4)\}, \{0.4(0.5), 0.6(0.3)\}, \{0.3(0.4)\}\rangle$
a_3	$\langle\{0.5(0.4), 0.6(0.5)\}, \{0.5(0.4), 0.7(0.5)\}, \{0.5(0.8)\}\rangle$	$\langle\{0.4(0.6), 0.7(0.4)\}, \{0.3(0.4), 0.4(0.6)\}, \{0.5(0.5)\}\rangle$
a_4	$\langle\{0.5(0.6)\}, \{0.5(0.5)\}, \{0.4(0.2), 0.6(0.5), 0.7(0.3)\}\rangle$	$\langle\{0.5(0.5), 0.7(0.5)\}, \{0.5(0.4)\}, \{0.4(0.6), 0.6(0.3)\}\rangle$

Table 3 Decision matrix of d_3

	c_1	c_2
a_1	$\langle\{0.5(0.4),0.7(0.4)\},\{0.5(0.6),0.7(0.4)\},\{0.6(0.3)\}\rangle$	$\langle\{0.4(0.6)\},\{0.6(0.4),0.7(0.3)\},\{0.5(0.3),0.7(0.6)\}\rangle$
a_2	$\langle\{0.5(0.4),0.6(0.4)\},\{0.5(0.4)\},\{0.4(0.6),0.6(0.4)\}\rangle$	$\langle\{0.4(0.5),0.5(0.3)\},\{0.5(0.5),0.6(0.4)\},\{0.4(0.6)\}\rangle$
a_3	$\langle\{0.3(0.4),0.5(0.4)\},\{0.4(0.4),0.5(0.6)\},\{0.3(0.4)\}\rangle$	$\langle\{0.6(0.4)\},\{0.5(0.3)\},\{0.4(0.5),0.5(0.4)\}\rangle$
a_4	$\langle\{0.4(0.4),0.5(0.6)\},\{0.4(0.4)\},\{0.5(0.4),0.6(0.2)\}\rangle$	$\langle\{0.5(0.6),0.7(0.4)\},\{0.4(0.3)\},\{0.3(0.4),0.5(0.5)\}\rangle$
	c_3	c_4
a_1	$\langle\{0.4(0.5)\},\{0.3(0.4)\},\{0.3(0.2),0.5(0.5),0.6(0.3)\}\rangle$	$\langle\{0.3(0.4),0.5(0.6)\},\{0.7(0.6)\},\{0.5(0.4),0.6(0.6)\}\rangle$
a_2	$\langle\{0.7(0.4)\},\{0.4(0.6),0.7(0.3)\},\{0.4(0.5),0.5(0.5)\}\rangle$	$\langle\{0.4(0.6)\},\{0.4(0.5)\},\{0.4(0.3),0.5(0.5),0.6(0.2)\}\rangle$
a_3	$\langle\{0.4(0.5),0.6(0.5)\},\{0.4(0.5)\},\{0.6(0.4),0.8(0.5)\}\rangle$	$\langle\{0.4(0.4),0.5(0.4)\},\{0.5(0.6)\},\{0.3(0.5),0.5(0.5)\}\rangle$
a_4	$\langle\{0.6(0.5),0.8(0.5)\},\{0.4(0.6),0.6(0.4)\},\{0.5(0.3)\}\rangle$	$\langle\{0.8(0.5)\},\{0.4(0.5),0.6(0.3),0.8(0.2)\},\{0.3(0.4)\}\rangle$

Table 4 Criteria weights $w_k^l(k=1,2,3,4;l=1,2,3)$ in each decision matrix

	w_1^l	w_2^l	w_3^l	w_4^l
R^1	0.2751	0.2730	0.2181	0.2338
R^2	0.2255	0.3346	0.2033	0.2366
R^3	0.1676	0.1810	0.2334	0.4180

Step 1 Normalize the evaluation information.

Taking into account all of the criteria are the benefit type and some incomplete probability distribution exists in the provided evaluation values, then the decision matrices need to be normalized using Eq. (3). Because of the space limitation, the normalized decision matrices are omitted here.

Step 2 Determine the criteria weights in each decision matrix $R^{l\prime}=\left(z_{jkl}^{\prime}\right)_{n\times m}(l=1,2,3)$.

Using Eqs. (18) and (19), the criteria weights in each decision matrix can be obtained, as shown in Table 4.

Step 3 Determine the weights associated to DMs.

According to the procedures of determining DMs weights described in Sect. 5, the weight for DMs can be obtained as $v_l=(0.3539,0.3204,0.3257)$.

Step 4 Calculate the collective weights of criteria.

Based on the acquired criteria weights $w_k^l(k=1,2,3,4;l=1,2,3)$ in individual decision matrix and the DM weights $v_l(l=1,2,3)$, the collective weight vector of criteria can be calculated using Eq. (24) as $w_k=(0.2242,0.2628,0.2184,0.2946)$.

Step 5 Obtain the collective evaluation information.

The evaluation information provided by individual DM can be aggregated by utilizing the PMVNNWA operator, and the comprehensive evaluation information can be acquired, as shown in Table 5.

Step 6 List all of the possible permutations for the alternatives.

Because four alternatives are provided, $4!=24$ permutations can be listed in the following, where P_q represents the qth permutation.

$P_1=(a_1,a_2,a_3,a_4),\ P_2=(a_1,a_2,a_4,a_3),$
$P_3=(a_1,a_3,a_2,a_4),\ P_4=(a_1,a_3,a_4,a_2),$
$P_5=(a_1,a_4,a_2,a_3),$
$P_6=(a_1,a_4,a_3,a_2),\ P_7=(a_2,a_1,a_3,a_4),$
$P_8=(a_2,a_1,a_4,a_3),\ P_9=(a_2,a_3,a_1,a_4),$
$P_{10}=(a_2,a_3,a_4,a_1),$
$P_{11}=(a_2,a_4,a_1,a_3),\ P_{12}=(a_2,a_4,a_3,a_1),$
$P_{13}=(a_3,a_1,a_2,a_4),\ P_{14}=(a_3,a_1,a_4,a_2),$
$P_{15}=(a_3,a_2,a_1,a_4),$
$P_{16}=(a_3,a_2,a_4,a_1),\ P_{17}=(a_3,a_4,a_1,a_2)$
$P_{18}=(a_3,a_4,a_2,a_1),\ P_{19}=(a_4,a_1,a_2,a_3),$
$P_{20}=(a_4,a_1,a_3,a_2),$
$P_{21}=(a_4,a_2,a_1,a_3),\ P_{22}=(a_4,a_2,a_3,a_1),$
$P_{23}=(a_4,a_3,a_1,a_2)$ and $P_{24}=(a_4,a_3,a_2,a_1)$.

Step 7 Obtain all of the concordance/discordance indices for the pairwise alternatives under each criterion.

By using Eq. (12), the concordance/discordance index $\varphi_q^k(a_\alpha,a_\beta)$ for the pairwise alternatives (a_α,a_β) $(a_\alpha,a_\beta\in\{a_1,a_2,a_3,a_4\})$ under each criterion $c_k(k=1,2,3,4)$ can be obtained, and the total concordance/discordance indices are shown in Table 6.

Step 8 Compute the comprehensive concordance/discordance index.

Table 5 Comprehensive evaluation information

	c_1
a_1	$\langle\{0.50(0.3), 0.57(0.22), 0.58(0.27), 0.64(0.21)\}, \{0.43(0.25), 0.48(0.20), 0.49(0.30), 0.55(0.25)\}, \{0.41(0.27), 0.47(0.23), 0.52(0.23), 0.46(0.27)\}\rangle$
a_2	$\langle\{0.44(0.27), 0.49(0.24), 0.48(0.26), 0.52(0.23)\}, \{0.46(0.47), 0.53(0.53)\}, \{0.29(0.18), 0.33(0.14), 0.36(0.20), 0.41(0.16)\}, 0.41(0.18), 0.47(0.14)\}\rangle$
a_3	$\langle\{0.41(0.30), 0.48(0.22), 0.47(0.27), 0.53(0.21)\}, \{0.46(0.23), 0.49(0.26), 0.49(0.24), 0.53(0.27)\} \{0.39(0.24), 0.41(0.25), 0.48(0.26), 0.45(0.25)\}\rangle$
a_4	$\langle\{0.47(0.25), 0.51(0.24), 0.50(0.26), 0.53(0.25)\}, \{0.34(0.33), 0.43(0.35), 0.50(0.31)\}, \{0.42(0.28), 0.45(0.21), 0.53(0.29), 0.56(0.22)\}\rangle$
	c_2
a_1	$\langle\{0.40(0.26), 0.51(0.25), 0.49(0.25), 0.58(0.24)\}, \{0.56(0.27), 0.59(0.24), 0.60(0.26), 0.63(0.23)\}, \{0.39(0.23), 0.43(0.29), 0.42(0.21), 0.47(0.27)\}\rangle$
a_2	$\langle\{0.51(0.53), 0.54(0.47)\},\{0.49(0.25), 0.52(0.22), 0.57(0.28), 0.60(0.25)\}, \{0.43(0.18), 0.46(0.18), 0.48(0.17), 0.50(0.16)\}, 0.53(0.16), 0.55(0.15)\}\rangle$
a_3	$\langle\{0.54(0.26), 0.60(0.25), 0.63(0.25), 0.68(0.24)\},\{0.50(0.48), 0.56(0.52)\}, \{0.43(0.26), 0.46(0.24), 0.46(0.26), 0.50(0.24)\}\rangle$
a_4	$\langle\{0.61(0.54), 0.67(0.46)\},\{0.43(0.27), 0.50(0.26), 0.46(0.24), 0.53(0.23)\}, \{0.42(0.23), 0.50(0.24), 0.45(0.26), 0.53(0.27)\}\rangle$
	c_3
a_1	$\langle\{0.56(0.24), 0.62(0.24), 0.59(0.26), 0.64(0.26)\},\{0.33(0.25), 0.36(0.24), 0.37(0.26), 0.41(0.25)\}, \{0.36(0.33), 0.42(0.36), 0.45(0.31)\}\rangle$
a_2	$\langle\{0.65(0.24), 0.69(0.27), 0.67(0.23), 0.71(0.26)\},\{0.43(0.31), 0.52(0.23), 0.46(0.27), 0.55(0.19)\}, \{0.43(0.26), 0.46(0.25), 0.50(0.25), 0.53(0.24)\}\rangle$
a_3	$\langle\{0.51(0.26), 0.54(0.26), 0.57(0.24), 0.60(0.24)\},\{0.43(0.26), 0.46(0.24), 0.48(0.26), 0.52(0.24)\}, \{0.49(0.25), 0.54(0.26), 0.57(0.24), 0.62(0.25)\}\rangle$
a_4	$\langle\{0.57(0.24), 0.66(0.28), 0.66(0.22), 0.73(0.26)\},\{0.43(0.54), 0.49(0.46)\}, \{0.47(0.16), 0.53(0.17), 0.56(0.16), 0.50(0.17), 0.57(0.18), 0.59(0.17)\}\rangle$
	c_4
a_1	$\langle\{0.48(0.47), 0.57(0.53)\},\{0.40(0.51), 0.47(0.49)\}, \{0.47(0.16), 0.50(0.15), 0.53(0.15), 0.49(0.19)\}, 0.54(0.18), 0.56(0.18)\}\rangle$
a_2	$\langle\{0.51(0.27), 0.62(0.26), 0.54(0.24), 0.64(0.23)\},\{0.40(0.25), 0.46(0.28), 0.46(0.22), 0.53(0.25)\}, \{0.39(0.34), 0.42(0.37), 0.45(0.30)\}\rangle$
a_3	$\langle\{0.48(0.28), 0.58(0.23), 0.51(0.26), 0.61(0.23)\},\{0.42(0.25), 0.45(0.24), 0.47(0.26), 0.50(0.25)\}, \{0.42(0.27), 0.50(0.26), 0.45(0.24), 0.53(0.23)\}\rangle$
a_4	$\langle\{0.66(0.27), 0.73(0.24), 0.71(0.26), 0.77(0.24)\},\{0.43(0.38), 0.49(0.33), 0.54(0.29)\}, \{0.36(0.27), 0.41(0.24), 0.39(0.26), 0.45(0.23)\}\rangle$

Table 6 Concordance/discordance indices

	c_1	c_2	c_3	c_4
$\varphi_q^k(a_1, a_2)$	0.0429	−0.0886	−0.0082	0.0312
$\varphi_q^k(a_1, a_3)$	0.0113	0.0353	0.0149	0.0446
$\varphi_q^k(a_1, a_4)$	0.0171	−0.1154	0.0260	0.0312
$\varphi_q^k(a_2, a_1)$	−0.0429	0.0886	0.0082	−0.0312
$\varphi_q^k(a_2, a_3)$	−0.0315	0.1239	0.0231	0.0134
$\varphi_q^k(a_2, a_4)$	−0.0257	−0.0268	0.0342	−0.0001
$\varphi_q^k(a_3, a_1)$	−0.0113	−0.0353	−0.0149	−0.0446
$\varphi_q^k(a_3, a_2)$	0.0315	−0.1239	−0.0231	−0.0134
$\varphi_q^k(a_3, a_4)$	0.0058	−0.1507	0.0111	−0.0134
$\varphi_q^k(a_4, a_1)$	−0.0171	0.1154	−0.0260	−0.0312
$\varphi_q^k(a_4, a_2)$	0.0257	0.0268	−0.0342	0.0001
$\varphi_q^k(a_4, a_3)$	−0.0058	0.1507	−0.0111	0.0134

Equation (14) can be employed to calculate the comprehensive concordance/discordance index φ_q for each permutation P_q. For example, the comprehensive concordance/discordance index φ_1 for the permutation P_1 can be obtained as

$$\begin{aligned}\varphi_1 = &\sum_{k=1}^{4} w_k\left(\varphi_1^k(a_1,a_2)+\varphi_1^k(a_1,a_3)+\varphi_1^k(a_1,a_4)+\varphi_1^k(a_2,a_3)\right.\\ &\left.+\varphi_1^k(a_2,a_4)+\varphi_1^k(a_3,a_4)\right)\\ &=0.2242\times(0.0429+0.0113+0.0171-0.0315-0.0257+0.0058)\\ &=0.2628\times(-0.0886+0.0353-0.1154+0.1239-0.0268-0.1507)\\ &=0.2184\times(-0.0082+0.0149+0.0260+0.0231+0.0342+0.0111)\\ &=0.2946\times(0.0312+0.0446+0.0312+0.0134-0.0001-0.0134)\\ &=-0.0003\end{aligned}$$

And the comprehensive concordance/discordance index for other permutations P_q can be obtained in the same way.

$$\begin{aligned}&\varphi_2 = 0.0793,\ \varphi_3 = -0.0693,\ \varphi_4 = -0.0586,\\ &\varphi_5 = 0.0900,\ \varphi_6 = 0.0211,\ \varphi_7 = 0.0122,\ \varphi_8 = 0.0918,\\ &\varphi_9 = -0.0443,\ \varphi_{10} = -0.0211,\ \varphi_{11} = 0.1150,\\ &\varphi_{12} = 0.0586,\ \varphi_{13} = -0.1257,\ \varphi_{14} = -0.1150,\ \varphi_{15} = -0.1132,\\ &\varphi_{16} = -0.0900,\ \varphi_{17} = -0.0918,\ \varphi_{18} = -0.0793,\\ &\varphi_{19} = 0.1132,\ \varphi_{20} = 0.0443,\ \varphi_{21} = 0.1257,\ \varphi_{22} = 0.0693,\\ &\varphi_{23} = -0.0122 \text{ and } \varphi_{24} = 0.0003.\end{aligned}$$

Step 9 Determine the final ranking of all alternatives.

By using Eq. (15), the optimal ranking can be obtained, $\varphi^* = \max_{l=1}^{24}\{\varphi_l\} = \varphi_{21}$, and $P^* = P_{21} = (a_4, a_2, a_1, a_3)$.

Therefore, the final ranking of all alternatives is identified as $a_4 \succ a_2 \succ a_1 \succ a_3$, with an optimum logistics provider a_4.

Moreover, when $CE^*(\phi_1, \phi_2) = CE_2^*(\phi_1, \phi_2)$ is used in the above steps, the ranking result $a_4 \succ a_2 \succ a_1 \succ a_3$ can also be obtained.

6.2 Comparative analysis and discussion

In order to further verify the feasibility and validity of the proposed method, the following comparative studies are conducted using other extant methods and the discussion is based on the same illustrative example described above in the context of MVNNs and SNNs.

In the method developed by Peng et al. [55], two aggregation operators, including MVNPWA and MVNPWG, are employed to fuse evaluation information, and then the alternatives are ranked utilizing the score function and accuracy function.

Wu et al.'s method [52] defined the simplified neutrosophic number prioritized weighted average (SNNPWA) and simplified neutrosophic number prioritized weighted geometric (SNNPWG) operators, and proposed two cross-entropy measurements for SNNs. This method can be adjusted to address MCGDM problems by using the proposed operators to aggregate individual and comprehensive evaluation values.

Peng et al.'s method [38] defined a novel operations and provided a comparison method for SNNs after discussing the drawbacks of the existing studies of SNNs, and then several SNN aggregation operators were proposed to fuse evaluation information, such as generalized simplified neutrosophic number weighted average (GSNNWA) operator and generalized simplified neutrosophic number weighted geometric (GSNNWG) operator. Now, the above illustrative example is addressed utilizing the three extant methods.

First, PMVNNs should be transformed into MVNNs and SNNs. To better retain the fuzziness of original information, PMVNNs can be converted into MVNNs when multiplying the value of degree by its corresponding probability. And SNNs can be obtained by calculating the average values of all possible truth-membership, indeterminacy-membership and falsity-membership degrees in MVNNs, respectively. Let $a = \langle \cup\{t_a(p_t)\}, \cup\{i_a(p_i)\}, \cup\{f_a(p_f)\}\rangle$ be a PMVNN, then $\tilde{a} = \langle \cup\{t_a \times p_t\}, \cup\{i_a \times p_i\}, \cup\{f_a \times p_f\}\rangle$ is a MVNN, and $\tilde{\tilde{a}} = \left\langle \frac{\sum t_a \times p_t}{\#t_a}, \frac{\sum i_a \times p_i}{\#i_a}, \frac{\sum f_a \times p_f}{\#f_a}\right\rangle$ is a SNN. For example, the PMVNN $a = \langle\{0.3(0.4), 0.5(0.4)\}, \{0.4(0.4), 0.5(0.6)\}, \{0.3(0.4)\}\rangle$ can

Table 7 Rankings results acquired utilizing different methods

Methods	Rankings results
Peng et al.'s method with the MVNPWA operator [55]	$a_4 \succ a_3 \succ a_1 \succ a_2$
Peng et al.'s method with the MVNPWG operator [55]	$a_4 \succ a_3 \succ a_2 \succ a_1$
Wu et al.'s method with the SNNPWA operator [52]	$a_3 \succ a_1 \succ a_2 \succ a_4$
Wu et al.'s method with the SNNPWG operator [52]	$a_3 \succ a_2 \succ a_1 \succ a_4$
Peng et al.'s method with the GSNNWA operator [38]	$a_4 \succ a_3 \succ a_1 \succ a_2$
Peng et al.'s method with the GSNNWG operator [38]	$a_4 \succ a_3 \succ a_1 \succ a_2$
The proposed method	$a_4 \succ a_2 \succ a_1 \succ a_3$

be transformed into the MVNN $\tilde{a} = \langle\{0.12, 0.20\}, \{0.16, 0.30\}, \{0.12\}\rangle$ and the SNN $\tilde{\tilde{a}} = \langle 0.16, 0.23, 0.12\rangle$.

The ranking results acquired utilizing different methods are shown in Table 7.

As it is shown in Table 7, the ranking results derived from the three extant methods are inconsistent with the results obtained by the method proposed in this paper. The main differences between the three extant methods and the proposed method are summarized as follows.

First, Peng et al.'s method [55] employs MVNNs, Wu et al.'s method [52] and Peng et al.'s method [38] use SNNs to depict decision-making information, while the proposed method introduces a new descriptor, that is, PMVNNs, to present actual decision-making information. Comparing with MVNNs and SNNs, PMVNNs can characterize the features, such as probability degree, importance, weight or belief degree, of all given possible values according to the principle of probability distribution. Second, there are essential differences in operations and aggregation operators between the three extant methods and the proposed method. The operations of MVNNs are defined based on the Einstein operations, and the power aggregation operators are proposed to fuse information in Ref. [55], and the operations of SNNs are defined based on the Archimedean t-conorm and t-norm in Ref. [38], as well as two prioritized aggregation operators are proposed in Ref. [52], while the operations of PMVNNs in this paper are defined based on the convex combination operations and the corresponding weighted average operator is developed to deal with evaluation information. Third, the importance of criteria and DMs is subjectively provided in advance in Refs. [38, 55], and the criteria' weights are determined by the prioritized operator in Ref. [52], while the proposed method construct two objective cross-entropy-based models to calculate the optimal weights, which eliminates DMs' subjectivity and is greatly feasible and useful in practical problems. Finally, there are essential difference in determining the final ranking between the three extant methods and the proposed method. Peng et al.'s method [55] and [38] employed the score function and accuracy function to compare different alternatives, while the proposed method incorporates the cross-entropy measurement and closeness coefficient of TOPSIS into the QUALIFLEX method, and a novel ranking method is developed to obtain the final alternatives' ranking. When comparing different alternatives presented by MVNNs or PMVNNs, the ranking results obtained by the proposed method using the cross-entropy measurement and dominance theory are more convincing than the score function and accuracy function because of the existence of hesitant values. And compared with Wu et al.'s method [52], which simply used the cross-entropy to determine the final ranking, the proposed method is more comprehensive and reliable because the QUALIFLEX method can systematically compare the pairwise alternatives based on the dominance theory.

Based on the above analysis, the advantages of the proposed method are summarized as follows:

1. This paper introduces PMVNSs to represent DMs' evaluation information. As an extension of SNNs and MVNNs, PMVNSs are more capable of describing uncertain, incomplete, inconsistent and hesitant decision-making information and reflecting the probabilities, importance and weights of all provided values.
2. The proposed method integrates the aggregation operators and the TOPSIS-based QUALIFLEX method, thereby establishing a robust and innovative model to address MCGDM problems. After the individual evaluation information is aggregated, the developed QUALIFLEX method is used to rank all alternatives, reducing the computational load required when only the aggregation operators are employed to handle MCGDM problems involved hesitant values. Moreover, compared with the score function and accuracy function for PMVNSs, the ranking results obtained by the novel TOPSIS-based QUALIFLEX method using the dominance theory are more reliable.
3. This paper develops two objective cross-entropy-based methods for DMs and criteria weights determination, which eliminates the subjectivity existed in the weights information are directly provided in advance. This is

greatly useful in situation where weights information cannot be identified explicitly and can guarantee the accuracy of the final results to some extent.

4. Many factors involved in complicated MCGDM problems, such as the information description, information fusion, information measurement, completely unknown weight information and ranking principle, are considered synthetically in the proposed method.

It is undeniable that some tedious calculations are required when the proposed method is employed to address decision-making problems with a large number of alternatives. Nevertheless, the computing workload can be greatly reduced with the assistance of programming tools such as MATLAB.

7 Conclusion

Taking into account the existing shortage of MVNSs, this paper proposed the concept of PMVNSs by introducing probability distribution to character the features of all provided values. Based on the related studies of convex combination operations, the operation and the corresponding aggregation operator for PMVNSs were proposed. Moreover, a novel TOPSIS-based QUALIFLEX method was developed based on the cross-entropy measures of PMVNSs. And then, an innovative MCGDM approach was established. Finally, the feasibility and effectiveness of the proposed method were tested through an illustrative example of logistics outsourcing, and the comparative analysis demonstrated that the proposed method can provide more precise outcomes than other existing methods.

The main contributions of this research are summarized as follows. First, MVNSs and probability distribution are combined using PMVNSs, which can reliably depict the uncertain, incomplete, inconsistent and hesitant decision-making information and reflect the distribution characteristics of all provided values. Second, two cross-entropy-based methods are developed to objectively acquire the weight information of DMs and criteria, which eliminates DMs' subjectivity and has great power to address practical problems with undeterminable weight information. Finally, an innovative method is established by integrating the proposed aggregation operator and developed QUALIFLEX method, which not only reduces the workload of calculation, but also successfully imposes the effective ranking function of the QUALIFLEX method.

Future research will focus on applying the proposed approach to more practical decision-making problems, such as green product development and medical treatment options selection. Moreover, we will consider conducting some meaningful studies on the concept of neutrosophic probability [67], which has some similar characteristics with PMVNSs.

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References

1. Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–356
2. Yager RR (1977) Multiple objective decision-making using fuzzy sets. Int J Man Mach Stud 9(4):375–382
3. Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20:87–96
4. Bagočius V, Zavadskas EK, Turskis Z (2014) Sequence determining of construction of the offshore wind farm construction applying permutation method. Ekon Manag 17(3):50–61
5. Kildienė S, Zavadskas EK, Tamošaitienė J (2014) Complex assessment model for advanced technology deployment. J Civil Eng Manag 20(2):280–290
6. Yu SM, Wang J, Wang JQ (2016) An extended TODIM approach with intuitionistic linguistic numbers. Int Tran Oper Res. doi:10.1111/itor.12363
7. Torra V, Narukawa Y (2009) On hesitant fuzzy sets and decision. In: The 18th IEEE international conference on fuzzy systems, pp 1378–1382
8. Rodríguez RM, Bedregal B, Bustince H, Dong YC, Farhadinia B, Kahraman C, Martinez L, Torra V, Xu YJ, Xu ZS, Herrera F (2016) A position and perspective analysis of hesitant fuzzy sets on information fusion in decision making. Towards high quality progress. Inf Fusion 29:89–97
9. Farhadinia B (2013) Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets. Inf Sci 240(11):129–144
10. Meng FY, Chen XH (2015) Correlation coefficients of hesitant fuzzy sets and their application based on fuzzy measures. Cogn Comput 7(4):445–463
11. Xu ZS, Xia MM (2012) Hesitant fuzzy entropy and cross-entropy and their use in multiattribute decision-making. Int J Intell Syst 27(9):799–822
12. Smarandache F (1998) Neutrosophy: neutrosophic probability, set, and logic. American Research Press, Rehoboth, pp 1–105
13. Smarandache F (1999) A unifying field in logics. Neutrosophy: neutrosophic probability, set and logic. American Research Press, Rehoboth, pp 1–141
14. Zavadskas EK, Baušys R, Lazauskas M (2015) Sustainable assessment of alternative sites for the construction of a waste incineration plant by applying WASPAS method with single-valued neutrosophic set. Sustainability 7(12):15923–15936
15. Wang HB, Smarandache F, Zhang YQ, Sunderraman R (2005) Interval neutrosophic sets and logic: theory and applications in computing. Hexis, Phoenix
16. Broumi S, Deli I, Smarandache F (2015) N-valued interval neutrosophic sets and their application in medical diagnosis. In: Critical review, center for mathematics of uncertainty, Creighton University, USA, vol 10, pp 46–69
17. Broumi S, Smarandache F, Talea M, Bakali A (2016) Decision-making method based on the interval valued neutrosophic graph. Future Technol 1–8
18. Broumi S, Talea M, Bakali A, Smarandache F (2016) On strong interval valued neutrosophic graphs. Crit Rev, 12, pp 49–71
19. Deli I (2015) NPN-soft sets theory and applications. Ann Fuzzy Math Inf 10(6):847–862

20. Deli I, Broumi S (2015) Neutrosophic soft matrices and NSM-decision making. J Intell Fuzzy Syst 28(5):2233–2241
21. Broumi S, Smarandache F (2015) Interval-valued neutrosophic soft rough set. Int J Comput Math. doi:10.1155/2015/232919
22. Deli I (2015) Interval-valued neutrosophic soft sets and its decision making. Int J Mach Learn Cybernet. doi:10.1007/s13042-015-0461-3
23. Smarandache F (2013) N-valued refined neutrosophic logic and its applications in physics. Progress in Physics 4:143–146
24. Broumi S, Smarandache F, Maji PK (2014) Intuitionistic neutrosphic soft set over rings. Math Stat 2(3):120–126
25. Smarandache F (2015) Symbolic neutrosophic theory. Europanova asbl, Brussels, pp 1–195
26. Deli I, Ali M, Smarandache F (2015) Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. In: International conference on advanced mechatronic systems. IEEE, pp 249–254
27. Deli I, Şubaş, Yusuf, Smaradache F, Ali M (2016) Interval valued bipolar neutrosophic sets and their application in pattern recognition
28. Tian ZP, Wang J, Wang JQ, Zhang HY (2016) Simplified neutrosophic linguistic multi-criteria group decision-making approach to green product development. Group Decis Negot. doi:10.1007/s10726-016-9479-5
29. Tian ZP, Wang J, Wang JQ, Zhang HY (2016) Multi-criteria decision-making based on generalized prioritized aggregation operators under simplified neutrosophic uncertain linguistic environment. Int J Mach Learn Cybernet. doi:10.1007/s13042-016-0552-9
30. Broumi S, Talea M, Bakali A, Smarandache F (2016) Single valued neutrosophic graphs. J New Theory 10:86–101
31. Broumi S, Bakali A, Talea M, Smarandache F (2016) Isolated single valued neutrosophic graphs. Neutrosophic Sets Syst 11:74–78
32. Broumi S, Talea M, Bakali A, Smarandache F (2016) On bipolar single valued neutrosophic graphs. J New Theory 11:84–102
33. Broumi S, Smarandache F, Talea M, Bakali A (2016) An introduction to bipolar single valued neutrosophic graph theory. Appl Mech Mater 841:184–191
34. Broumi S, Talea M, Bakali A, Smarandache F (2016) Interval valued neutrosophic graphs. Crit Rev 12:5–33
35. Broumi S, Talea M, Smarandache F, Bakali A (2016) Single valued neutrosophic graphs: degree, order and size. In: IEEE world congress on computational intelligence, pp 2444–2451
36. Liu PD, Tang GL (2016) Some power generalized aggregation operators based on the interval neutrosophic sets and their application to decision making. J Intell Fuzzy Syst 30(5):2517–2528
37. Şahin R, Liu PD (2015) Maximizing deviation method for neutrosophic multiple attribute decision making with incomplete weight information. Neural Comput Appl 27(7):2017–2029
38. Peng JJ, Wang JQ, Wang J, Zhang HY, Chen XH (2016) Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems. Int J Syst Sci 47(10):2342–2358
39. Broumi S, Smarandache F (2005) New distance and similarity measures of interval neutrosophic sets. In: International conference on information fusion. IEEE, pp 1–7
40. Broumi S, Deli I, Smarandache F (2009) Distance and similarity measures of interval neutrosophic soft sets. In: 2014 17th International Conference on information fusion (FUSION). IEEE, pp 1–7
41. Ye J (2015) Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses. Artif Intell Med 63(3):171–179
42. Uluçay V, Deli I, Şahin M (2016) Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. Neural Comput Appl. doi:10.1007/s00521-016-2479-1
43. Broumi S, Smarandache F (2014) Correlation coefficient of interval neutrosophic set. Appl Mech Mater 436:511–517
44. Şahin R, Liu PD (2016) Correlation coefficient of single-valued neutrosophic hesitant fuzzy sets and its applications in decision making. Neural Comput Appl. doi:10.1007/s00521-015-2163-x
45. Broumi S, Deli I (2016) Correlation measure for neutrosophic refined sets and its application in medical diagnosis. Palest J Math 5(1):135–143
46. Ye J (2014) Single valued neutrosophic cross-entropy for multicriteria decision making problems. Appl Math Model 38(3):1170–1175
47. Tian ZP, Zhang HY, Wang J, Wang JQ, Chen XH (2016) Multicriteria decision-making method based on a cross-entropy with interval neutrosophic sets. Int J Syst Sci 47(15):3598–3608
48. Şahin R (2015) Cross-entropy measure on interval neutrosophic sets and its applications in multicriteria decision making. Neural Comput Appl. doi:10.1007/s00521-015-2131-5
49. Şahin R, Küçük A (2015) Subsethood measure for single valued neutrosophic sets. J Intell Fuzzy Syst 29(2):525–530
50. Ye J (2014) A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. J Intell Fuzzy Syst 26(5):2459–2466
51. Ye J (2014) Vector similarity measures of simplified neutrosophic sets and their application in multicriteria decision making. Int J Fuzzy Syst 16(2):204–211
52. Wu XH, Wang J, Peng JJ, Chen XH (2016) Cross-entropy and prioritized aggregation operator with simplified neutrosophic sets and their application in multi-criteria decision-making problems. Int J Fuzzy Syst. doi:10.1007/s40815-016-0180-2
53. Wang JQ, Li XE (2015) TODIM method with multi-valued neutrosophic sets. Control Decis 30(6):1139–1142
54. Ye J (2014) Multiple-attribute decision-making method under a single-valued neutrosophic hesitant fuzzy environment. J Intell Syst 24(1):23–36
55. Peng JJ, Wang JQ, Wu XH, Wang J, Chen XH (2015) Multivalued neutrosophic sets and power aggregation operators with their applications in multi-criteria group decision-making problems. Int J Comput Intell Syst 8(2):345–363
56. Liu PD, Zhang LL, Liu X, Wang P (2016) Multi-valued neutrosophic number Bonferroni mean operators with their applications in multiple attribute group decision making. Int J Inf Technol Decis Mak 15(5):1181–1210
57. Ji P, Zhang HY, Wang JQ (2016) A projection-based TODIM method under multi-valued neutrosophic environments and its application in personnel selection. Neural Comput Appl. doi:10.1007/s00521-016-2436-z
58. Peng JJ, Wang JQ, Wu XH (2016) An extension of the ELECTRE approach with multi-valued neutrosophic information. Neural Comput Appl. doi:10.1007/s00521-016-2411-8
59. Peng JJ, Wang JQ, Yang WE (2017) A multi-valued neutrosophic qualitative flexible approach based on likelihood for multi-criteria decision-making problems. Int J Syst Sci 48(2):425–435
60. Delgado M, Verdegay JL, Vila MA (1993) On aggregation operations of linguistic labels. Int J Intell Syst 8(3):351–370
61. Wei CP, Zhao N, Tang XJ (2014) Operators and comparisons of hesitant fuzzy linguistic term sets. IEEE Trans Fuzzy Syst 22(3):575–585
62. Paelinck JHP (1978) Qualiflex: a flexible multiple-criteria method. Econ Lett 1(3):193–197
63. Chen SJ, Hwang CL (1992) Fuzzy multiple attribute decision making: methods and applications. Springer, Berlin

64. Tian ZP, Wang J, Wang JQ, Zhang HY (2016) An improved MULTIMOORA approach for multi-criteria decision-making based on interdependent inputs of simplified neutrosophic linguistic information. Neural Comput Appl. doi:10.1007/s00521-016-2378-5
65. Yu L, Lai KK (2011) A distance-based group decision-making methodology for multi-person multi-criteria emergency decision support. Decis Support Syst 51(2):307–315
66. Wang J, Wang JQ, Zhang HY (2016) A likelihood-based TODIM approach based on multi-hesitant fuzzy linguistic information for evaluation in logistics outsourcing. Comput Ind Eng 99:287–299
67. Smarandache F (2013) Introduction to neutrosophic measure, neutrosophic integral, and neutrosophic Probability. Infinite study, pp 1–140