The classical spin-rotation coupling

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Abstract

This paper is prepared to show that a rigid body which accelerates circularly relative to a fixed point must simultaneously accelerate angularly relative to its center of mass. Formulae which coupling of the angular momentum and kinetic energy due to the induced spin motion of the rigid body to the angular momentum and kinetic energy due to the rotational motion of the same spinning rigid body have been derived. The paper also bringing to light the nature of the force which causes the induction of spin motion in a rigid body and a formula which coupling of this highlighted force to the force which causes the rotation of the rigid body has been also derived.

1 Introduction

One can define the problem by the following statement: "division of the total angular momentum into its orbital and spin parts is especially useful because it is often true (at least to a good approximation) that the two parts are *separately* conserved." [**1**] —John Taylor. The statement briefs the common understanding within scientific community about spin-rotation relation for a rigid body in circular motion. Hence, here we are going to prove that the negation of this statement is what is true.

We beginning with the distinction between rotational and spinning motion therefore we define *rotation* with the motion of the rigid body circularly around a fixed point or a center of rotation which is different from its center of mass while the *spinning* is the rotation of the rigid body around its center of mass. Another thing is that the analysis is going to be with planar rigid body in planar motion over Euclidean space.

2 Analysis

2.1 Coupling of the spinning and rotational angular momentum

Referring to Figure 1, a rigid body "A" of a mass m is free to rotate relative to its center of mass CM as it is also can be, simultaneously, free to rotate relative to the point O. Thus, it is pivoted at these two points. It is known that the total angular momentum $\mathbf{L}$ of the rigid body "A in its circular motion is given by[**1**]:

$$\mathbf{L} = \mathbf{r}_{CM} \times \mathbf{p} + \sum_i \boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i m_i \tag{1}$$

where $\mathbf{r}_{CM}$ is the vector position of the center of mass of the rigid body relative to the axis of rotation O, $\mathbf{p}$ is the linear momentum and $\boldsymbol{\rho}_i$ is the vector position of the element mass m_i relative to the center of mass of the rigid body. The first term is the angular momentum (relative to O) of the motion of the center of mass. The second is the angular momentum of the motion relative to the center of mass. Thus, we can re-express Equation (1) to say[**1**]

$$\mathbf{L} = \mathbf{L}_{\text{motion of CM}} + \mathbf{L}_{\text{motion relative to CM}} \tag{2}$$

Since the mass is constrained to a circle, the tangential velocity of the mass of the rigid body "A is $\boldsymbol{\omega} \times \mathbf{r}_{CM}$ —where $\boldsymbol{\omega}$ is the rotational angular velocity of the rigid body relative to the axis of rotation O which is perpendicular to the plane of the motion— and since $\mathbf{p} = m\boldsymbol{\omega}\times\mathbf{r}_{CM}$, the total angular momentum equation (Equation (1)) becomes (assuming the motion is planar, hence both axises of rotation, O and CM, are parallel):

$$\mathbf{L} = \mathbf{r}_{CM} \times (m\boldsymbol{\omega} \times \mathbf{r}_{CM}) \; + \; \sum_i \boldsymbol{\rho}_i \times (\boldsymbol{\Omega} \times \boldsymbol{\rho}_i) m_i \tag{3}$$

where $\boldsymbol{\Omega}$ is the rigid body spinning angular velocity (arbitrary) relative to its center of mass. Taking the first term in the RHS and using the position vector equation[1]

$$\mathbf{r}_i = \mathbf{r}_{CM} + \boldsymbol{\rho}_i \ . \tag{4}$$

Therefore, one finds

$$\begin{aligned}
\mathbf{r}_{CM} \times (m\boldsymbol{\omega} \times \mathbf{r}_{CM}) &= (\mathbf{r}_i - \boldsymbol{\rho}_i) \times (m\boldsymbol{\omega} \times (\mathbf{r}_i - \boldsymbol{\rho}_i)) \ , \\
&= \mathbf{r}_i \times (m\boldsymbol{\omega} \times \mathbf{r}_i) - \boldsymbol{\rho}_i \times (m\boldsymbol{\omega} \times \mathbf{r}_i) - \mathbf{r}_i \times (m\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \\
&\quad + \boldsymbol{\rho}_i \times (m\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \ , \\
&= \mathbf{r}_i \times (m\boldsymbol{\omega} \times \mathbf{r}_i) - \boldsymbol{\rho}_i \times (m\boldsymbol{\omega} \times (\mathbf{r}_{CM} + \boldsymbol{\rho}_i)) \\
&\quad - (\mathbf{r}_{CM} + \boldsymbol{\rho}_i) \times (m\boldsymbol{\omega} \times \boldsymbol{\rho}_i) + \boldsymbol{\rho}_i \times (m\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \ , \\
&= \mathbf{r}_i \times (m\boldsymbol{\omega} \times \mathbf{r}_i) - \boldsymbol{\rho}_i \times (m\boldsymbol{\omega} \times \mathbf{r}_{CM}) - \boldsymbol{\rho}_i \times (m\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \\
&\quad - \mathbf{r}_{CM} \times (m\boldsymbol{\omega} \times \boldsymbol{\rho}_i) - \boldsymbol{\rho}_i \times (m\boldsymbol{\omega} \times \boldsymbol{\rho}_i) + \boldsymbol{\rho}_i \times (m\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \ , \\
&= \mathbf{r}_i \times (m\boldsymbol{\omega} \times \mathbf{r}_i) - \boldsymbol{\rho}_i \times (m\boldsymbol{\omega} \times \mathbf{r}_{CM}) - \boldsymbol{\rho}_i \times (m\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \\
&\quad - \mathbf{r}_{CM} \times (m\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \ ,
\end{aligned}$$

since $m = \sum_i m_i$, where m_i is a mass element, hence we can write

$$\begin{aligned}
\mathbf{r}_{CM} \times (m\boldsymbol{\omega} \times \mathbf{r}_{CM}) &= \sum_i \mathbf{r}_i \times (m_i\boldsymbol{\omega} \times \mathbf{r}_i) - \sum_i m_i\boldsymbol{\rho}_i \times (\boldsymbol{\omega} \times \mathbf{r}_{CM}) \\
&\quad - \sum_i \boldsymbol{\rho}_i \times (m_i\boldsymbol{\omega} \times \boldsymbol{\rho}_i) - \mathbf{r}_{CM} \times \left(\boldsymbol{\omega} \times \sum_i m_i\boldsymbol{\rho}_i\right) \ ,
\end{aligned}$$

and because $\boldsymbol{\rho}_i$ is the vector position of a mass element m_i relative to the center of mass, then from the definition of the center of mass we have[**2**]

$$\sum_i m_i\boldsymbol{\rho}_i = 0 \ , \tag{5}$$

which implies that

$$\mathbf{r}_{CM} \times (m\boldsymbol{\omega} \times \mathbf{r}_{CM}) = \sum_i \mathbf{r}_i \times (m_i\boldsymbol{\omega} \times \mathbf{r}_i) - \sum_i \boldsymbol{\rho}_i \times (m_i\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \ , \tag{6}$$

[1] This substitution is the main tool which brings us to another level of analysis of these formulae and the results follow directly from it.

using the identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\circ\mathbf{C})\,\mathbf{B} - (\mathbf{A}\circ\mathbf{B})\,\mathbf{C} \tag{7}$$

and using the facts that $\boldsymbol{\rho}_i$ and $\boldsymbol{\omega}$ are mutually orthogonal and so are $\mathbf{r}_i$ and $\boldsymbol{\omega}$. Therefore, one will find

$$\begin{aligned}\mathbf{r}_{CM}\times(m\boldsymbol{\omega}\times\mathbf{r}_{CM}) &= \sum_i m_i r_i^2\boldsymbol{\omega} - \sum_i m_i\rho_i^2\boldsymbol{\omega}\ ,\\ &= I_O\boldsymbol{\omega} - I_{CM}\boldsymbol{\omega}\ ,\end{aligned} \tag{8}$$

where

$$I_O = \sum_i m_i r_i^2\ , \tag{9}$$

is the moment of inertia relative to the axis of rotation O which is a perpendicular distance $\mathbf{r}_{CM}$ from the centre of mass and

$$I_{CM} = \sum_i m_i\rho_i^2\ , \tag{10}$$

is the moment of inertia of the rigid body relative to its center of mass[**3**].

Thus, Equation (3), the total angular momentum becomes

$$\begin{aligned}\mathbf{L} &= I_O\boldsymbol{\omega} - I_{CM}\boldsymbol{\omega} + \sum_i \boldsymbol{\rho}_i\times(\boldsymbol{\Omega}\times\boldsymbol{\rho}_i)m_i\ ,\\ &= I_O\boldsymbol{\omega} - I_{CM}\boldsymbol{\omega} + I_{CM}\boldsymbol{\Omega}\ .\end{aligned} \tag{11}$$

The term $-I_{CM}\boldsymbol{\omega}$ is an additional angular momentum term relative to the center of mass of the rigid body (spinning) and occurs due to the rigid body rotational or circular motion relative to the axis of rotation O. The term $I_{CM}\boldsymbol{\Omega}$ can be consider as the *initial* spinning angular momentum that the rigid body acquired before it start its circular motion and since $\boldsymbol{\Omega}$ is arbitrary, so that it can be zero and have not to be a mandatory term of Equation (11). Hence, we can write

$$\begin{aligned}\mathbf{L} &= I_O\boldsymbol{\omega} - I_{CM}\boldsymbol{\omega}\ ,\\ &= \mathbf{L}_R + \mathbf{L}_S\ .\end{aligned} \tag{12}$$

where $\mathbf{L} = \mathbf{r}_{CM}\times(m\boldsymbol{\omega}\times\mathbf{r}_{CM})$ is the *total* angular momentum, $\mathbf{L}_R = I_O\boldsymbol{\omega}$ is the *rotational* angular momentum and $\mathbf{L}_S = I_{CM}(-\boldsymbol{\omega})$ is the *spinning* angular momentum.

Since the total angular momentum (Equation (12)) is conserved, that implies the rotational and spinning angular momentum are mutually exchangeable in order to conserve it, that is

$$\mathbf{L} = \downarrow\uparrow\mathbf{L}_R + \ \uparrow\downarrow\mathbf{L}_S \tag{13}$$

This property (Equation (13)) negate the above statement which we have begin with. Another thing we can notice is that if we simplify the term $\mathbf{r}_{CM} \times (m\boldsymbol{\omega} \times \mathbf{r}_{CM})$ in Equation (8) using the identity (7) and rearrange it, that yields

$$I_O\boldsymbol{\omega} = \left[(\mathbf{r}_{CM} \circ \mathbf{r}_{CM})\, m\boldsymbol{\omega} - (\mathbf{r}_{CM} \circ m\boldsymbol{\omega})\, \mathbf{r}_{CM}\right] + I_{CM}\boldsymbol{\omega} \ ,$$

since motion is planar then $\mathbf{r}_{CM}$ and $\boldsymbol{\omega}$ are mutually orthogonal, so that

$$I_O\boldsymbol{\omega} = mr_{CM}^2 \ \boldsymbol{\omega} + I_{CM}\boldsymbol{\omega} \ ,$$

dotted with $\boldsymbol{\omega}$ and then divide by ω^2, we have

$$I_O = mr_{CM}^2 + I_{CM} \ . \tag{14}$$

which is nothing other than the *parallel axis theorem*[**3**].

2.2 Coupling of the spinning and rotational kinetic energy

It is known that the kinetic energy T of the rigid body "A in its rotational motion relative to the axis of rotation O is given by[**4**]:

$$T = \tfrac{1}{2}m\left(\boldsymbol{\omega} \times \mathbf{r}_{CM} \circ \boldsymbol{\omega} \times \mathbf{r}_{CM}\right) + \tfrac{1}{2}I_{CM}\left(\boldsymbol{\Omega} \circ \boldsymbol{\Omega}\right) \tag{15}$$

where $\boldsymbol{\omega} \times \mathbf{r}_{CM}$ is the tangential velocity of the center of mass of the rigid body relative to the axis of rotation O and $\boldsymbol{\Omega}$ is an arbitrary spinning velocity relative to the center of mass of the rigid body. Taking the first term in the RHS and using the identity

$$(\mathbf{A} \times \mathbf{B} \circ \mathbf{C} \times \mathbf{D}) = (\mathbf{A} \circ \mathbf{C})\,(\mathbf{B} \circ \mathbf{D}) - (\mathbf{A} \circ \mathbf{D})\,(\mathbf{B} \circ \mathbf{C}) \ , \tag{16}$$

one obtains

$$\tfrac{1}{2}m\left(\boldsymbol{\omega} \times \mathbf{r}_{CM} \circ \boldsymbol{\omega} \times \mathbf{r}_{CM}\right) = \tfrac{1}{2}m\left(\boldsymbol{\omega} \circ \boldsymbol{\omega}\right)\left(\mathbf{r}_{CM} \circ \mathbf{r}_{CM}\right) \ , \tag{17}$$

substitute Equation (4), using the fact that $\boldsymbol{\omega}$ and $\mathbf{r}_{CM}$ are mutually orthogonal, we have

$$\begin{aligned}
\tfrac{1}{2}m\left(\boldsymbol{\omega} \circ \boldsymbol{\omega}\right)\left(\mathbf{r}_{CM} \circ \mathbf{r}_{CM}\right) &= \tfrac{1}{2}m\left(\boldsymbol{\omega} \circ \boldsymbol{\omega}\right)\left(\mathbf{r}_i - \boldsymbol{\rho}_i \circ \mathbf{r}_i - \boldsymbol{\rho}_i\right) \ , \\
&= \tfrac{1}{2}m\left(\boldsymbol{\omega} \circ \boldsymbol{\omega}\right)\left[(\mathbf{r}_i \circ \mathbf{r}_i) - 2\,(\mathbf{r}_i \circ \boldsymbol{\rho}_i) + (\boldsymbol{\rho}_i \circ \boldsymbol{\rho}_i)\right] , \\
&= \tfrac{1}{2}\sum_i m_i\,(\mathbf{r}_i \circ \mathbf{r}_i)\,(\boldsymbol{\omega} \circ \boldsymbol{\omega}) - \sum_i m_i\,(\mathbf{r}_i \circ \boldsymbol{\rho}_i)\,(\boldsymbol{\omega} \circ \boldsymbol{\omega}) \\
&\quad + \tfrac{1}{2}\sum_i m_i\,(\boldsymbol{\rho}_i \circ \boldsymbol{\rho}_i)\,(\boldsymbol{\omega} \circ \boldsymbol{\omega}) \ , \\
&= \tfrac{1}{2}\sum_i m_i\,(\mathbf{r}_i \circ \mathbf{r}_i)\,(\boldsymbol{\omega} \circ \boldsymbol{\omega}) - \sum_i m_i\,(\mathbf{r}_{CM} + \boldsymbol{\rho}_i \circ \boldsymbol{\rho}_i)\,(\boldsymbol{\omega} \circ \boldsymbol{\omega}) \\
&\quad + \tfrac{1}{2}\sum_i m_i\,(\boldsymbol{\rho}_i \circ \boldsymbol{\rho}_i)\,(\boldsymbol{\omega} \circ \boldsymbol{\omega}) \ ,
\end{aligned}$$

$$\tfrac{1}{2}m\left(\boldsymbol{\omega}\circ\boldsymbol{\omega}\right)\left(\mathbf{r}_{CM}\circ\mathbf{r}_{CM}\right)=\tfrac{1}{2}\sum_i m_i\left(\mathbf{r}_i\circ\mathbf{r}_i\right)\left(\boldsymbol{\omega}\circ\boldsymbol{\omega}\right)-\sum_i m_i\left(\boldsymbol{\rho}_i\circ\boldsymbol{\rho}_i\right)\left(\boldsymbol{\omega}\circ\boldsymbol{\omega}\right)$$
$$-\left(\mathbf{r}_{CM}\circ\sum_i m_i\boldsymbol{\rho}_i\right)\left(\boldsymbol{\omega}\circ\boldsymbol{\omega}\right)+\tfrac{1}{2}\sum_i m_i\left(\boldsymbol{\rho}_i\circ\boldsymbol{\rho}_i\right)\left(\boldsymbol{\omega}\circ\boldsymbol{\omega}\right)\ ,$$

using Equation (5), (9) and (10), we obtain

$$\tfrac{1}{2}m\left(\boldsymbol{\omega}\circ\boldsymbol{\omega}\right)\left(\mathbf{r}_{CM}\circ\mathbf{r}_{CM}\right)=\tfrac{1}{2}I_O\left(\boldsymbol{\omega}\circ\boldsymbol{\omega}\right)-\tfrac{1}{2}I_{CM}\left(\boldsymbol{\omega}\circ\boldsymbol{\omega}\right)\ . \tag{18}$$

An important thing has to be notice in Equation (18) when we rearrange it as following:

$$\tfrac{1}{2}I_O\left(\boldsymbol{\omega}\circ\boldsymbol{\omega}\right)=\tfrac{1}{2}m\left(\boldsymbol{\omega}\circ\boldsymbol{\omega}\right)\left(\mathbf{r}_{CM}\circ\mathbf{r}_{CM}\right)+\tfrac{1}{2}I_{CM}\left(\boldsymbol{\omega}\circ\boldsymbol{\omega}\right)\ ,$$

doing the dot product and then divide into $\frac{1}{2}\omega^2$, we will obtain $I_O = mr_{CM}^2 + I_{CM}$, which again is the *parallel axis theorem*.

Since there is no negative kinetic energy therefore one can understand that the negative sign is assign to the direction instead of the magnitude. Thus, substituting Equation (18) back into Equation (15), the kinetic energy formula becomes

$$T=\tfrac{1}{2}I_O\left(\boldsymbol{\omega}\circ\boldsymbol{\omega}\right)+\tfrac{1}{2}I_{CM}\left(-\boldsymbol{\omega}\circ-\boldsymbol{\omega}\right)+\tfrac{1}{2}I_{CM}\left(\boldsymbol{\Omega}\circ\boldsymbol{\Omega}\right)\ . \tag{19}$$

The term $\frac{1}{2}I_{CM}\left(\boldsymbol{\Omega}\circ\boldsymbol{\Omega}\right)$ can be consider as the *initial* spinning kinetic energy which is the rigid body have before it start its circular motion and since $\boldsymbol{\Omega}$ is arbitrary, so that it can be zero and have not to be a mandatory term of Equation (19). Hence we can write

$$\begin{aligned} T &= \tfrac{1}{2}I_O\left(\boldsymbol{\omega}\circ\boldsymbol{\omega}\right)+\tfrac{1}{2}I_{CM}\left(-\boldsymbol{\omega}\circ-\boldsymbol{\omega}\right)\ ,\\ &= T_R + T_S\ . \end{aligned} \tag{20}$$

where $T=\frac{1}{2}m\left(\boldsymbol{\omega}\times\mathbf{r}_{CM}\circ\boldsymbol{\omega}\times\mathbf{r}_{CM}\right)$ is the *total* kinetic energy, $T_R=\frac{1}{2}I_O\left(\boldsymbol{\omega}\circ\boldsymbol{\omega}\right)$ is the *rotational* kinetic energy and $T_S=\frac{1}{2}I_{CM}\left(-\boldsymbol{\omega}\circ-\boldsymbol{\omega}\right)$ is the *spinning* kinetic energy.

2.3 Coupling of the acting forces

At this section we will explore the coupling between the force causes the rotation of the rigid body "*A*" and the force causes its spin. Referring to Figure 1 if an external force $\mathbf{F}$ acts on the center of mass of the rigid body "*A*" such that it causes it to angularly accelerate with angular acceleration $\dot{\boldsymbol{\omega}}=\left(d\boldsymbol{\omega}/dt\right)$ relative to the axis of rotation O —where $\boldsymbol{\omega}$ is the angular velocity of rotation of the center of mass of the rigid body relative to the axis of rotation O— and since the mass m is constrained to a circle, the tangential acceleration of the mass of the rigid body "A" is $\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM}$, and since $\mathbf{F} = m\boldsymbol{a}$, the total torque equation is given by:

$$\begin{aligned} \boldsymbol{\tau} &= \mathbf{r}_{CM} \times \mathbf{F} \ , \\ &= \mathbf{r}_{CM} \times (m\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM}) \ . \end{aligned} \tag{21}$$

where $\boldsymbol{\tau}$ is the total torque and $\mathbf{r}_{CM}$ is the vector position of the center of mass of the rigid body relative to the axis of rotation O. Substituting Equation (4), we find

$$\begin{aligned} \mathbf{r}_{CM} \times (m\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM}) &= (\mathbf{r}_i - \boldsymbol{\rho}_i) \times (m\dot{\boldsymbol{\omega}} \times (\mathbf{r}_i - \boldsymbol{\rho}_i)) \ , \\ &= \sum_i \mathbf{r}_i \times (m_i\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) - \sum_i \mathbf{r}_i \times (m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_i) \\ &\quad - \sum_i \boldsymbol{\rho}_i \times (m_i\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) \ + \ \sum_i \boldsymbol{\rho}_i \times (m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_i) \ , \\ &= \sum_i \mathbf{r}_i \times (m_i\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) - \sum_i (\mathbf{r}_{CM} + \boldsymbol{\rho}_i) \times (m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_i) \\ &\quad - \sum_i \boldsymbol{\rho}_i \times (m_i\dot{\boldsymbol{\omega}} \times (\mathbf{r}_{CM} + \boldsymbol{\rho}_i)) \ + \ \sum_i \boldsymbol{\rho}_i \times (m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_i) \ , \\ &= \sum_i \mathbf{r}_i \times (m_i\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) - \mathbf{r}_{CM} \times \left(\dot{\boldsymbol{\omega}} \times \sum_i m_i \boldsymbol{\rho}_i \right) \\ &\quad - \sum_i \boldsymbol{\rho}_i \times (m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_i) \ - \ \sum_i m_i \boldsymbol{\rho}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM}) \\ &\quad - \sum_i \boldsymbol{\rho}_i \times (m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_i) \ + \ \sum_i \boldsymbol{\rho}_i \times (m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_i) \ , \end{aligned}$$

using Equation (5), one finds

$$\mathbf{r}_{CM} \times (m\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM}) = \sum_i \mathbf{r}_i \times (m_i\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) - \sum_i \boldsymbol{\rho}_i \times (m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_i) \ , \tag{22}$$

using the identity (7) accompany with the facts that $\boldsymbol{\rho}_i$ and $\dot{\boldsymbol{\omega}}$ are mutually orthogonal and so are $\mathbf{r}_i$ and $\dot{\boldsymbol{\omega}}$. Then simplify using Equation (9) and (10). Therefore, one will have

$$\begin{aligned} \mathbf{r}_{CM} \times (m\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM}) &= \sum_i m_i r_i^2 \dot{\boldsymbol{\omega}} - \sum_i m_i \rho_i^2 \dot{\boldsymbol{\omega}} \ , \\ &= I_O\dot{\boldsymbol{\omega}} - I_{CM}\dot{\boldsymbol{\omega}} \ . \end{aligned} \tag{23}$$

We can write it as:

$$\boldsymbol{\tau} = \boldsymbol{\tau_R} + \boldsymbol{\tau_S} \tag{24}$$

where $\boldsymbol{\tau} = \mathbf{r}_{CM} \times (m\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM})$ is the *total* torque. $\boldsymbol{\tau_R} = I_O\dot{\boldsymbol{\omega}}$ is the *rotation* torque and $\boldsymbol{\tau_S} = -I_{CM}\dot{\boldsymbol{\omega}}$ is the *spin* torque.

If we rearrange Equation (23) as following:

$$I_O\dot{\boldsymbol{\omega}} = \mathbf{r}_{CM} \times (m\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM}) + I_{CM}\dot{\boldsymbol{\omega}} \ ,$$

and using the identity (7) with the fact that $\mathbf{r}_{CM}$ and $\dot{\boldsymbol{\omega}}$ are mutually orthogonal, we obtain

$$I_O\dot{\boldsymbol{\omega}} = mr_{CM}^2 \ \dot{\boldsymbol{\omega}} + I_{CM}\dot{\boldsymbol{\omega}} \ ,$$

dotted with $\dot{\boldsymbol{\omega}}$ and divide into $\dot{\omega}^2$, we again obtain $I_O = mr_{CM}^2 + I_{CM}$, the *parallel axis theorem.*

2.4 The nature of the force which causes the spin torque

To find out the nature of the force behinds the spin torque, $\boldsymbol{\tau_s}$, we are going to take the kinetic approach to find the same term that assigned to it and which appears in Equation (23), that is, $-I_{CM}\dot{\boldsymbol{\omega}}$. Referring to Figure 2, we have a space-fixed coordinate system, S, which is a coordinate system with the origin fixed in space at point O, and with space-fixed directions for the axes. We have also a body-fixed coordinate system, S', with an arbitrary point Q (reference point) on the rigid body is selected as the coordinate origin. Therefore, the quantities in the reference systems S and S' are related as follows[**5**]:

$$\mathbf{r}_i = \mathbf{r}_{CM} + \mathbf{r}'_i \ . \tag{25}$$

where $\mathbf{r}_i$ is the position vector of point i in the space-fixed reference system S. $\mathbf{r}'_i$ is the position vector of point i in the body-fixed reference system S'. $\mathbf{r}_{CM}$ is the position vector of the reference point Q in the space-fixed reference system S.

First change in position vector:

Taking the first change in position vector (Equation (25)) with respect to time, that yields

$$\dot{\mathbf{r}}_i = \dot{\mathbf{r}}_{CM} + \dot{\mathbf{r}}'_i \ ,$$

$$\boldsymbol{v}_i + \boldsymbol{\omega}_1 \times \mathbf{r}_i = (\boldsymbol{v}_{CM} + \boldsymbol{\omega}_2 \times \mathbf{r}_{CM}) + (\boldsymbol{v}'_i + \boldsymbol{\omega}_3 \times \mathbf{r}'_i) \ ,$$

where $\boldsymbol{v}_i$ is the velocity of point i relative to the origin of S-frame, $\boldsymbol{v}_{CM}$ is the velocity of the origin of S'-frame relative to the origin of S-frame, and $\boldsymbol{v}'_i$ is the velocity of point i relative to the origin of S'-frame. The angular velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$ are due to the rotation of the vectors position $\mathbf{r}_i, \mathbf{r}_{CM}$ and $\mathbf{r}'_i$, respectively. Since the motion is happening to a rigid body, therefore we have $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2 = \boldsymbol{\omega}_3 = \boldsymbol{\omega}$. Hence, we can write

$$\boldsymbol{v}_i + \boldsymbol{\omega} \times \mathbf{r}_i = (\boldsymbol{v}_{CM} + \boldsymbol{\omega} \times \mathbf{r}_{CM}) + (\boldsymbol{v}'_i + \boldsymbol{\omega} \times \mathbf{r}'_i) \ . \tag{26}$$

With the help of Equation (25), Equation (26) yields

$$\boldsymbol{v}_{CM} - \boldsymbol{v}_i + \boldsymbol{v}'_i = -\boldsymbol{\omega} \times (\mathbf{r}_{CM} - \mathbf{r}_i + \mathbf{r}'_i) = 0 \ . \tag{27}$$

Therefore, we have

$$\boldsymbol{v}_{CM} = \boldsymbol{v}_i - \boldsymbol{v}'_i \ , \text{and} \tag{28}$$

$$\boldsymbol{\omega} \times \mathbf{r}_{CM} = \boldsymbol{\omega} \times \mathbf{r}_i - \boldsymbol{\omega} \times \mathbf{r}'_i \ . \tag{29}$$

Second change in position vector:

Taking the second change in position vector (Equation (25)) with respect to time, that yields

$$\begin{aligned} \ddot{\mathbf{r}}_i &= \ddot{\mathbf{r}}_{CM} + \ddot{\mathbf{r}}'_i \ , \\ \dot{\boldsymbol{v}}_i + \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + \boldsymbol{\omega} \times \dot{\mathbf{r}}_i &= (\dot{\boldsymbol{v}}_{CM} + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM} + \boldsymbol{\omega} \times \dot{\mathbf{r}}_{CM}) \\ &\quad + (\dot{\boldsymbol{v}}'_i + \dot{\boldsymbol{\omega}} \times \mathbf{r}'_i + \boldsymbol{\omega} \times \dot{\mathbf{r}}'_i) \ , \end{aligned}$$

substituting $\dot{\mathbf{r}}_i, \dot{\mathbf{r}}_{CM}$ and $\dot{\mathbf{r}}'_i$ from Equation (26), that gives[**6**]

$$\begin{aligned} &\dot{\boldsymbol{v}}_i + \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + \boldsymbol{\omega} \times \boldsymbol{v}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i) \\ &\qquad = (\dot{\boldsymbol{v}}_{CM} + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM} + \boldsymbol{\omega} \times \boldsymbol{v}_{CM} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{CM})) \\ &\qquad\quad + (\dot{\boldsymbol{v}}'_i + \dot{\boldsymbol{\omega}} \times \mathbf{r}'_i + \boldsymbol{\omega} \times \boldsymbol{v}'_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_i)) \ , \end{aligned}$$

and

$$\begin{aligned} &\boldsymbol{a}_i + \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + 2\boldsymbol{\omega} \times \boldsymbol{v}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i) \\ &\qquad = (\boldsymbol{a}_{CM} + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM} + 2\boldsymbol{\omega} \times \boldsymbol{v}_{CM} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{CM})) \\ &\qquad\quad + (\boldsymbol{a}'_i + \dot{\boldsymbol{\omega}} \times \mathbf{r}'_i + 2\boldsymbol{\omega} \times \boldsymbol{v}'_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_i)) \ . \end{aligned} \tag{30}$$

where $\boldsymbol{a}_i$ is the acceleration of point i relative to the origin of S-frame, $\boldsymbol{a}_{CM}$ is the acceleration of the origin of S'-frame relative to the origin of S-frame, and $\boldsymbol{a}'_i$ is the acceleration of point i relative to the origin of S'-frame. Regrouping by acceleration type, we obtain

$$\begin{aligned} 0 &= [\boldsymbol{a}_{CM} - (\boldsymbol{a}_i - \boldsymbol{a}'_i)] + [\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM} - (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i - \dot{\boldsymbol{\omega}} \times \mathbf{r}'_i)] \\ &\quad + [2\boldsymbol{\omega} \times \boldsymbol{v}_{CM} - (2\boldsymbol{\omega} \times \boldsymbol{v}_i - 2\boldsymbol{\omega} \times \boldsymbol{v}'_i)] \\ &\quad + [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{CM}) - \{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i) - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_i)\}] \ . \end{aligned} \tag{31}$$

With the aid of Equation (25), we find that, the grouped terms in the second and fourth square brackets in Equation (31) are equal to zeros, that is

$$\dot{\boldsymbol{\omega}} \times (\mathbf{r}_{CM} - \mathbf{r}_i + \mathbf{r}'_i) = 0 \ , \text{and} \tag{32}$$

$$\boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\mathbf{r}_{CM} - \mathbf{r}_i + \mathbf{r}'_i)] = 0 \ . \tag{33}$$

Equation (28) implies that, the grouped terms in the third square brackets in Equation (31) are equal to zero, that is

$$2\boldsymbol{\omega} \times [\boldsymbol{v}_{CM} - \boldsymbol{v}_i + \boldsymbol{v}'_i] = 0 \ . \tag{34}$$

Equations (32), (33) and (34) imply that, the grouped terms in the first square brackets in Equation (31) are also equal to zero, that is

$$\boldsymbol{a}_{CM} - \boldsymbol{a}_i + \boldsymbol{a}'_i = 0 \ . \tag{35}$$

Thus, Equations (32), (33), (34) and (35) can be rewritten as following:

$$\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM} = \dot{\boldsymbol{\omega}} \times \mathbf{r}_i - \dot{\boldsymbol{\omega}} \times \mathbf{r}'_i \ . \tag{36}$$

$$2\boldsymbol{\omega} \times \boldsymbol{v}_{CM} = 2\boldsymbol{\omega} \times \boldsymbol{v}_i - 2\boldsymbol{\omega} \times \boldsymbol{v}'_i \ . \tag{37}$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{CM}) = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i) - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_i) \ . \tag{38}$$

$$\boldsymbol{a}_{CM} = \boldsymbol{a}_i - \boldsymbol{a}'_i \ . \tag{39}$$

The sum of Equations (36) to (39) equals to Equation (30), that is

$$\begin{aligned} &\boldsymbol{a}_{CM} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{CM}) + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM} + 2\boldsymbol{\omega} \times \boldsymbol{v}_{CM} \\ &= \ \boldsymbol{a}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i) + \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + 2\boldsymbol{\omega} \times \boldsymbol{v}_i \\ &\quad + \\ &\quad (-\boldsymbol{a}'_i) + (-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_i)) + (-\dot{\boldsymbol{\omega}} \times \mathbf{r}'_i) + (-2\boldsymbol{\omega} \times \boldsymbol{v}'_i) \ . \end{aligned} \tag{40}$$

The forces that acting over the rigid body:

From Equations (36) to (40), and by multiplying by mass m and remembering that $m = \sum_i m_i$, we obtain

Tangential force:

$$m\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM} = \sum_i m_i \dot{\boldsymbol{\omega}} \times \mathbf{r}_i - \sum_i m_i \dot{\boldsymbol{\omega}} \times \mathbf{r}'_i \ . \tag{41}$$

Coriolis force:

$$2\boldsymbol{\omega} \times m\boldsymbol{v}_{CM} = 2\boldsymbol{\omega} \times \sum_i m_i \boldsymbol{v}_i - 2\boldsymbol{\omega} \times \sum_i m_i \boldsymbol{v}'_i \ . \tag{42}$$

Centripetal force:

$$\boldsymbol{\omega} \times m(\boldsymbol{\omega} \times \mathbf{r}_{CM}) = \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \sum_i m_i \mathbf{r}_i\right) - \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \sum_i m_i \mathbf{r}'_i\right) \ . \tag{43}$$

Rectilinear force:

$$m\boldsymbol{a}_{CM} = \sum_i m_i \boldsymbol{a}_i - \sum_i m_i \boldsymbol{a}'_i \ . \tag{44}$$

The combination of these forces gives the total force:

$$\overbrace{m[\boldsymbol{a}_{CM} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{CM}) + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM} + 2\boldsymbol{\omega} \times \boldsymbol{v}_{CM}]}^{Total force}$$

$$= \overbrace{\sum_i m_i[\boldsymbol{a}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i) + \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + 2\boldsymbol{\omega} \times \boldsymbol{v}_i]}^{\text{Active force}}$$

$$+ \tag{45}$$

$$\overbrace{\sum_i m_i[(-\boldsymbol{a}'_i) + (-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_i)) + (-\dot{\boldsymbol{\omega}} \times \mathbf{r}'_i) + (-2\boldsymbol{\omega} \times \boldsymbol{v}'_i)]}^{\text{Inertial force}} \ .$$

We find that, generally, every single force of the above forces (Equations (41) to (44)), is a synthesis of an *active* force and an *inertial* force. Similarly, the total force (Equation (45)).

If the coordinate origin (point Q) of the S'-frame has been chosen to be a center of mass of a rigid body —say the rigid body "A", that is, $\mathbf{r}'_i = \boldsymbol{\rho}_i$, where $\boldsymbol{\rho}_i$ is the position vector of the point i relative to the center of mass, and with the help of Equation (5) and its first and second derivative with respect to time, then, Equations (41) to (45) will give

Tangential force:

$$m\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM} = \sum_i m_i \dot{\boldsymbol{\omega}} \times \mathbf{r}_i \ . \tag{46}$$

Coriolis force:

$$2\boldsymbol{\omega} \times m\boldsymbol{v}_{CM} = 2\boldsymbol{\omega} \times \sum_i m_i \boldsymbol{v}_i \ . \tag{47}$$

Centripetal force:

$$\boldsymbol{\omega} \times m(\boldsymbol{\omega} \times \mathbf{r}_{CM}) = \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \sum_i m_i \mathbf{r}_i\right) \ . \tag{48}$$

Rectilinear force:

$$m\boldsymbol{a}_{CM} = \sum_i m_i \boldsymbol{a}_i \ . \tag{49}$$

The combination of these forces gives the total force:

$$m[\boldsymbol{a}_{CM} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{CM}) + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM} + 2\boldsymbol{\omega} \times \boldsymbol{v}_{CM}]$$

$$= \sum_i m_i[\boldsymbol{a}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i) + \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + 2\boldsymbol{\omega} \times \boldsymbol{v}_i] \ . \tag{50}$$

2.4.1 The contribution of the Euler force to the spin torque

If the rigid body "*A*" is *circularly* accelerating, then by cross multiplying Equation (46) with $\mathbf{r}_{CM}$, we will obtain the torque due to the tangential force, that is

$$\begin{aligned}
\mathbf{r}_{CM} \times (m\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM}) &= \sum_i (\mathbf{r}_i - \boldsymbol{\rho}_i) \times (m_i\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) \ , \\
&= \sum_i \mathbf{r}_i \times (m_i\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) - \sum_i \boldsymbol{\rho}_i \times (m_i\dot{\boldsymbol{\omega}} \times (\mathbf{r}_{CM} + \boldsymbol{\rho}_i)) \ , \\
&= \sum_i \mathbf{r}_i \times (m_i\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) - \sum_i m_i\boldsymbol{\rho}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM}) \\
&\quad - \sum_i \boldsymbol{\rho}_i \times (m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_i) \ ,
\end{aligned}$$

using Equation (5), we obtain

$$\mathbf{r}_{CM} \times (m\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM}) = \sum_i \mathbf{r}_i \times (m_i\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) - \sum_i \boldsymbol{\rho}_i \times (m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_i) \ . \tag{51}$$

Substituting Equation (25) into the second term of RHS of Equation (51) and then with the help of Equation (5), we find

$$\begin{aligned}
\sum_i \boldsymbol{\rho}_i \times [-(m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_i)] &= \sum_i \boldsymbol{\rho}_i \times [-m_i\dot{\boldsymbol{\omega}} \times (\boldsymbol{r}_i - \mathbf{r}_{CM})] \ , \\
&= \sum_i \boldsymbol{\rho}_i \times [-(m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{r}_i)] \ , \\
&= \sum_i \boldsymbol{\rho}_i \times \mathbf{F}_i \ , \\
&= \boldsymbol{\tau}_{Euler} \ .
\end{aligned}$$

where $\boldsymbol{\tau}_{Euler}$ is an inertial torque which occurs due to the Euler force[**7**], $\mathbf{F}_i = -m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{r}_i = -m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_i$, which acts on the mass element m_i. Therefore, we find

$$\boldsymbol{\tau}_{Euler} = -\sum_i \boldsymbol{\rho}_i \times (m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_i) = -I_{CM}\dot{\boldsymbol{\omega}} \ . \tag{52}$$

This force causes the spinning of the mass element m_i relative to the center of mass of the rigid body in a direction counter to the direction of rotation of the rigid body relative to the axis of rotation O. Hence, Equation (51) becomes

$$\begin{aligned} \mathbf{r}_{CM} \times (m\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM}) &= I_O\dot{\boldsymbol{\omega}} - I_{CM}\dot{\boldsymbol{\omega}} \ , \\ &= \boldsymbol{\tau}_{\boldsymbol{R}} \ + \ \boldsymbol{\tau}_{Euler} \ . \end{aligned} \tag{53}$$

Equation (53) is identically Equation (23) which derived earlier in section (2.3), which implies that $\boldsymbol{\tau}_{Euler} = \boldsymbol{\tau}_{\boldsymbol{S}}$, that is, the counter rotation of the rigid body relative to its center of mass occurs due to the Euler inertial force[2] .

Retrieving the angular momentum formula:

If the rigid body "A" is *circularly* moving, then by cross multiplying Equation (29) with $\mathbf{r}_{CM}$ and then multiply by m, using the fact that $m = \sum_i m_i$, then we will obtain

$$\mathbf{r}_{CM} \times (m\boldsymbol{\omega} \times \mathbf{r}_{CM}) = \mathbf{r}_{CM} \times \left(\boldsymbol{\omega} \times \sum_i m_i\mathbf{r}_i\right) - \mathbf{r}_{CM} \times \left(\boldsymbol{\omega} \times \sum_i m_i\boldsymbol{\rho}_i\right) \ ,$$

using Equation (5), that gives

$$\begin{aligned} \mathbf{r}_{CM} \times (m\boldsymbol{\omega} \times \mathbf{r}_{CM}) &= \sum_i (\mathbf{r}_i - \boldsymbol{\rho}_i) \times (m_i\boldsymbol{\omega} \times \boldsymbol{r}_i) \ , \\ &= \sum_i \mathbf{r}_i \times (m_i\boldsymbol{\omega} \times \boldsymbol{r}_i) - \sum_i \boldsymbol{\rho}_i \times (m_i\boldsymbol{\omega} \times (\mathbf{r}_{CM} + \boldsymbol{\rho}_i)) \ , \\ &= \sum_i \mathbf{r}_i \times (m_i\boldsymbol{\omega} \times \boldsymbol{r}_i) - \sum_i m_i\boldsymbol{\rho}_i \times (\boldsymbol{\omega} \times \mathbf{r}_{CM}) \\ &\quad - \sum_i \boldsymbol{\rho}_i \times (m_i\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \ , \end{aligned}$$

which reduces by Equation (5) to

$$\mathbf{r}_{CM} \times (m\boldsymbol{\omega} \times \mathbf{r}_{CM}) = \sum_i \mathbf{r}_i \times (m_i\boldsymbol{\omega} \times \boldsymbol{r}_i) - \sum_i \boldsymbol{\rho}_i \times (m_i\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \ . \tag{54}$$

Equation (54), identically, is Equation (6), the angular momentums coupling formula which derived earlier in section (2.1).

2.4.2 The contribution of the Coriolis force to the spin torque

If the rigid body "A" is radial translate while rotating, then by cross multiplying Equation (47) with $\mathbf{r}_{CM}$, we will obtain the torque due to the Coriolis

[2] In this case, Euler force acts like a relative field of force.

force, that is

$$\begin{aligned}\mathbf{r}_{CM} \times (2\boldsymbol{\omega} \times m\boldsymbol{v}_{CM}) &= \mathbf{r}_{CM} \times \left(2\boldsymbol{\omega} \times \sum_i m_i \boldsymbol{v}_i\right) \ , \\ &= \sum_i (\mathbf{r}_i - \boldsymbol{\rho}_i) \times (2\boldsymbol{\omega} \times m_i \boldsymbol{v}_i) \ , \\ &= \sum_i \mathbf{r}_i \times (2\boldsymbol{\omega} \times m_i \boldsymbol{v}_i) - \sum_i \boldsymbol{\rho}_i \times (2\boldsymbol{\omega} \times m_i \boldsymbol{v}_i) \ . \end{aligned} \tag{55}$$

Thus, the Coriolis inertial force contributes to the spinning of the rigid body with the inertial torque

$$\boldsymbol{\tau}_{Cor} = -\sum_i \boldsymbol{\rho}_i \times (2\boldsymbol{\omega} \times m_i \boldsymbol{v}_i) \tag{56}$$

where $\boldsymbol{\tau}_{Cor}$ is an inertial torque which occurs due to the Coriolis force, $\mathbf{F}_{Cor,i} = -2\boldsymbol{\omega} \times m_i \boldsymbol{v}_i$, which acts on the mass element m_i. This force, also, causes the spinning of the mass element m_i relative to the center of mass of the rigid body in a direction counter to the direction of rotation of the rigid body relative to the axis of rotation O.

2.4.3 The contribution of the centripetal force to the spin torque

If the rigid body "A" is rotating, then by cross multiplying Equation (48) with $\mathbf{r}_{CM}$, we will obtain the torque due to the centripetal force, that is

$$\begin{aligned}\mathbf{r}_{CM} \times [\boldsymbol{\omega} \times (m\boldsymbol{\omega} \times \mathbf{r}_{CM})] &= \mathbf{r}_{CM} \times \left[\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \sum_i m_i \mathbf{r}_i\right)\right] \ , \\ &= \sum_i (\mathbf{r}_i - \boldsymbol{\rho}_i) \times [\boldsymbol{\omega} \times (m_i \boldsymbol{\omega} \times \mathbf{r}_i)] \ , \\ &= \sum_i \mathbf{r}_i \times [\boldsymbol{\omega} \times (m_i \boldsymbol{\omega} \times \mathbf{r}_i)] \\ &\qquad - \sum_i \boldsymbol{\rho}_i \times [\boldsymbol{\omega} \times (m_i \boldsymbol{\omega} \times \mathbf{r}_i)] \ , \end{aligned} \tag{57}$$

using the identity (7), remembering that the motion is planar. Therefore, one finds

$$\mathbf{r}_{CM} \times [-m\omega^2 \mathbf{r}_{CM}] = \sum_i \mathbf{r}_i \times [-m_i \omega^2 \mathbf{r}_i] - \sum_i \boldsymbol{\rho}_i \times [-m_i \omega^2 \mathbf{r}_i] \ .$$

Since the cross product of a vector with itself is zero, so we have

$$\mathbf{r}_{CM} \times [-m\omega^2 \mathbf{r}_{CM}] = 0 \ , \text{and}$$

$$\sum_i \mathbf{r}_i \times [-m_i \omega^2 \mathbf{r}_i] = 0 \ .$$

Which implies that

$$-\sum_i \boldsymbol{\rho}_i \times [-m_i\omega^2 \mathbf{r}_i] = 0 \ .$$

Thus, the centripetal force does *not* contribute to the spinning of the rigid body.

2.4.4 The contribution of the rectilinear force to the spin torque

If the rigid body "*A*" is *rectilinearly* accelerating, then by cross multiplying Equation (49) with $\mathbf{r}_{CM}$, we will obtain the torque due to the rectilinear force, that is

$$\begin{aligned} \mathbf{r}_{CM} \times m\boldsymbol{a}_{CM} &= \mathbf{r}_{CM} \times \sum_i m_i \boldsymbol{a}_i \ , \\ &= \sum_i (\mathbf{r}_i - \boldsymbol{\rho}_i) \times m_i \boldsymbol{a}_i \ , \\ &= \sum_i \mathbf{r}_i \times m_i \boldsymbol{a}_i - \sum_i \boldsymbol{\rho}_i \times m_i \boldsymbol{a}_i \ . \end{aligned} \tag{58}$$

Using the facts that $\mathbf{r}_{CM}$ and $\boldsymbol{a}_{CM}$ are parallel to each other and so are $\mathbf{r}_i$ and $\boldsymbol{a}_i$. Therefore, one finds

$$\mathbf{r}_{CM} \times m\boldsymbol{a}_{CM} = 0 \ \text{, and}$$
$$\sum_i \mathbf{r}_i \times m_i \boldsymbol{a}_i = 0 \ .$$

Which implies that

$$-\sum_i \boldsymbol{\rho}_i \times m_i \boldsymbol{a}_i = 0 \ .$$

Hence, the rectilinear force does *not* contribute to the spinning of the rigid body. Now, we can write the equation of motion which describes all the forces that causing rotation coupled with spinning as follows:

$$\begin{aligned} &\overbrace{\mathbf{r}_{CM} \times (m\dot{\boldsymbol{\omega}} \times \mathbf{r}_{CM}) + \mathbf{r}_{CM} \times (2\boldsymbol{\omega} \times m\boldsymbol{v}_{CM})}^{\boldsymbol{\tau}} \\ &= \overbrace{\sum_i \mathbf{r}_i \times (m_i\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) + \sum_i \mathbf{r}_i \times (2\boldsymbol{\omega} \times m_i \boldsymbol{v}_i)}^{\boldsymbol{\tau_R}} \\ &\quad - \overbrace{\left[\sum_i \boldsymbol{\rho}_i \times (m_i\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_i) + \sum_i \boldsymbol{\rho}_i \times (2\boldsymbol{\omega} \times m_i \boldsymbol{v}_i)\right]}^{\boldsymbol{\tau_S}} \ . \end{aligned} \tag{59}$$

where $\boldsymbol{\tau}$ is the total torque, $\boldsymbol{\tau_R}$ is the rotation torque or the *active* torque, and $\boldsymbol{\tau_S}$ is the spin torque or the *inertial* torque.

3 Conclusion

A rigid body that angularly moves in a curvilinear path, will spin under the influence of inertial force, exclusively, the Euler and Coriolis forces.

In opposite to the translational motion where inertia presents as resistance of the mass to motion, the mass when moves curvilinearly it spins under the influence of its inertia .

The inertial force supplies a curvilinearly moving rigid body with an additional kinetic energy and angular momentum and these are independent from the kinetic energy and angular momentum which have been supplied by the active force.

The orbital angular momentum of a rigid body which undergoes circular acceleration is mutually exchangeable with its spin angular momentum and that happens in order to conserve the total angular momentum of the rigid body.

The angular motion of the center of mass of a rigid body relative to a fixed point is equivalent to the collective angular motion of its mass elements relative to the that fixed point and relative to the center of mass. This is the spin-rotation coupling theorem which has been summarized from the preceding analysis.

The parallel axis theorem coupling of the orbital dynamics of a circularly accelerated rigid body to its spin dynamics. It reduces a moment of inertia of a rigid body to a moment of inertia of a point mass.

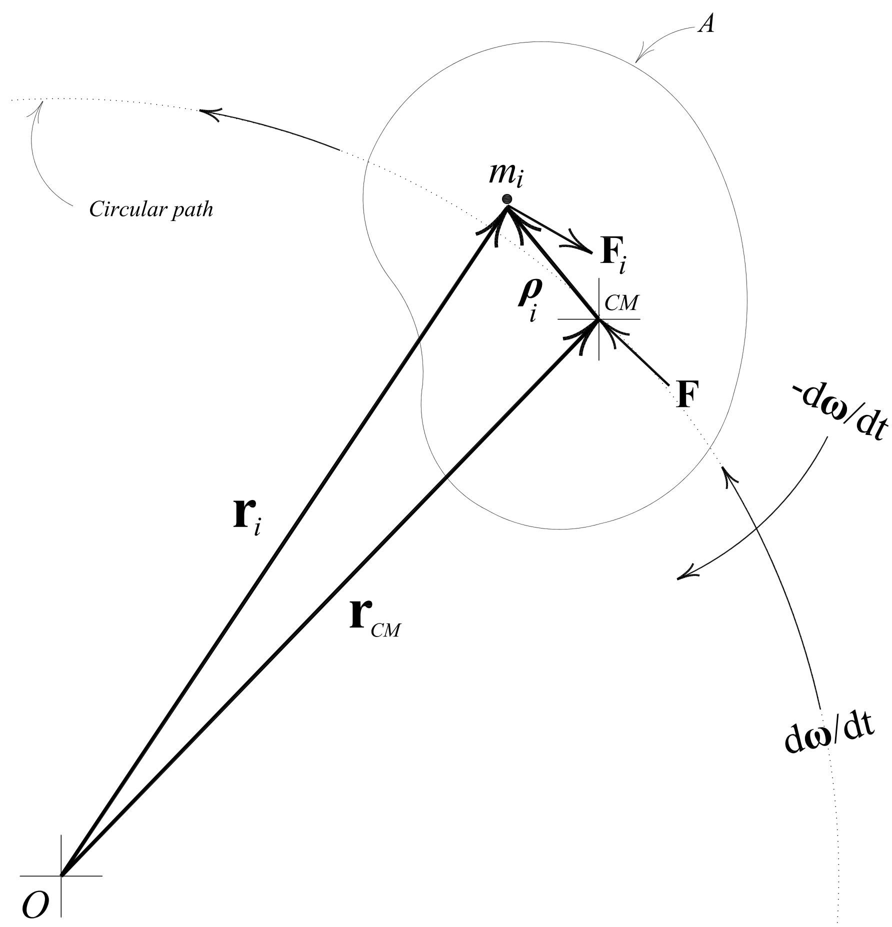

Figure 1. Rotation of the rigid body "A" relative to the fixed point O.

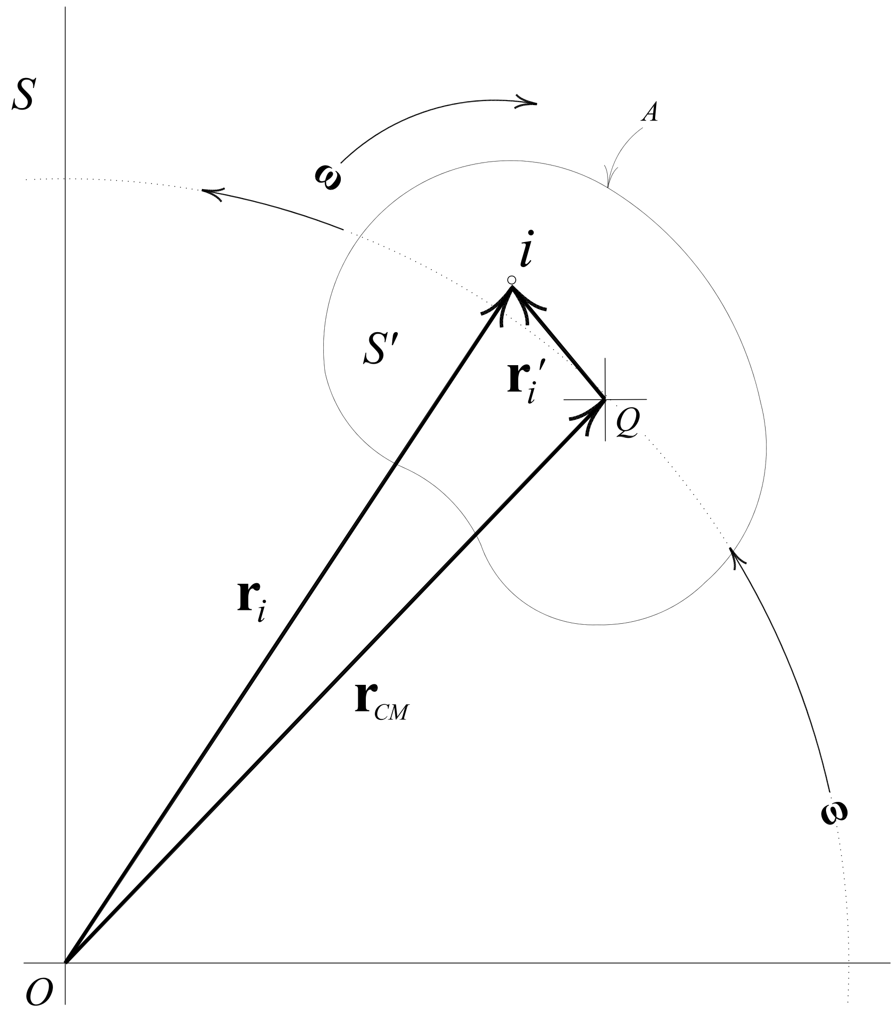

Figure 2. Body-fixed S' and space-fixed coordinate systems S.

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