Spin-rotation coupling of the angularly accelerated rigid body

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Abstract

This paper is prepared to show that a rigid body which undergoes an angular acceleration relative to a fixed point must simultaneously accelerate angularly relative to its center of mass. Formulae which coupling of the angular momentum and kinetic energy due to this spin motion to the angular momentum and kinetic energy due to the rotational motion of the same spinning rigid body have been derived. The paper also bringing to light the nature of the force which causes this spin motion and the formula which coupling of this highlighted force to the force which causes the rotation of the rigid body has been also derived.

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I. Introduction

One can define the problem by the following statement: "division of the total angular momentum into its orbital and spin parts is especially useful because it is often true (at least to a good approximation) that the two parts are *separately* conserved." [1] —John Taylor. The statement briefs the common understanding within scientific community about spin-rotation relation for a rigid body in circular motion. Hence, here we are going to prove that the negation of this statement is what is true.

We beginning with the distinction between rotational and spinning motion therefore we define *rotation* with the motion of the rigid body circularly around a fixed point or a center of rotation which is different from its center of mass while the *spinning* is the rotation of the rigid body around its center of mass. Another thing is that the analysis is going to be with planar rigid body in planar motion over Euclidean space.

II. Analysis

II.1. Coupling of the spinning and rotational angular momentum

Referring to figure(1), a rigid body "A" of a mass m is free to rotate relative to its center of mass CM as it is also can be, simultaneously, free to rotate relative to the point O. Thus, it is pivoted at these two points. It is known that the total angular momentum $\boldsymbol{L}$ of the rigid body "A" in its circular motion is given by[2]:

$$\boldsymbol{L} = \boldsymbol{r}_o \times \mathbf{p} \ + \ \sum_i \boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i m_i \tag{1}$$

where $\boldsymbol{r}_o$ is the vector position of the center of mass of the rigid body relative to the axis of rotation O, $\mathbf{p}$ is the linear momentum and $\boldsymbol{\rho}_i$ is the vector position of the element mass m_i relative to the center of mass of the rigid body. The first term is the angular momentum (relative to O) of the motion of the center of mass. The second is the angular momentum of the motion relative to the center of mass. Thus, we can re-express equation(1) to say[2]

$$\boldsymbol{L} = \boldsymbol{L}_{\text{motion of CM}} + \boldsymbol{L}_{\text{motion relative to CM}} \tag{2}$$

Since the mass is constrained to a circle, the tangential velocity of the mass of the rigid body "A" is $\omega\hat{k} \times \boldsymbol{r}_O$ —where $\omega\hat{k}$ is the rotational angular velocity of the rigid body relative to the axis of rotation O, $\hat{k}$ is a unit vector perpendicular to the plane of the motion— and since $\mathbf{p} = m\omega\hat{k} \times \boldsymbol{r}_O$, the total angular momentum equation (equation(1)) becomes (assuming the motion is planar hence both axises of rotation, O and CM, are parallel):

$$\boldsymbol{L} = \boldsymbol{r}_O \times (m\omega\hat{k} \times \boldsymbol{r}_O) \ + \ \sum_i \boldsymbol{\rho}_i \times (\Omega\hat{k} \times \boldsymbol{\rho}_i) m_i \tag{3}$$

where $\Omega\hat{k}$ is the rigid body spinning angular velocity (arbitrary) relative to its center of mass. Taking the first term in the RHS and using the fact that $\boldsymbol{r}_O = \boldsymbol{r}_i - \boldsymbol{\rho}_i$, one finds

$$\begin{aligned}
\boldsymbol{r}_O \times (m\omega\hat{k} \times \boldsymbol{r}_O) &= (\boldsymbol{r}_i - \boldsymbol{\rho}_i) \times (m\omega\hat{k} \times (\boldsymbol{r}_i - \boldsymbol{\rho}_i)) \ , \\
&= (\boldsymbol{r}_i - \boldsymbol{\rho}_i) \times (m\omega\hat{k} \times \boldsymbol{r}_i - m\omega\hat{k} \times \boldsymbol{\rho}_i) \ , \\
&= (\boldsymbol{r}_i - \boldsymbol{\rho}_i) \times (m\omega\hat{k} \times \boldsymbol{r}_i) - (\boldsymbol{r}_i - \boldsymbol{\rho}_i) \times (m\omega\hat{k} \times \boldsymbol{\rho}_i) \ , \\
&= \boldsymbol{r}_i \times (m\omega\hat{k} \times \boldsymbol{r}_i) - \boldsymbol{\rho}_i \times (m\omega\hat{k} \times \boldsymbol{r}_i) - \boldsymbol{r}_i \times (m\omega\hat{k} \times \boldsymbol{\rho}_i) \\
&\quad + \boldsymbol{\rho}_i \times (m\omega\hat{k} \times \boldsymbol{\rho}_i) \ ,
\end{aligned}$$

since $m = \sum_i m_i$, therefore

$$\begin{aligned}
\boldsymbol{r}_O \times (m\omega\hat{k} \times \boldsymbol{r}_O) = &\sum_i \boldsymbol{r}_i \times (m_i\omega\hat{k} \times \boldsymbol{r}_i) - \sum_i \boldsymbol{\rho}_i \times (m_i\omega\hat{k} \times \boldsymbol{r}_i) \\
&- \sum_i \boldsymbol{r}_i \times (m_i\omega\hat{k} \times \boldsymbol{\rho}_i) + \sum_i \boldsymbol{\rho}_i \times (m_i\omega\hat{k} \times \boldsymbol{\rho}_i) \ ,
\end{aligned}$$

farther mathematical simplification can be done by doing the cross product using the Cartesian coordinate system and converting the summation to integration over the whole body[3][4]. Therefore, one will find

$$\begin{aligned}
\boldsymbol{r}_O \times (m\omega\hat{k} \times \boldsymbol{r}_O) = I_O\omega\hat{k} + I_{CM}\omega\hat{k} - &\sum_i \boldsymbol{\rho}_i \times (m_i\omega\hat{k} \times \boldsymbol{r}_i) \\
- &\sum_i \boldsymbol{r}_i \times (m_i\omega\hat{k} \times \boldsymbol{\rho}_i) \ ,
\end{aligned} \tag{4}$$

where I_O is the moment of inertia relative to the axis of rotation O which is a perpendicular distance $\boldsymbol{r}_O$ from the centre of mass and I_{CM} is the moment of inertia of the rigid body relative to its center of mass. Continue the mathematical simplification using the identity $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \circ \boldsymbol{C})\,\boldsymbol{B} - (\boldsymbol{A} \circ \boldsymbol{B})\,\boldsymbol{C}$, we finds

$$\begin{aligned}
\boldsymbol{r}_O \times (m\omega\hat{k} \times \boldsymbol{r}_O) = I_O\omega\hat{k} + I_{CM}\omega\hat{k} - &\sum_i \left[(\boldsymbol{\rho_i} \circ \boldsymbol{r_i})\, m_i\omega\hat{k} - \left(\boldsymbol{\rho_i} \circ m_i\omega\hat{k}\right) \boldsymbol{r_i} \right] \\
- &\sum_i \left[(\boldsymbol{r_i} \circ \boldsymbol{\rho_i})\, m_i\omega\hat{k} - \left(\boldsymbol{r_i} \circ m_i\omega\hat{k}\right) \boldsymbol{\rho_i} \right] \ ,
\end{aligned}$$

since $\boldsymbol{\rho}_i$ and $\hat{k}$ are mutually orthogonal and so are $\boldsymbol{r}_i$ and $\hat{k}$ then we have

$$\begin{aligned}
\boldsymbol{r}_O \times (m\omega\hat{k} \times \boldsymbol{r}_O) &= I_O\omega\hat{k} + I_{CM}\omega\hat{k} - \sum_i (\boldsymbol{\rho}_i \circ \boldsymbol{r}_i)\, m_i\omega\hat{k} - \sum_i (\boldsymbol{r_i} \circ \boldsymbol{\rho}_i)\, m_i\omega\hat{k}\ , \\
&= I_O\omega\hat{k} + I_{CM}\omega\hat{k} - 2\sum_i (\boldsymbol{r_i} \circ \boldsymbol{\rho}_i)\, m_i\omega\hat{k}\ , \\
&= I_O\omega\hat{k} + I_{CM}\omega\hat{k} - 2\sum_i ((\boldsymbol{r}_O + \boldsymbol{\rho}_i) \circ \boldsymbol{\rho}_i)\, m_i\omega\hat{k}\ , \\
&= I_O\omega\hat{k} + I_{CM}\omega\hat{k} - 2\sum_i (\boldsymbol{r}_O \circ \boldsymbol{\rho}_i)\, m_i\omega\hat{k} - 2\sum_i (\boldsymbol{\rho}_i \circ \boldsymbol{\rho}_i)\, m_i\omega\hat{k}\ , \\
&= I_O\omega\hat{k} + I_{CM}\omega\hat{k} - 2\sum_i m_i\rho_i\; r_o \cos\gamma\; \omega\hat{k} - 2\sum_i m_i\rho_i^2\; \omega\hat{k}\ ,
\end{aligned}$$

where γ is the angle between $\boldsymbol{r}_O$ and $\boldsymbol{\rho}_i$, $\sum_i m_i\rho_i = 0$ (property of the center of mass), and $\sum_i m_i\rho_i^2 = I_{CM}$. Hence we have

$$\boldsymbol{r}_O \times (m\omega\hat{k} \times \boldsymbol{r}_O) = I_O\omega\hat{k} - I_{CM}\omega\hat{k}\ . \tag{5}$$

Thus the total angular momentum (equation(3)) becomes

$$\begin{aligned}
\boldsymbol{L} &= I_O\omega\hat{k} - I_{CM}\omega\hat{k} + \sum_i \boldsymbol{\rho}_i \times (\Omega\hat{k} \times \boldsymbol{\rho}_i) m_i\ , \\
&= I_O\omega\hat{k} - I_{CM}\omega\hat{k} + I_{CM}\Omega\hat{k}\ .
\end{aligned} \tag{6}$$

where $-I_{CM}\omega\hat{k}$ is an additional angular momentum term relative to the center of mass of the rigid body (spinning) and occurs due to the rigid body rotational or circular motion relative to the axis of rotation O.

The term $I_{CM}\Omega\hat{k}$ can be consider as the *initial* spinning angular momentum that the rigid body acquired before it start its circular motion and since $\Omega\hat{k}$ is arbitrary, so that it can be zero and therefore have not to be a mandatory term of equation(6). Hence, we can rewrite equation(6) as:

$$\begin{aligned}
\boldsymbol{L} &= I_O\omega\hat{k} - I_{CM}\omega\hat{k}\ , \\
&= \boldsymbol{L_r} + \boldsymbol{L_s}\ .
\end{aligned} \tag{7}$$

where $\boldsymbol{L} = \boldsymbol{r}_O \times (m\omega\hat{k} \times \boldsymbol{r}_O)$ is the *total* angular momentum, $\boldsymbol{L_r} = I_O\omega\hat{k}$ is the *rotational* angular momentum and $\boldsymbol{L_s} = -I_{CM}\omega\hat{k}$ is the *spinning* angular momentum.

Since the total angular momentum (equation(7)) is conserved then that implies the rotational and spinning angular momentum are mutually exchangeable in order to conserve it, that is

$$\boldsymbol{L} = \downarrow\uparrow \boldsymbol{L_r} + \uparrow\downarrow \boldsymbol{L_s} \tag{8}$$

This property (equation(8)) negate the above statement which we have begin with.

Another thing we can notice is that if we simplify the term $\boldsymbol{r}_O \times (m\omega\hat{k} \times \boldsymbol{r}_O)$ in equation(5) using the identity $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \circ \boldsymbol{C})\,\boldsymbol{B} - (\boldsymbol{A} \circ \boldsymbol{B})\,\boldsymbol{C}$ and rearrange it, that yields

$$I_O\omega\hat{k} = \Big[(\boldsymbol{r}_O \circ \boldsymbol{r}_O)\, m\omega\hat{k} - \Big(\boldsymbol{r}_O \circ m\omega\hat{k}\Big)\, \boldsymbol{r}_O\Big] + I_{CM}\omega\hat{k}\ ,$$

since motion is planar then $\boldsymbol{r}_o$ and $\hat{k}$ are mutually orthogonal, so that

$$I_O\omega\hat{k} = mr_o^2\,\omega\hat{k} + I_{CM}\omega\hat{k} \ ,$$

dividing by ω and dotted with $\hat{k}$, we have

$$I_o = mr_o^2 + I_{CM} \ . \tag{9}$$

which is nothing other than the *parallel axis theorem*. Thus the theorem display the coupling of spin and rotation moment of inertia which leads to the coupling of spinning and rotational motion of a rigid body when moves circularly.

II.2. Coupling of the effected forces

At this section we will explore the coupling between the force causes the rotation of the rigid body "A" and the force causes its spin. Referring to figure(1) if an external force $\mathbf{F}$ acts on the center of mass of the rigid body "A" such that it causes it to angularly accelerate with angular acceleration $\alpha\hat{k} = (d\omega/dt)\hat{k}$ relative to the axis of rotation O —where ω is the angular velocity of rotation of the center of mass of the rigid body relative to the axis of rotation O— and since the mass m is constrained to a circle, the tangential acceleration of the mass of the rigid body "A" is $\alpha\hat{k} \times \boldsymbol{r}_o$ and since $\mathbf{F} = m\boldsymbol{a}$, the total torque equation is given by:

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{r}_O \times \mathbf{F} \ , \\ &= \boldsymbol{r}_O \times (m\alpha\hat{k} \times \boldsymbol{r}_O) \ . \end{aligned} \tag{10}$$

where $\boldsymbol{\tau}$ is the total torque and $\boldsymbol{r}_o$ is the vector position of the center of mass of the rigid body relative to the axis of rotation O. From figure(1) we have $\boldsymbol{r}_o = \boldsymbol{r}_i - \boldsymbol{\rho}_i$. Hence one can write

$$\begin{aligned} \boldsymbol{\tau} &= (\boldsymbol{r}_i - \boldsymbol{\rho}_i) \times (m\alpha\hat{k} \times (\boldsymbol{r}_i - \boldsymbol{\rho}_i)) \ , \\ &= \boldsymbol{r}_i \times (m\alpha\hat{k} \times (\boldsymbol{r}_i - \boldsymbol{\rho}_i)) - \boldsymbol{\rho}_i \times (m\alpha\hat{k} \times (\boldsymbol{r}_i - \boldsymbol{\rho}_i)) \ , \\ &= \boldsymbol{r}_i \times (m\alpha\hat{k} \times \boldsymbol{r}_i) \ - \ \boldsymbol{r}_i \times (m\alpha\hat{k} \times \boldsymbol{\rho}_i) \\ &\quad - \ \boldsymbol{\rho}_i \times (m\alpha\hat{k} \times \boldsymbol{r}_i) \ + \ \boldsymbol{\rho}_i \times (m\alpha\hat{k} \times \boldsymbol{\rho}_i) \ , \end{aligned}$$

since $m = \sum\limits_i m_i$, therefore we have

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{r}_i \times \left(\sum_i m_i\alpha\hat{k} \times \boldsymbol{r}_i\right) \ - \ \boldsymbol{r}_i \times \left(\sum_i m_i\alpha\hat{k} \times \boldsymbol{\rho}_i\right) \\ &\quad - \ \boldsymbol{\rho}_i \times \left(\sum_i m_i\alpha\hat{k} \times \boldsymbol{r}_i\right) \ + \ \boldsymbol{\rho}_i \times \left(\sum_i m_i\alpha\hat{k} \times \boldsymbol{\rho}_i\right) \ , \\ &= \boldsymbol{r}_i \times \left(\sum_i m_i\alpha\hat{k} \times \boldsymbol{r}_i\right) \ - \ (\boldsymbol{r}_O + \boldsymbol{\rho}_i) \times \left(\sum_i m_i\alpha\hat{k} \times \boldsymbol{\rho}_i\right) \\ &\quad - \ \boldsymbol{\rho}_i \times \left(\sum_i m_i\alpha\hat{k} \times (\boldsymbol{r}_O + \boldsymbol{\rho}_i)\right) \ + \ \boldsymbol{\rho}_i \times \left(\sum_i m_i\alpha\hat{k} \times \boldsymbol{\rho}_i\right) \ , \end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\tau} = \sum_i \boldsymbol{r}_i \times (m_i \alpha \hat{k} \times \boldsymbol{r}_i) \; - \; \boldsymbol{r}_O \times \left(\alpha \hat{k} \times \sum_i m_i \boldsymbol{\rho}_i \right) \\
- \; \sum_i \boldsymbol{\rho}_i \times (m_i \alpha \hat{k} \times \boldsymbol{\rho}_i) \; - \; \sum_i m_i \boldsymbol{\rho}_i \times (\alpha \hat{k} \times \boldsymbol{r}_O) \\
- \; \sum_i \boldsymbol{\rho}_i \times (m_i \alpha \hat{k} \times \boldsymbol{\rho}_i) \; + \; \sum_i \boldsymbol{\rho}_i \times (m_i \alpha \hat{k} \times \boldsymbol{\rho}_i) \; ,
\end{aligned}
$$

from the properties of the center of mass we have $\sum_i m_i \boldsymbol{\rho}_i = 0$. Therefore, one finds

$$
\boldsymbol{r}_O \times (m \alpha \hat{k} \times \boldsymbol{r}_O) = \sum_i \boldsymbol{r}_i \times (m_i \alpha \hat{k} \times \boldsymbol{r}_i) \; - \; \sum_i \boldsymbol{\rho}_i \times (m_i \alpha \hat{k} \times \boldsymbol{\rho}_i) \; . \tag{11}
$$

Equation(11) can be mathematically simplified by doing the cross product using the Cartesian coordinate system and converting the summation to integration over the whole body. Therefore one will find

$$
\boldsymbol{r}_O \times (m \alpha \hat{k} \times \boldsymbol{r}_O) = I_O \alpha \hat{k} \; - \; I_{CM} \alpha \hat{k} \tag{12}
$$

where I_O is the moment of inertia relative to the axis of rotation O which is a perpendicular distance $\boldsymbol{r}_O$ from the centre of mass and I_{CM} is the moment of inertia of the rigid body relative to its center of mass. We can write it as:

$$
\boldsymbol{\tau} = \boldsymbol{\tau_r} + \boldsymbol{\tau_s} \tag{13}
$$

where $\boldsymbol{\tau} = \boldsymbol{r}_O \times (m \alpha \hat{k} \times \boldsymbol{r}_O)$ is the *total* torque. $\boldsymbol{\tau_r} = I_O \alpha \hat{k}$ is the *rotation* torque and $\boldsymbol{\tau_s} = -I_{CM} \alpha \hat{k}$ is the *spin* torque. Thus the total torque $\boldsymbol{\tau}$ is a synthesis of two torques; the rotation torque $\boldsymbol{\tau_r}$ and the spin torque $\boldsymbol{\tau_s}$.

There is another thing that has to be notice in equation(12) when we rearrange it as following:

$$
I_O \alpha \hat{k} = \boldsymbol{r}_O \times (m \alpha \hat{k} \times \boldsymbol{r}_O) + I_{CM} \alpha \hat{k} \; ,
$$

and using the identity $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \circ \boldsymbol{C})\, \boldsymbol{B} - (\boldsymbol{A} \circ \boldsymbol{B})\, \boldsymbol{C}$ to obtain

$$
I_O \alpha \hat{k} = \left[(\boldsymbol{r}_O \circ \boldsymbol{r}_O)\, m \alpha \hat{k} - \left(\boldsymbol{r}_O \circ m \alpha \hat{k} \right) \boldsymbol{r}_O \right] + I_{CM} \alpha \hat{k} \; ,
$$

since motion is planar then $\boldsymbol{r}_O$ and $\hat{k}$ are mutually orthogonal, so that

$$
I_O \alpha \hat{k} = m r_O^2 \, \alpha \hat{k} + I_{CM} \alpha \hat{k} \; ,
$$

dividing by α and dotted with $\hat{k}$, we get

$$
I_O = m r_O^2 + I_{CM} \; . \tag{14}
$$

Again we obtained the *parallel axis theorem*.

2.2.1. How one can understand the negative sign of the spin torque $\boldsymbol{\tau_s}$

Referring to equations(12) and (13) we find $\boldsymbol{\tau_s} = -I_{CM}\alpha\hat{k}$ and the crucial question is; Does the negative sign assign to the magnitude α or to the direction $\hat{k}$? or in another words; Does the spin of the rigid body is de-acceleration in the same direction of its rotation or acceleration in opposite direction of its rotation?. To answer this question we are going to take another approach to find the same term that assigned to the spin $\boldsymbol{\tau_s}$.

Referring to figure(1) and since the rigid body "A" is angularly accelerates relative to the axis of rotation O therefore an inertial force $\mathbf{F}_i$ will occur at any element mass m_i that composes its total mass m. This inertial force is known as Euler force[5], $\mathbf{F}_{Euler}$, such that:

$$\begin{aligned}\mathbf{F}_i = \mathbf{F}_{Euler} &= -m_i\frac{d\omega}{dt}\hat{k} \times \boldsymbol{r}_i \ , \\ &= -m_i\alpha\hat{k} \times \boldsymbol{r}_i \ . \end{aligned} \tag{15}$$

where $\alpha\hat{k} = (d\omega/dt)\hat{k}$ is the same angular acceleration of rotation of the center of mass of the rigid body relative to the axis of rotation O and $\boldsymbol{r}_i$ is the vector position of the point where the acceleration is measured relative to the axis of the rotation O. We know that the Euler force is an inertial force which implies that it acts in a direction that opposite to the direction of the force applied (the active force). Therefore, the negative sign that occurs in equation(15) is assigned to the direction $\hat{k}$. Thus, we can rewrite it:

$$\mathbf{F}_i = m_i\alpha(-\hat{k}) \times \boldsymbol{r}_i \tag{16}$$

The torque that acts on any element mass m_i due to the Euler force $\mathbf{F}_i$ is given with:

$$\boldsymbol{\tau}_i = \boldsymbol{\rho}_i \times \mathbf{F}_i \tag{17}$$

where $\boldsymbol{\tau}_i$ is the torque of any element mass m_i relative to the center of mass of the rigid body "A" —which occurs due to Euler force $\mathbf{F}_i$— and $\boldsymbol{\rho}_i$ is the vector position of the same element mass m_i relative to the center of mass of the rigid body. Hence, the total *inertial* torque, $\boldsymbol{\tau}_{inr}$, that acts on the rigid body due to the Euler force $\mathbf{F}_{Euler}$ is given by:

$$\begin{aligned}\boldsymbol{\tau}_{inr} &= \sum_i \boldsymbol{\rho}_i \times \mathbf{F}_i \ , \\ &= \sum_i \boldsymbol{\rho}_i \times (m_i\alpha(-\hat{k}) \times \boldsymbol{r}_i) \ , \end{aligned}$$

from figure(1) we have $\boldsymbol{r}_i = \boldsymbol{r}_O + \boldsymbol{\rho}_i$. Hence one can write

$$\begin{aligned}\boldsymbol{\tau}_{inr} &= \sum_i \boldsymbol{\rho}_i \times \Big(m_i\alpha(-\hat{k}) \times \boldsymbol{r}_O + m_i\alpha(-\hat{k}) \times \boldsymbol{\rho}_i\Big) \ , \\ &= \sum_i m_i\boldsymbol{\rho}_i \times (\alpha(-\hat{k}) \times \boldsymbol{r}_O) + \sum_i \boldsymbol{\rho}_i \times (m_i\alpha(-\hat{k}) \times \boldsymbol{\rho}_i) \ , \end{aligned}$$

from the properties of the center of mass we know that $\sum_i m_i \boldsymbol{\rho}_i = 0$. Therefore, we obtain

$$\boldsymbol{\tau}_{inr} = \sum_i \boldsymbol{\rho}_i \times (m_i \alpha(-\hat{k}) \times \boldsymbol{\rho}_i) \ . \tag{18}$$

Equation(18) can be mathematically simplified by doing the cross product using the Cartesian coordinate system and converting the summation to integration over the whole body. Therefore, one will find

$$\boldsymbol{\tau}_{inr} = I_{CM} \alpha(-\hat{k}) \ . \tag{19}$$

where $\alpha(-\hat{k}) = (d\omega/dt)(-\hat{k})$ is the angular acceleration of the rigid body "A" relative to its center of mass, where it is equal in magnitude to the rigid body angular acceleration relative to the axis of rotation O. Therefore, the inertial torque $\boldsymbol{\tau}_{inr}$ acts over the rigid body "A" —where $\boldsymbol{\tau}_{inr} \neq 0$— such that it causes it to rotate relative to its center of mass[1] in a direction counter to the direction of rotation of its center of mass relative to the axis of rotation O.

Therefore, we retrieve the spin torque term of equation(12), that is; the term that associated with the total inertial torque $\boldsymbol{\tau}_{inr}$ (equation(19)) is identical with the one that associated with the spin torque $\boldsymbol{\tau_s}$ occurs as a second term in equation(12). Hence, one can rewrite equation(12), the total torque $\boldsymbol{\tau}$, and equation(7), the total angular momentum, in a more precise manner:

$$\boldsymbol{r}_O \times (m\alpha\hat{k} \times \boldsymbol{r}_O) = I_O \alpha \hat{k} \ + \ I_{CM} \alpha(-\hat{k}) \ , \tag{20}$$

where $I_{CM}\alpha(-\hat{k}) = \boldsymbol{\tau_s}$ and

$$\boldsymbol{L} = I_O \omega \hat{k} + I_{CM} \omega(-\hat{k}) \ , \tag{21}$$

where $I_{CM}\omega(-\hat{k}) = \boldsymbol{L_s}$. Thus, the negative sign points to the direction of spinning of the rigid body.

II.3. Coupling of the spinning and rotational kinetic energy

It is known that the kinetic energy T of the rigid body "A" in its rotational motion relative to the axis of rotation O is given by[6][7]:

$$T = \tfrac{1}{2} m \left(\omega\hat{k} \times \boldsymbol{r}_O \circ \omega\hat{k} \times \boldsymbol{r}_O \right) + \tfrac{1}{2} I_{CM} \left(\Omega\hat{k} \circ \Omega\hat{k} \right) \ , \tag{22}$$

where $\omega\hat{k} \times \boldsymbol{r}_O$ is the tangential velocity of the center of mass of the rigid body relative to the axis of rotation O and $\Omega\hat{k}$ is an arbitrary spinning velocity relative to the center of mass of the rigid body. When one uses the identity $(\boldsymbol{A} \times \boldsymbol{B} \circ \boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \circ \boldsymbol{C})(\boldsymbol{B} \circ \boldsymbol{D}) - (\boldsymbol{A} \circ \boldsymbol{D})(\boldsymbol{B} \circ \boldsymbol{C})$, he will get

$$T = \tfrac{1}{2} m \left[\left(\omega\hat{k} \circ \omega\hat{k} \right) (\boldsymbol{r}_O \circ \boldsymbol{r}_O) - \left(\omega\hat{k} \circ \boldsymbol{r}_O \right) \left(\boldsymbol{r}_O \circ \omega\hat{k} \right) \right] + \tfrac{1}{2} I_{CM} \left(\Omega\hat{k} \circ \Omega\hat{k} \right) \ ,$$

[1]The observation of this phenomenon can be obtain easily by rotating a metallic solid disk pivoted at its center or by rotating a vessel containing ice cubes floating on water and can be exercise using one's hands.

since $\hat{k}$ and $\boldsymbol{r}_O$ are mutually orthogonal, so that

$$
\begin{aligned}
T &= \tfrac{1}{2}m\left(\omega\hat{k}\circ\omega\hat{k}\right)\left(\boldsymbol{r}_O\circ\boldsymbol{r}_O\right)+\tfrac{1}{2}I_{CM}\left(\Omega\hat{k}\circ\Omega\hat{k}\right)\ , \qquad (23)\\
&= \tfrac{1}{2}m\left(\omega\hat{k}\circ\omega\hat{k}\right)\left(\boldsymbol{r}_i-\boldsymbol{\rho}_i\circ\boldsymbol{r}_i-\boldsymbol{\rho}_i\right)+\tfrac{1}{2}I_{CM}\left(\Omega\hat{k}\circ\Omega\hat{k}\right)\ ,\\
&= \tfrac{1}{2}m\left(\omega\hat{k}\circ\omega\hat{k}\right)\left[\left(\boldsymbol{r}_i\circ\boldsymbol{r}_i\right)-2\left(\boldsymbol{r}_i\circ\boldsymbol{\rho}_i\right)+\left(\boldsymbol{\rho}_i\circ\boldsymbol{\rho}_i\right)\right]+\tfrac{1}{2}I_{CM}\left(\Omega\hat{k}\circ\Omega\hat{k}\right)\ ,\\
&= \tfrac{1}{2}\sum_i m_i\left(\boldsymbol{r}_i\circ\boldsymbol{r}_i\right)\left(\omega\hat{k}\circ\omega\hat{k}\right)-\sum_i m_i\left(\boldsymbol{r}_i\circ\boldsymbol{\rho}_i\right)\left(\omega\hat{k}\circ\omega\hat{k}\right)\\
&\quad+\tfrac{1}{2}\sum_i m_i\left(\boldsymbol{\rho}_i\circ\boldsymbol{\rho}_i\right)\left(\omega\hat{k}\circ\omega\hat{k}\right)+\tfrac{1}{2}I_{CM}\left(\Omega\hat{k}\circ\Omega\hat{k}\right)\ ,\\
&= \tfrac{1}{2}I_O\left(\omega\hat{k}\circ\omega\hat{k}\right)-\sum_i m_i\left(\boldsymbol{r}_i\circ\boldsymbol{\rho}_i\right)\left(\omega\hat{k}\circ\omega\hat{k}\right)+\tfrac{1}{2}I_{CM}\left(\omega\hat{k}\circ\omega\hat{k}\right)\\
&\quad+\tfrac{1}{2}I_{CM}\left(\Omega\hat{k}\circ\Omega\hat{k}\right)\ ,\\
&= \tfrac{1}{2}I_O\left(\omega\hat{k}\circ\omega\hat{k}\right)-\sum_i m_i\left(\boldsymbol{r}_O+\boldsymbol{\rho}_i\circ\boldsymbol{\rho}_i\right)\left(\omega\hat{k}\circ\omega\hat{k}\right)+\tfrac{1}{2}I_{CM}\left(\omega\hat{k}\circ\omega\hat{k}\right)\\
&\quad+\tfrac{1}{2}I_{CM}\left(\Omega\hat{k}\circ\Omega\hat{k}\right)\ ,\\
&= \tfrac{1}{2}I_O\left(\omega\hat{k}\circ\omega\hat{k}\right)-\sum_i m_i\left(\boldsymbol{\rho}_i\circ\boldsymbol{\rho}_i\right)\left(\omega\hat{k}\circ\omega\hat{k}\right)-\sum_i m_i\left(\boldsymbol{r}_O\circ\boldsymbol{\rho}_i\right)\left(\omega\hat{k}\circ\omega\hat{k}\right)\\
&\quad+\tfrac{1}{2}I_{CM}\left(\omega\hat{k}\circ\omega\hat{k}\right)+\tfrac{1}{2}I_{CM}\left(\Omega\hat{k}\circ\Omega\hat{k}\right)\ ,\\
&= \tfrac{1}{2}I_O\left(\omega\hat{k}\circ\omega\hat{k}\right)-\tfrac{1}{2}I_{CM}\left(\omega\hat{k}\circ\omega\hat{k}\right)-\sum_i m_i\rho_i\ r_o\cos\gamma\left(\omega\hat{k}\circ\omega\hat{k}\right)\\
&\quad+\tfrac{1}{2}I_{CM}\left(\Omega\hat{k}\circ\Omega\hat{k}\right)\ ,
\end{aligned}
$$

where γ is the angle between $\boldsymbol{r}_O$ and $\boldsymbol{\rho}_i$. Again $\sum_i m_i\rho_i = 0$. Therefore, we obtain

$$
T = \tfrac{1}{2}I_O\left(\omega\hat{k}\circ\omega\hat{k}\right)-\tfrac{1}{2}I_{CM}\left(\omega\hat{k}\circ\omega\hat{k}\right)+\tfrac{1}{2}I_{CM}\left(\Omega\hat{k}\circ\Omega\hat{k}\right)\ . \qquad (24)
$$

When subtracting equation(23) from (24), we again will obtain the *parallel axis theorem* and if we subtract equation(22) from (24) we obtain

$$
\tfrac{1}{2}m\left(\omega\hat{k}\times\boldsymbol{r}_O\circ\omega\hat{k}\times\boldsymbol{r}_O\right)=\tfrac{1}{2}I_O\left(\omega\hat{k}\circ\omega\hat{k}\right)-\tfrac{1}{2}I_{CM}\left(\omega\hat{k}\circ\omega\hat{k}\right)\ . \qquad (25)
$$

Since there is no negative kinetic energy therefore one can understand that the negative sign is assign to direction. Thus, the correct kinetic energy formula:

$$
T = \tfrac{1}{2}I_O\left(\omega\hat{k}\circ\omega\hat{k}\right)+\tfrac{1}{2}I_{CM}\left(\omega(-\hat{k})\circ\omega(-\hat{k})\right)+\tfrac{1}{2}I_{CM}\left(\Omega\hat{k}\circ\Omega\hat{k}\right)\ . \qquad (26)
$$

The term $\tfrac{1}{2}I_{CM}\left(\Omega\hat{k}\circ\Omega\hat{k}\right)$ can be consider as the *initial* spinning kinetic energy that the rigid body have before it start its circular motion and since $\Omega\hat{k}$ is arbitrary, so that it can be zero and therefore have not to be a mandatory term of equation(26). Hence we can write equation(26) as:

$$\begin{aligned} T &= \tfrac{1}{2} I_O \left(\omega\hat{k} \circ \omega\hat{k}\right) + \tfrac{1}{2} I_{CM} \left(\omega(-\hat{k}) \circ \omega(-\hat{k})\right) \ , \\ &= T_r + T_s \ . \end{aligned} \tag{27}$$

where $T = \frac{1}{2} m \left(\omega\hat{k} \times \boldsymbol{r}_O \circ \omega\hat{k} \times \boldsymbol{r}_O\right)$ is the *total* kinetic energy, $T_r = \frac{1}{2} I_O \left(\omega\hat{k} \circ \omega\hat{k}\right)$ is the *rotational* kinetic energy and $T_s = \frac{1}{2} I_{CM} \left(\omega(-\hat{k}) \circ \omega(-\hat{k})\right)$ is the *spinning* kinetic energy.

III. Conclusion

A rigid body that angularly accelerates in a circular path will spin under the influence of inertial force, specially the Euler force.

In opposite to the translational motion where inertia presents as resistance of the mass to motion, the mass when moves curvilineraly it spins under the influence of its inertia .

In the case of the angular acceleration of a rigid body in a circular path, both the active torque and the inertial (fictitious) torque that are causing the rigid body to rotate and spin —respectively— are real torques and occur due to the action of real forces and these forces are de-synthesis from one real origin, the total force which in return causes a total torque. In another words, the total torque de-synthesis to active torque (causes the rotation of the rigid body) and inertial torque (causes the spinning of the rigid body) and the three (total, active and inertial) are real torques.

The inertial force supplies an angularly accelerating rigid body with an additional kinetic energy and angular momentum and these are independent from the kinetic energy and angular momentum which are be supplied by the active force.

The parallel axis theorem coupling of the orbital motion of an angularly accelerated rigid body to its spin motion.

The orbital angular momentum of a rigid body which undergoes angular acceleration is mutually exchangeable with its spin angular momentum and that happens in order to conserve the total angular momentum of the rigid body.

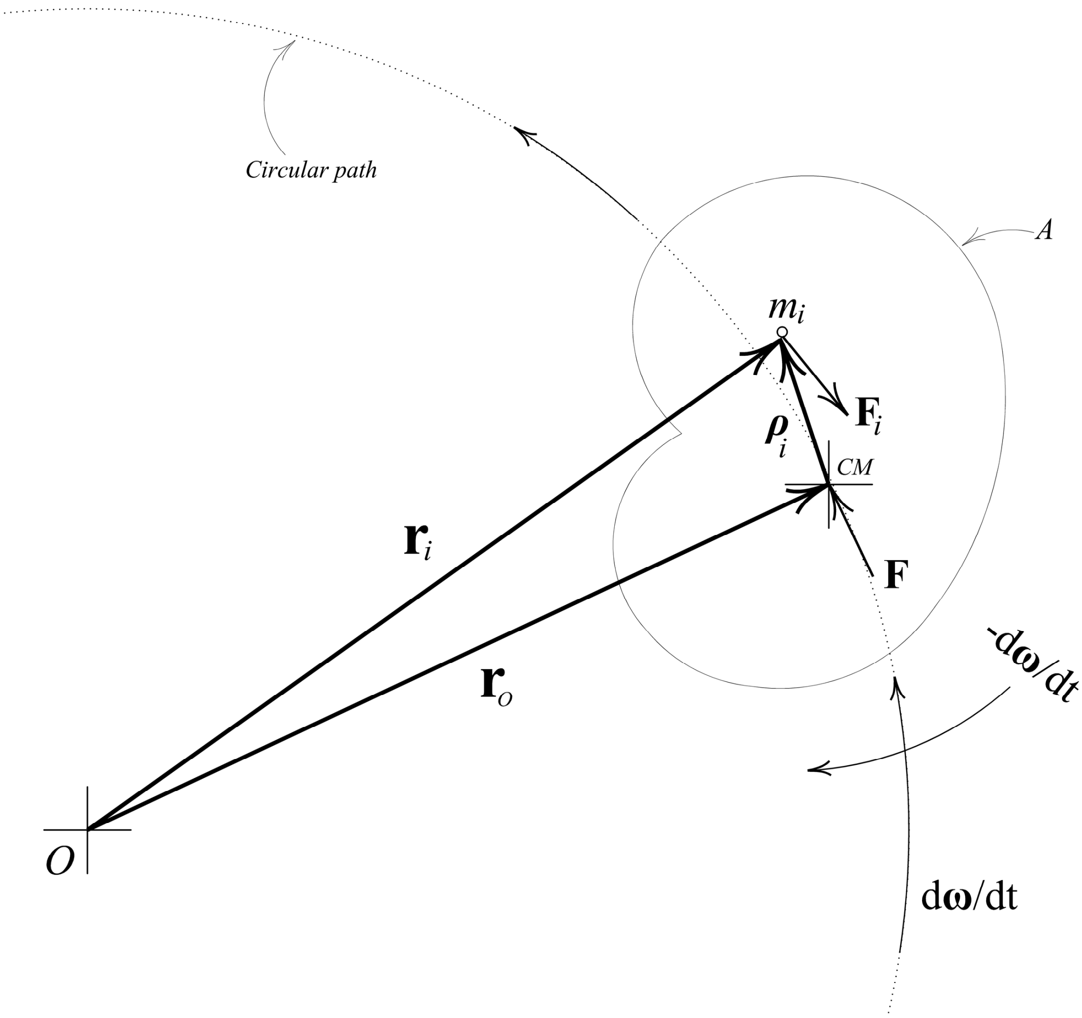

Figure 1: Rotation of the rigid body "A" relative to the fixed point O.

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