Quantization of a three-dimensional damped harmonic oscillator

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Abstract
An explicitly time independent Lagrangian functional of a three-dimensional damped harmonic oscillator has been proposed. I derive results for the motion of the three-dimensional damped harmonic oscillator with a pure imaginary three dimensional vector and oscillator's position-dependent friction coefficient. The Hamiltonians corresponding to the Lagrangian is also explicitly time independent. The choice of functional form of the friction coefficient on the oscillator position determines and plays a vital role in the form of the equation of motion classically and quantum mechanically. One choice of the form of the friction coefficient I made lead to breaking the symmetry of the isotropy of oscillations in the three dimensional space.

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Introduction

The equation of motion of a damped simple harmonic oscillator in one dimension- ($x-$ direction) - is

$$m\ddot{x} + b\dot{x} + kx = 0 \qquad (1)$$

where m is the mass of the particle, b the coefficient of friction of the medium in which the oscillator moves, which assumed to be constant, and finally k is the spring constant.

In three dimensions it may be written

$$m\ddot{\vec{r}} + b\dot{\vec{r}} + k\vec{r} = 0 \qquad (2)$$

Since 1930 there have been many efforts to find an explicit time independent Lagrangian and Hamiltonian functionals of the damped harmonic oscillator.

A Lagrangian that lead to the equation of motion of the DHO equation (1) was found to be

$$L(t,x,\dot{x}) = e^{\frac{b}{m}t}\left(-\frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2\right)$$

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It is explicitly time dependent so do the Hamiltonian.
The nonexistence of these functionals made it difficult to canonically quantize the equation of motion of the damped harmonic oscillator.

The Model

The classical model of the damped harmonic oscillator: the Lagrangian

Here, I proposed an explicit time independent Lagrangian for a three dimensional damped harmonic oscillator.
Consider the following Lagrangian functional

$$L = \frac{1}{2} m\dot{\vec{r}}^2 + i\vec{b}\cdot\dot{\vec{r}} - \frac{1}{2}k\vec{r}^2 \qquad (3)$$

where $i \equiv \sqrt{-1}$ is the imaginary unit and $\vec{b} = \vec{b}(\vec{r}(t))$ is the coefficient of friction of the medium in which the oscillator moves. The coefficient is assumed to be a real three dimensional vector and is a function of the displacement vector of the oscillator from its equilibrium position. $\vec{b} = \vec{b}(\vec{r}(t))$ is measured in $kgms^{-1}$ (unit of momentum).

The above Lagrangian functional in equation (3) is of the type

$$L = T - U$$

Where U is a velocity dependent potential. In the model above, $U = i\vec{b}\cdot\dot{\vec{r}} - \frac{1}{2}k\vec{r}^2$ and $\dot{\vec{r}}$ is the kinetic velocity of the oscillator.

The equation of motion of the damped harmonic oscillator derived from the Lagrangian functional in equation (3), for this model has the following mathematical form

$$m\ddot{\vec{r}} + i(\vec{\nabla}\times\vec{b})\times\dot{\vec{r}} + k\vec{r} = 0 \qquad (4)$$

The canonical conjugate momentum derived from the Lagrangian in equation (3) is given by

$$\vec{p} = m\dot{\vec{r}} + i\vec{b} \qquad (5)$$

The second terms in this expression plays the role of a potential momentum.

The mechanical momentum is given by

$$m\dot{\vec{r}} = \vec{p} - i\vec{b}$$

and the mechanical “velocity” is given by

$$\dot{\vec{r}} = \frac{1}{m}(\vec{p} - i\vec{b}) \qquad (6)$$

The classical model of the damped harmonic oscillator: the Hamiltonian

The Hamiltonian functional is

$$H = \frac{1}{2m}(\vec{p} - i\vec{b})^2 + \frac{1}{2}k\vec{r}^{\,2} \qquad (7)$$

where I have substituted the kinematical velocity in equation (6) and the Lagrangian functional in equation (3) in the general definition of the Hamiltonian function defined as:

$$H = \sum_s p_s \dot{q}_s - L$$

The Hamiltonian functional in equation (7) is that of a simple harmonic oscillator with the canonical momentum being shifted by an amount equals to the coefficient of friction $\vec{b} = \vec{b}(\vec{r}(t))$ of the medium in which the oscillator moves times the imaginary unit $i = \sqrt{-1}$. The friction perturbed the Hamiltonian of the simple harmonic oscillator.

Since $\vec{b}$ is not explicitly dependent on the time t, then L does not depend on t explicitly as can be seen in equation (3) and the Hamiltonian functional also is not explicitly dependent on time t and it is a constant, that is

$$\frac{dH}{dt} = \frac{d}{dt}[\frac{1}{2m}(\vec{p} - i\vec{b})^2 + \frac{1}{2}k\vec{r}^{\,2}] = \frac{1}{m}(\vec{p} - i\vec{b})\cdot(\dot{\vec{p}} - i\dot{\vec{b}}) + k\vec{r}\cdot\dot{\vec{r}} = \frac{1}{m}(m\dot{\vec{r}})\cdot(m\ddot{\vec{r}}) + k\vec{r}\cdot\dot{\vec{r}}$$

$$= (\dot{\vec{r}})\cdot(m\ddot{\vec{r}}) + k\vec{r}\cdot\dot{\vec{r}} = (\dot{\vec{r}})\cdot[(m\ddot{\vec{r}}) + k\vec{r}] = (\dot{\vec{r}})\cdot[-i(\vec{\nabla}\times\vec{b})\times\dot{\vec{r}}] = 0$$

Where the vector identity

$$\vec{A}\cdot[(\vec{\nabla}\times\vec{B})\times\vec{A}] = 0$$

has been used and the term in the square brackets is substituted from the equation of motion: equation (4).

It is easy to see that in this case the total energy of the system is conserved.

$$H = \text{constant} \qquad (8)$$

Discussion

Here, I give an example of explicit position dependence of the coefficient $\vec{b}$ and determine the equation of motion of the DHO for that case.

I choose,

$$\vec{b} = -b_0 y\hat{i} + b_0 x\hat{j} \qquad (9)$$

Then,

$$\vec{\nabla}\times\vec{b} = 2b_0\hat{k}$$

where b_0 is a constant which has unit of kgs^{-1}.

With this choice, the Lagrangian in equation (3) is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + ib_0(-x\dot{y} + \dot{x}y) - \frac{1}{2}k(x^2 + y^2 + z^2)$$

The corresponding Euler-Lagrange's equations after dividing both sides by the mass of the particle are written

$$0=-\omega_0{}^2x-2i\omega_b\dot{y}-\ddot{x}$$
$$0=-\omega_0{}^2y+2i\omega_b\dot{x}-\ddot{y} \qquad (10)$$
$$0=-\omega_0{}^2z-\ddot{z}$$

where I introduced the natural angular frequency ω_0 of the oscillator where $\omega_0=\sqrt{k/m}$, and ω_b is an oscillation frequency (measured in Hz) defined as

$$\omega_b=b_0/m$$

and x,y,z are the components of the displacement of the particle from the equilibrium position.

Equations of motion also can be directly deduced from the equation of motion in it vector form in equation (4) after substituting the condition in equation (9), and then setting each component to zero.

The first and the second equations in equation (10) are a system of the coupled linear differential equations of the second order, and hence we can try a solution of type

$$x=x_0e^{i(\omega t-\varphi)}, \qquad y=y_0e^{i(\omega t-\varphi)} \qquad (11)$$

where x_0,y_0 are the amplitudes of oscillation, ω the frequency in Hz and φ and α are phase factors, respectively.

The general solution of the last equation in equation (10) is

$$z=z_0\cos(\omega_0t-\delta), \qquad (12)$$

that is the oscillation in $z-$ direction takes place with the natural angular frequency ω_0 of the free oscillation, z_0 the amplitude of oscillation and δ is a phase factor.

Inserting the presupposed solutions in equation (11) in the first two equations in equation (10), we get

$$\begin{aligned}0&=(\omega^2+\omega_0{}^2)x_0+2\omega_b\omega y_0\\0&=-(\omega^2+\omega_0{}^2)y_0+2\omega_b\omega x_0\end{aligned} \qquad (13)$$

Then the system (13) changes to the system of the algebraic equations which is written in matrix form as

$$\begin{pmatrix}\omega^2+\omega_0{}^2 & 2\omega_b\omega\\ 2\omega_b\omega & -(\omega^2+\omega_0{}^2)\end{pmatrix}\begin{pmatrix}x_0\\ y_0\end{pmatrix}=0$$

So the secular equation is

$$\begin{vmatrix}\omega^2+\omega_0{}^2 & 2\omega_b\omega\\ 2\omega_b\omega & -(\omega^2+\omega_0{}^2)\end{vmatrix}=-(\omega^2+\omega_0{}^2)^2-4(\omega_b\omega)^2=0.$$

This equation is equivalent to two equations,

$$\omega^2+2i\omega_b\omega+\omega_0{}^2=0;$$
$$\omega^2-2i\omega_b\omega+\omega_0{}^2=0,$$

which has four roots

$$\begin{aligned}\omega_1 &= i[\omega_b + \sqrt{\omega_b{}^2 + \omega_0{}^2}\,],\\ \omega_2 &= i[\omega_b - \sqrt{\omega_b{}^2 + \omega_0{}^2}\,],\\ \omega_3 &= i[-\omega_b + \sqrt{\omega_b{}^2 + \omega_0{}^2}\,],\\ \omega_4 &= i[-\omega_b - \sqrt{\omega_b{}^2 + \omega_0{}^2}\,].\end{aligned} \qquad (14)$$

The only root that is compatible with the decay behavior of the damped harmonic oscillator for the trial solutions in equation (11) is the first one in equation (14)

$$\omega_1 = i[\omega_b + \sqrt{\omega_b{}^2 + \omega_0{}^2}\,] \qquad (15)$$

Hence the damped harmonic oscillator exhibits two Eigen frequencies ω_0 and ω_1. Note that first mode does not depend on the coefficient of friction of the medium at all, and the second mode is caused by the coefficient of friction of the medium.

In the case $\vec{b} = -b_0 y\hat{i} + b_0 x\hat{j}$, the oscillator is showing a damping behavior in the $x-$ and $y-$directions -equation (18) - with the frequency ω_1 -equation (15) - and an oscillatory behavior with the frequency ω_0 -equation (12)- in the $z-$direction.

It can be clearly seen that the choice in equation (9) from the beginning has no $z-$component; this broke the symmetry of the equality of the directions in space and made the oscillator to oscillate simple harmonically in the $z-$direction and decay in the other directions.

For a weak friction,

$$\omega_b << \omega_0,$$

the frequency ω_1 is approximated to the form

$$\omega_1 = i(\omega_b + \omega_0) \qquad (16)$$

while in an opposite case of a strong friction,

$$\omega_b >> \omega_0,$$

we have

$$\omega_1 \approx i[\omega_b + \omega_b(1 + \frac{1}{2}\frac{\omega_0{}^2}{\omega_b{}^2})] = i[2\omega_b + \frac{\omega_0{}^2}{\omega_b}] \qquad (17)$$

Other choices of $\vec{b}$ as explicit function of the position of the oscillator give similar behavior as the example above.
The position vector of the oscillator at any time for the case above may now be written as

$$\vec{r}(t) = x_0 e^{-(\omega_b + \sqrt{\omega_b{}^2 + \omega_0{}^2})t - i\varphi}\hat{i} + y_0 e^{-(\omega_b + \sqrt{\omega_b{}^2 + \omega_0{}^2})t - i\varphi}\hat{j} + z_0\cos(\omega_0 t - \delta)\hat{k} \qquad (18)$$

This is symmetric about the $z-$ direction.

The Quantum Hamiltonian of the Damped harmonic oscillator (DHO)

The Quantum Hamiltonian of DHO system as given by equation (7) is

$$\hat{H} = \frac{1}{2m}(\hat{\vec{p}} - i\vec{b})^2 + \frac{1}{2}k\hat{\vec{r}}^2$$

Can be expanded

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} - \frac{1}{2m}i(\hat{\vec{p}}\cdot\vec{b} + \vec{b}\cdot\hat{\vec{p}}) - \frac{1}{2m}\vec{b}^2 + \frac{1}{2}k\hat{\vec{r}}^2 \quad (19)$$

For any function $g(\vec{r})$ holds

$$(\hat{\vec{p}}\cdot\vec{b} - \vec{b}\cdot\hat{\vec{p}})g(\vec{r}) = -i\hbar(\vec{b}\cdot\vec{\nabla}g + g\vec{\nabla}\cdot\vec{b} - \vec{b}\cdot\vec{\nabla}g) = -i\hbar g\vec{\nabla}\cdot\vec{b}$$

This expression always vanishes in the present case since

$$\vec{\nabla}\cdot\vec{b} = 0 \qquad (20)$$

where I assumed $\vec{\nabla}\times\vec{b} \neq 0$ in the equation of motion (4).
Accordingly, holds

$$\hat{\vec{p}}\cdot\vec{b}g = \vec{b}\cdot\hat{\vec{p}}g \qquad (21)$$

and, consequently,

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} - \frac{1}{m}i(\vec{b}\cdot\hat{\vec{p}}) - \frac{1}{2m}\vec{b}^2 + \frac{1}{2}k\hat{\vec{r}}^2 \qquad (22)$$

Using Cartesian coordinates and $\vec{b} = -b_0 y\hat{i} + b_0 x\hat{j}$ yield,

$$\hat{H} = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{ib_0}{m}(-\hat{y}\hat{p}_x + \hat{x}\hat{p}_y) - \frac{{b_0}^2}{2m}(\hat{x}^2 + \hat{y}^2) + \frac{1}{2}k(\hat{x}^2 + \hat{y}^2 + \hat{z}^2)$$

After some manipulations it may be written

$$\hat{H} = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + i\omega_b(-\hat{y}\hat{p}_x + \hat{x}\hat{p}_y) - \frac{m{\omega_b}^2}{2}(\hat{x}^2 + \hat{y}^2) + \frac{m{\omega_0}^2}{2}(\hat{x}^2 + \hat{y}^2 + \hat{z}^2) \quad (23)$$

Rearranging equation (23) and use the expression for the angular momentum operator in the $z-$ direction:

$$\hat{L}_z = -\hat{y}\hat{p}_x + \hat{x}\hat{p}_y = -i\hbar\left(-\hat{y}\frac{\partial}{\partial x} + \hat{x}\frac{\partial}{\partial y}\right) \qquad (24)$$

Equation (23) becomes

$$\hat{H} = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + i\omega_b\hat{L}_z + \frac{m}{2}({\omega_0}^2 - {\omega_b}^2)(\hat{x}^2 + \hat{y}^2) + \frac{m{\omega_0}^2}{2}\hat{z}^2 \quad (25)$$

Observe that the third term in the RHS of equation (23) vanishes when:

$$\omega_b = \omega_0 \qquad (26)$$

The Hamiltonian in equation (25) may be separated into two parts

$$\hat{H} = \left[\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + i\omega_b\hat{L}_z + \frac{m}{2}({\omega_0}^2 - {\omega_b}^2)(\hat{x}^2 + \hat{y}^2)\right] + \left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m{\omega_0}^2}{2}\hat{z}^2\right] (27)$$

The three terms in the first square bracket in the RHS in the Hamiltonian (27) describes two identical oscillators along the $x-$ and $y-$directions which are coupled through the angular momentum operator $\hat{L}_z$, whereas the last term describe a free oscillator in the $z-$direction. The square of the frequency of the each oscillator in equation (27) is shift by ${\omega_b}^2$. Equation (27) has the same symmetry about the $z-$direction as the classical equation of motion in equation (18).

The Hamiltonian in equation (27) may be written

$$\hat{H} = \hat{H}(x,y) + \hat{H}(z) \qquad (28)$$

Where

$$\hat{H}(x,y) = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + i\omega_b \hat{L}_z + \frac{m}{2}(\omega_0{}^2 - \omega_b{}^2)(\hat{x}^2 + \hat{y}^2)$$

This may be further be written

$$\hat{H}(x,y) = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \omega_b \hbar\left(-\hat{y}\frac{\partial}{\partial x} + \hat{x}\frac{\partial}{\partial y}\right) + \frac{m}{2}(\omega_0{}^2 - \omega_b{}^2)(\hat{x}^2 + \hat{y}^2) \quad (29)$$

and

$$\hat{H}(z) = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_0{}^2}{2}\hat{z}^2 \qquad (30)$$

Schrödinger equation in the case $\vec{b} = -b_0 y\hat{i} + b_0 x\hat{j}$

To obtain the stationary states, i.e., the solutions of

$$\hat{H}\Psi(x,y,z) = E\Psi(x,y,z)$$

where the Hamiltonian $\hat{H}$ is given by equation (27). For this purpose we use the wave function in the form

$$\Psi(x,y,z) = \psi(x,y)\cdot Z(z) \qquad (31)$$

The Hamiltonian $\hat{H}(z)$ in equation (30) has the well known Eigen functions and Eigen values. It is Schrödinger equation is the well known:

$$\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_0{}^2}{2}\hat{z}^2\right]Z(z) = E_z Z(z) \qquad (32)$$

We also want to describe the stationary states corresponding to the Hamiltonian $\hat{H}(x,y)$ in equation (29). It is Schrödinger equation may be written

$$\left[\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \omega_b \hbar\left(-\hat{y}\frac{\partial}{\partial x} + \hat{x}\frac{\partial}{\partial y}\right) + \frac{m}{2}(\omega_0{}^2 - \omega_b{}^2)(\hat{x}^2 + \hat{y}^2)\right]\psi(x,y) = (E - E_z)\psi(x,y) \quad (33)$$

Accordingly, we seek stationary states which are simultaneous Eigen states of the Hamiltonian of the two-dimensional isotropic harmonic oscillator

$$\hat{H} = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{m}{2}(\omega_0{}^2 - \omega_b{}^2)(\hat{x}^2 + \hat{y}^2) \qquad (34)$$

and its Schrödinger equation

$$\left[\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{m}{2}(\omega_0{}^2 - \omega_b{}^2)(\hat{x}^2 + \hat{y}^2)\right]\psi(x,y) = (E - E_z)\psi(x,y) \qquad (35)$$

as well as of the angular momentum operator $\hat{L}_z$. To obtain these Eigen states we introduce the dimensionless variables of the harmonic oscillator

$$X = \sqrt{\frac{m\omega_0}{\hbar}}x, \quad \text{and} \quad Y = \sqrt{\frac{m\omega_0}{\hbar}}y \qquad (36)$$

Equation (29) is

$$\hat{H}(x,y)=\frac{-\hbar^2}{2m}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})+\omega_b\hbar(-\hat{y}\frac{\partial}{\partial x}+\hat{x}\frac{\partial}{\partial y})+\frac{m}{2}({\omega_0}^2-{\omega_b}^2)(\hat{x}^2+\hat{y}^2)$$

can then be written

$$\frac{1}{\hbar\omega_0}\hat{H}(X,Y)=-\frac{1}{2}(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2})+\varpi(-\hat{Y}\frac{\partial}{\partial X}+\hat{X}\frac{\partial}{\partial Y})+\frac{1}{2}(1-\varpi^2)(\hat{X}^2+\hat{Y}^2) \quad (37)$$

where $\varpi=\frac{\omega_b}{\omega_0}$ is a dimensionless constant.

By employing the following annihilation and creation operators

$$\begin{aligned}\hat{d}_X&=\sqrt{\frac{1}{2}}[\sqrt{1-\varpi^2}X+\frac{\partial}{\partial X}]; \quad \hat{d}_Y=\sqrt{\frac{1}{2}}[\sqrt{1-\varpi^2}Y+\frac{\partial}{\partial Y}];\\ \hat{d}_X{}^\dagger&=\sqrt{\frac{1}{2}}[\sqrt{1-\varpi^2}X-\frac{\partial}{\partial X}]; \quad \hat{d}_Y{}^\dagger=\sqrt{\frac{1}{2}}[\sqrt{1-\varpi^2}Y-\frac{\partial}{\partial Y}]\end{aligned} \quad (38)$$

and the identity of operators which can be readily proven

$$(-\hat{Y}\frac{\partial}{\partial X}+\hat{X}\frac{\partial}{\partial Y})=\frac{1}{\sqrt{1-\varpi^2}}(\hat{d}_X{}^\dagger\hat{d}_Y-\hat{d}_Y{}^\dagger\hat{d}_X) \quad (39)$$

We obtain for equation (37)

$$\frac{1}{\hbar\omega_0}\hat{H}(X,Y)=(\sqrt{1-\varpi^2})\cdot\hat{1}+\hat{d}_X{}^\dagger\hat{d}_X+\hat{d}_Y{}^\dagger\hat{d}_Y+\frac{\varpi}{\sqrt{1-\varpi^2}}(\hat{d}_X{}^\dagger\hat{d}_Y-\hat{d}_Y{}^\dagger\hat{d}_X) \quad (40)$$

We note that the operator in equation (39) leaves the total number of vibrational quanta constant, since one phonon in the $Y-$ direction is annihilated and one in the $X-$ direction is created by the first term in the RHS and one phonon in the $X-$ direction is annihilated and one in the $Y-$ direction is created by the second term in equation (39). We, therefore, attempt to express Eigen states in terms of vibrational wave functions

$$\psi(n,m;X,Y)=\frac{(d_X{}^\dagger)^{n+m}}{\sqrt{(n+m)!}}\frac{(d_Y{}^\dagger)^{n-m}}{\sqrt{(n-m)!}}\psi(0,0;X,Y) \quad (41)$$

where $\psi(0,0;X,Y)$ is the wave function for the state with zero vibrational quanta for the $X-$ direction as well as for the $Y-$ direction oscillator. Equation (41) represents a state with $n+m$ quanta in the $X-$ direction oscillator and $n-m$ quanta in the $Y-$ direction oscillator, the total vibrational energy being

$$\hbar\omega(2n+\sqrt{1-\varpi^2}) \quad (42)$$

where the second term in equation (42) arises from the factor in front of the identity operator in equation (40).

The states (41) are not Eigen states of $\hat{L}_z$. There *no* way -via rotation- of the wave function $\psi(n,m;X,Y)$ that would make it both be an Eigen state of the two parts of the Hamiltonian in equation (40). I have remarked that as the breaking of symmetry for simple harmonic oscillations in space.

Conclusion:

The existence of many choices of the functional form of the friction coefficient on the oscillator coordinates lead to different equations of motion in the classical mechanics as well as different Eigen states in quantum mechanics. I have solved the Hamiltonian for the stationary states of DHO for one choice of the friction coefficient. Once the friction coefficient is set to zero the oscillator becomes a SHO in 3-dimensions and consequently the symmetry of motion in space and the quantum mechanics of a 3-D SHO are restored.

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