PROOF FOR THE FOUR COLOR THEOREM (4CT)

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Abstract: The Four Color Theorem (4CT) is the theorem stating that no more than four colors are required to color each part of a plane divided into finite parts so that no two adjacent parts have the same color. It was proven in 1976 by Kenneth Appel and Wolfgang Haken, but in this paper, we will prove 4CT simply without computer resources.

1. What is the Four Color Theorem (4CT)?

The 4CT states that states that no more than four colors are required to color each part of a plane divided into finite parts(map) so that no two adjacent parts have the same color.

2. Definition of Terms Used for Proof

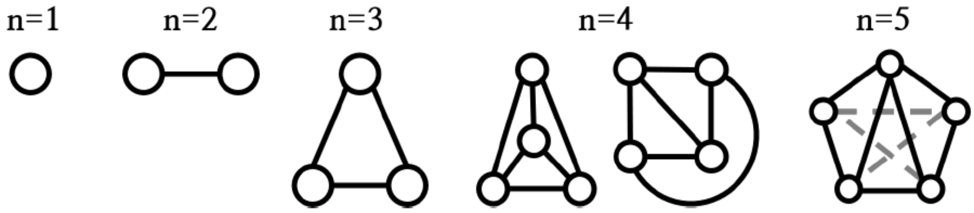

FIGURE 1. [Figure 1]

Points that represent each figure in the picture, which we will call nodes, are marked. A figure represented by Node A is called "figure of Node A".
Among the figures prepared by dividing the picture, two adjacent figures are marked by connecting each node with lines. The lines connecting the nodes are called 'stects'[1], and a stect connecting two nodes, Node A and Node B, is referred to as 'Stect AB'.
A stect may have any form of straight lines and curves, and is drawn over a line dividing the picture into figures.

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[1] "Stect," a compound word of the words "intersect" and "string," refers to a line connecting a node and another in a graph. While a link having the form of a straight line connects a node and another in the conventional graphs structure, a stect is a new concept that is differentiated from a link because a stect may connect one node and another not only with a straight line but also with a curve (enabling nonlinear connection).

A stect having the form of a straight line is called a straight stect, while one having the form of a curve is called a nonlinear stect.
A picture consisting of only nodes and stects is referred to as a graph.
A graph consisting of only nodes and linear stects is called a linear graph, while all other graphs, consisting of at least one nonlinear stect, are called nonlinear graphs.
If the number of nodes (=the number of figures into which a picture is divided) is n, for all the figures to be colored differently, a node among the n nodes must be connected with the other n−1 nodes by using stects. In other words, to violate the condition of the 4CT (i.e., if the 4CT is false), if n≥5, all the nodes of the graph should be connected with each other. Nevertheless, a stect should not intersect another stect at a position that is not a node. As shown in [Figure 1], when n=5, it is impossible to make a structure where a node is connected with the other four nodes other than the node itself, and none of the stects intersects with another stect at a position that is not a node.
The following terms are used in the present proof:

(1) Closed node: A node that is confined in a figure consisting of other nodes and stects so that the node may not be connected with another node any more.

(2) Homogenous node: A node that shares the color with a closed node for being not adjacent to the closed node in a graph.

(3) Successful node: A node that is connected with all the other n−1 nodes other than the node itself.

(4) Connection strength: Degree of connection by means of stects, representing how close a node is to other nodes.

A convex set refers to a set of elements, X, including all elements. In a graph having the form of a convex set, a convex cell is defined as a figure consisting of nodes and stects that may incorporate all other nodes in the inside.
A convex cell having a triangular form is called a triangular convex cell, and a graph where nodes and stects are closed in a triangular convex cell is referred to as 'a graph having the form of a triangular convex cell'.

3. Steps of Proof

The proof will be carried out in the following steps by showing

(1) that any picture may be expressed with linear graphs;

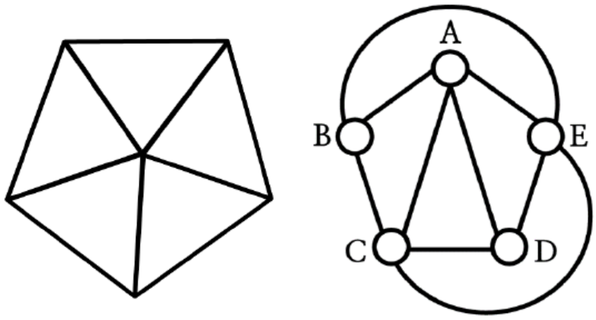
(2) that, when a picture is expressed with graphs, the graph having the greatest number of stects (i.e., the sum of the connection strength of all nodes in the graph is largest) is a graph having the form of a triangular convex cell;

(3) that a graph having the form of a triangular convex cell may be colored with four colors regardless of how many nodes are added to the graph;

(4) that an infinite number of nodes may be colored with four colors; and

(5) that a graph having the greatest number of adjacent sides may also be colored with four colors. Q.E.D

4. Method of Expressing a Picture with a Nonlinear Graph



(A) [Figure 2-1] (B) [Figure 2-2]

FIGURE 2. [Figure 2]

(1) Nodes that represent each of the figures in a picture are marked, and the nodes of the graphs sharing an adjacent side are connected with each other by using stects.

(2) In the order of connection strength of the nodes of the graph, attempts are made to make a successful node that is connected with the other n−1 nodes. The stects added in the procedure of making the attempts are deleted after completion of the coloring, because they are not those that connect the nodes of adjacent figures in the original picture but those that are added as a reference in the coloring procedure.

The picture of [Figure 2−1] is here in converted to a linear graph such as the figure in [Figure 2−2] by using the method described above.

First, Nodes A, B, C, D, and E, representing each of the figures, are marked in the picture of [Figure 2−1]. Since a figure represented a node is adjacent to the figures of two nodes at both sides, Stects AB, BC, CD, DE, and AE are marked.

Since the connection strength of all the nodes in the graphs is the same as 2, to make Node A a successful node (i.e. to be connected with the other four nodes), Stects AC and AD are drawn.

Subsequently, the nodes now having the highest connection strength are Nodes C and D having connection strength of 3.

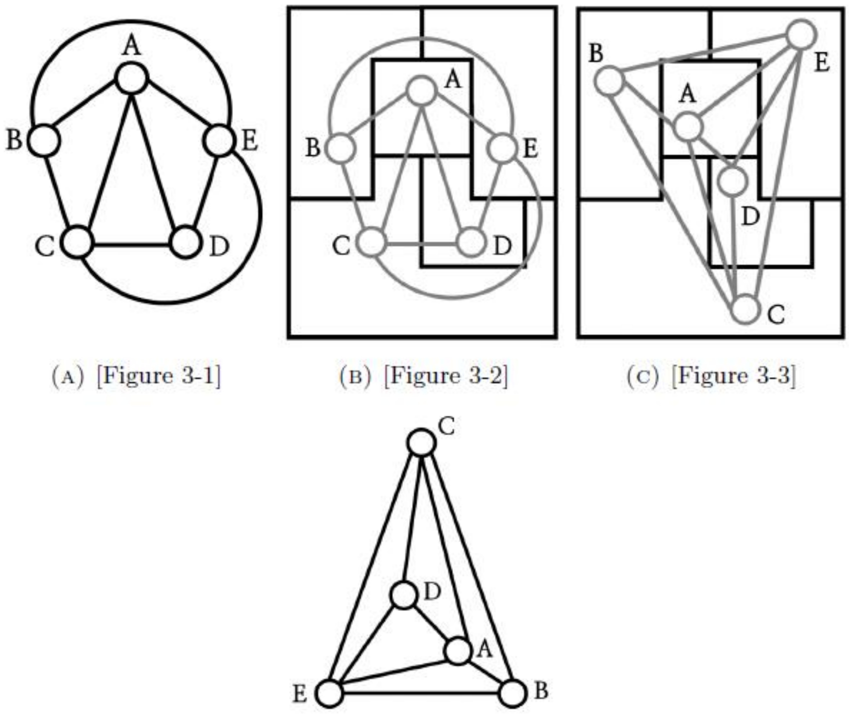
For Node C that is already connected with Nodes B, A, and D to be a successful node, it should be connected to Node E. Since Stect AD prevents Node C from being connected with Node E in the pentagon ABCDE, a stect should be drawn to the outside of the pentagon, as Stect CE shown in [Figure 2−2]. For Node D to be a successful node, it should be connected to Node B. However, since Node C is confined in the figure ABCD, it may not reach Node B that is outside the figure. Therefore, the figure represented by Node D may not be adjacent to the figure represented by Node B.

Subsequently, Node E, now having the highest connection strength of 3, should be connected to Node B, and thus Stect BE is drawn to connect.

Although Node B should be connected to Node D, Node B may not be connected with a stect because Node D is a closed node.

The resulting graphs are shown in [Figure 2-2].

5. Method of Expressing a Picture with a Nonlinear Graph



(A) [Figure 3-1] (B) [Figure 3-2] (C) [Figure 3-3]

(D) [Figure 3-4]

FIGURE 3. [Figure 3]

The graph in [Figure 3-1] may be expressed with the picture in [Figure 3-2].
Conversion of the graph in [Figure 3-1] to a linear graph requires that Stects BE and CE be linear. However, since a stect in a figure should not intersect another stect at a position that is not a node, the node configuration shown in the picture in [Figure 3-2] may not be expressed as a linear graph.
Nodes A and D representing Figures A and D in [Figure 3-3] may be transposed, and the region consisting of all three nodes and stects inside Stects BC, BE, and CE may be transformed to linear triangles in order to draw a linear graph having the form of a triangular convex cell shown in [Figure 3-4].
Therefore, a graph of any picture may be transformed to a linear graph by transposing the nodes representing individual figures to which a picture is divided and then converting into linear triangles the region consisting of all three nodes and stects inside the graph with the three outermost stects as three sides of an outer triangle of a triangular convex cell.
Therefore, any picture may be expressed by using only nodes and linear stects.

6. Derivation of Triangular Convex Cell

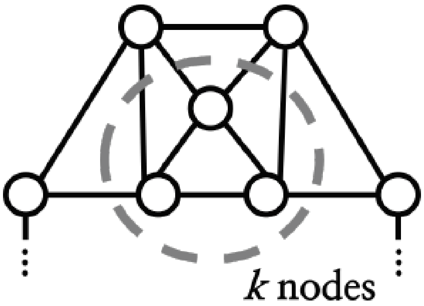


FIGURE 4. [Figure 4]

If the total number of nodes in a graph is denoted as v, the total number of stects as e, and the number of faces made by nodes and stects in the divided graph as f, Euler's formula, $v - e + f = 1$, is valid.

Assume that k nodes exist inside a triangular convex cell, as shown in [Figure 4].

If the number of triangles is f, the sum of all the internal angles is 180f°.

Since a triangular convex cell is a $(n - k)$−gon, the sum of the internal angles of a triangular convex cell is $180(n - k - 2)°$. In addition, the angle generated by each of the nodes is 360, the sum of the angles generated by the k nodes is 360k°.

Therefore, the following equation is obtained:

$$180f = 180(v - k - 2) + 360k$$

This gives $180f = 180(v - k - 2) + 360k$

$$\begin{cases} v - e + f = 1 \\ f = k + v - 2 \end{cases}$$

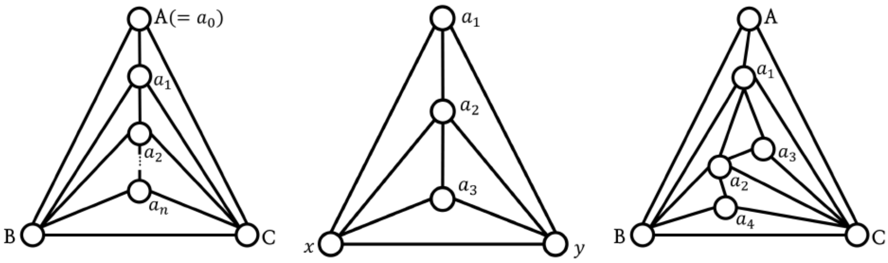
The two equations gives the relation, $2v = 3 + e - k$.

When v is a fixed number, e should be as large as possible (since the connections should be made in a way that makes the connection strength of a graph as high as possible, the number of stects (connections) should be as great as possible in a graph), which requires f to become as large as possible.

For e to be a maximum, $-k$ should be a minimum. Therefore, k should be as large as possible, which means that all the nodes should be connected in the shape of a triangle, a figure that may not be divided further. Hence, among linear graphs, the form of graph having the highest connection strength at individual nodes is a triangular convex cell.

As described above, when figures prepared by dividing a picture are adjacent to the maximum degree, the figures may be expressed as a linear graph consisting of only nodes and linear stects. Any picture may be expressed as a linear graph consisting of only nodes and linear stects by deleting nodes or stects when necessary.

7. Graph Having the Form of a Triangular Convex Cell May be Colored with Four Colors Regardless of How Many Nodes Are Added to the Graph



(A) [Figure 5-1] (B) [Figure 5-2] (C) [Figure 5-3]

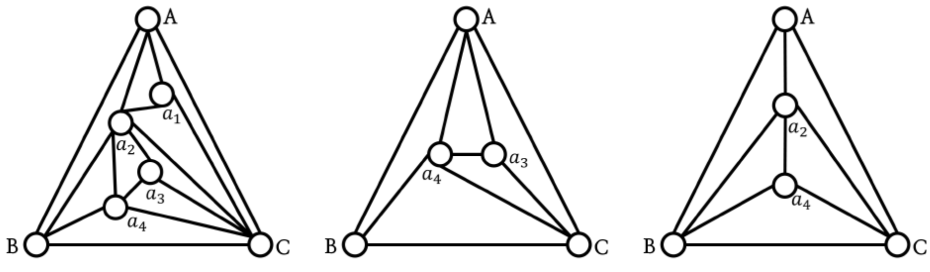
FIGURE 5. [Figure 5]

For a graph having the form of the triangular convex cell shown in [Figure 5−1], since the a_{2k} type nodes, such as a_2, a_4 , $a_6\cdots$, including a_0, are not adjacent with each other, the nodes may be colored with one color. Similarly, the a_{2k-1} type nodes, such as a_1, a_3 , $a_5\cdots$, are not adjacent with each other, and thus they also may be colored with a same color. Therefore, the graph may be colored by using four colors, which are the color for Node A (=the color for the a_{2k} type nodes), the color for the a_{2k-1} type nodes, the color for Node B, and the color for Node C.

If Node a_3 is added into triangle $a_2$xy, the figure represented by Node a_3 may be colored with the same color for the figure represented by a_1. This rule is applied.

For a graph having the form of the triangular convex cell shown in [Figure 5−2], the graph may be colored with four colors, the color for the figures represented by Nodes A and a_2, the color for the figures represented by Nodes B and a_3, the color for the figure represented by Node C, and the color for the figures represented by Nodes a_1 and a_4. Therefore, any graph having the form of a triangular convex cell may be colored with four colors regardless of how many nodes are added into the triangular convex cell.

8. Graph Having the Form of a Triangular Convex Cell May be Colored with Four Colors Regardless of How Many Nodes Are Added to the Graph



(A) [Figure 6-1] (B) [Figure 6-2] (C) [Figure 6-3]

FIGURE 6. [Figure 6]

The graph having the form of the triangular convex cell shown in [Figure 6−1] becomes the graph shown in [Figure 6−2] after deleting Nodes a_1 and a_2 as well as all the stects

connected to Nodes a_1 and a_2. The graph shown in [Figure 6−1] becomes the graph shown in [Figure 6−3] after deleting Nodes a_1 and a_3 as well as all the stects connected to Nodes a_1 and a_3.
In the same manner, a graph having the form of the triangular convex cell shown in [Figure 6−1] may be expressed as graphs of various forms by removing nodes and stects according to the need.
Similarly, any pictures may be expressed as a graph having the form of a triangular convex cell by removing nodes and stects according to the need from a graph having the form of a triangular convex cell in which the number of nodes is $\lim_{k\to\infty} k$.
In addition, since any graphs having the form a triangular convex cell may be colored with four colors by meeting the conditions of the 4CT, regardless of how many nodes are added to the triangular convex cell, as proved above, the 4CT is valid even when $k = \infty$. Therefore, a huge graph having the form a triangular convex cell of which k is ∞ may be first colored, and the nodes and stects of the graph may be deleted according to the need to express any picture. Thus, any picture may be colored with four colors.

Appendix A.
Proof of Euler's Theorem (v−e+f=1)

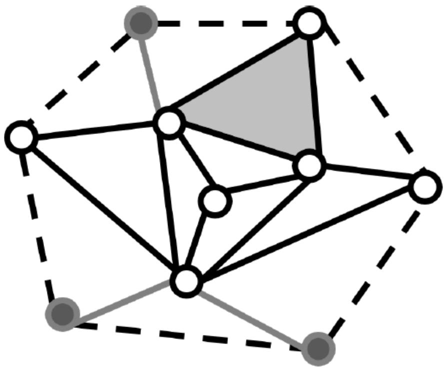


FIGURE 7. [Figure A]

In the graph shown in [Figure A], it is assumed that the number of points is denoted as v, the number of lines as e, and the number of faces consisting of the points and lines in the graph as f.

(1) If the dotted lines in the graphs shown in [Figure A] are removed one by one, the triangles consisting of the points and lines in the graph are removed one by one, resulting in a decrease of e and f values by 1 each time. Therefore, the value of $(v - e + f)$ remains unchanged.

(2) If the gray−colored points and lines are removed from the graphs shown in [Figure A], in other words, if the points and lines that do not constitute a face are removed, the values for v and e are decreased by 1 each time. Therefore, the value of $(v - e + f)$ remains unchanged.

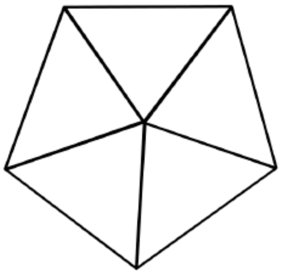
(3) If the gray−colored points and lines constituting the gray−colored face are removed, v is decreased by 1, e is decreased by 2, and f is decreased by 1 each time. Therefore,

the value of (v − e + f) remains unchanged.

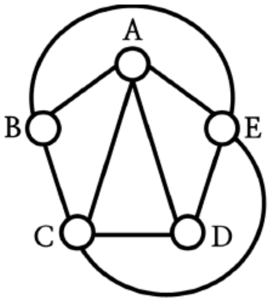
(4) If the points and lines are removed continuously in the same manner, a triangle consisting of three points and three lines is left. Since $v = 3$, $e = 3$, $f = 1$ for a triangle, the theorem, $v - e + f = 1$, is valid $(3 - 3 + 1 = 1)$.

If one internal line in [Figure A] is removed, two faces are merged into one face, resulting in a decrease of e and f values, respectively, by 1. Therefore, the value of $(v - e + f)$ remains unchanged.

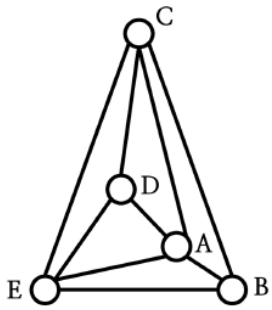
Appendix B.
Application Example



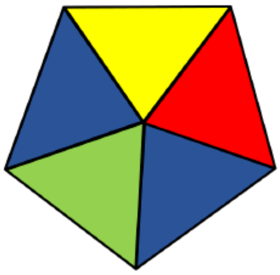
(A) [Figure B-1]



(B) [Figure B-2]



(C) [Figure B-3]



(D) [Figure B-4]

FIGURE 8. [Figure B]

According to the method described above, the picture of [Figure B−1] will be colored by using four colors.

In the graph, Stects AC, AD, BE, and CE will be removed after completing the coloring because they are temporary stects for the coloring.

The graph shown in [Figure B−2] is converted into a linear graph having the form of a triangular convex cell shown in [Figure B−3], where in the linear graph has Nodes B, C, and E as the three vertices and Stects BC, BE, and CE as the three sides.

Since the Nodes B and D in the graph shown in [Figure B−3] are homogenous nodes, the picture may be colored with a total of four colors, the color for the figure represented by

Node A, the color for the figures represented by Nodes B and D, the color for the figure represented by Node C, and the color for the figure represented by Node E.