An approximation to $\pi(n)$ through the sum of consecutive prime numbers

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Abstract

In this paper it is proved that the sum of consecutive prime numbers under the square root of a given natural number is asymptotically equivalent to the prime counting function $\pi(n)$. Also, it is proved another asymptotic equivalence between the sum of the first $\lfloor \sqrt{n} \rfloor$ prime numbers and the prime counting function $\pi(n)$.

Theorem 1. Let the prime counting function until a given natural number be

$$\pi(n) = \# \{ p \in P \mid p \leq n \}$$  \hspace{1cm} (1)

Let the sum of consecutive prime numbers under the square root of a given natural number be

$$\Upsilon(n) = \sum_{p \leq \sqrt{n}} p$$  \hspace{1cm} (2)

It can be stated that

$$\Upsilon(n) \sim \pi(n)$$  \hspace{1cm} (3)

Which can be expressed also stating that

$$\lim_{n \to \infty} \frac{\Upsilon(n)}{\pi(n)} = 1$$  \hspace{1cm} (4)
Proof

By partial summation

\[ \Upsilon(n) = (\lfloor \sqrt{n} \rfloor \pi(\sqrt{n})) - \sum_{m=2}^{\lfloor \sqrt{n} \rfloor - 1} \pi(m) \]  \hspace{1cm} (5)\]

Where \( \lfloor \sqrt{n} \rfloor \) denotes the integer part of \( \sqrt{n} \).

By the Prime Number Theorem with error term, there exists a constant \( C \) such that

\[ \left| \pi(x) - \frac{x}{\log x} \right| \leq C \frac{x}{\log^2 x} \quad \text{for} \quad x \geq 2 \]  \hspace{1cm} (6)\]

Therefore, substituting \( \pi(\sqrt{n}) \) and \( \pi(m) \) by the application of the Prime Number Theorem on (6)

\[ \Upsilon(n) = (\lfloor \sqrt{n} \rfloor \frac{\sqrt{n}}{\log(\sqrt{n})}) - \sum_{m=2}^{\lfloor \sqrt{n} \rfloor - 1} \frac{m}{\log(m)} + O \left( \frac{n}{\log^2(\sqrt{n})} \right) \]  \hspace{1cm} (7)\]

Applying Riemann Sums theory to the sum on the right of (5)

\[ \sum_{m=2}^{\lfloor \sqrt{n} \rfloor - 1} \frac{m}{\log(m)} = \int_{2}^{\lfloor \sqrt{n} \rfloor} \frac{x}{\log(x)} \, dx + O \left( \frac{n}{\log^2(\sqrt{n})} \right) \]  \hspace{1cm} (8)\]

Solving the integral by partial integration, we have that

\[ \int_{2}^{\lfloor \sqrt{n} \rfloor} \frac{x}{\log(x)} \, dx = \left[ \frac{x^2}{2 \log(x)} \right]_{2}^{\lfloor \sqrt{n} \rfloor} + \int_{2}^{\lfloor \sqrt{n} \rfloor} \frac{x}{2 \log^2(x)} \, dx = \]

\[ = \frac{n}{2 \ln \lfloor \sqrt{n} \rfloor} + O \left( \frac{n}{\log^2(\sqrt{n})} \right) \]

It is easy to see that

\[ \frac{n}{2 \log \lfloor \sqrt{n} \rfloor} \sim \frac{n}{\log n} \]  \hspace{1cm} (10)\]
Thus
\[
\sum_{m=2}^{\lfloor \sqrt{n} \rfloor^{-1}} \frac{m}{\log(m)} \sim \frac{n}{\log(n)} + O\left(\frac{n}{\log^2(\sqrt{n})}\right)
\] (11)

Regarding the left product on (5) it can be seen that
\[
\lfloor \sqrt{n} \rfloor \sqrt{n} \log(\sqrt{n}) \sim \frac{n}{\log(\sqrt{n})} = \frac{n}{\frac{1}{2} \log(n)} = \frac{2n}{\log(n)}
\] (12)

Substituting (11) and (12) on (5), we have that
\[
\Upsilon(n) \sim \frac{2n}{\log(n)} - \frac{n}{\log(n)} + O\left(\frac{n}{\log^2(\sqrt{n})}\right)
\] (13)

As
\[
\frac{2n}{\log(n)} - \frac{n}{\log(n)} = \frac{n}{\log(n)}
\] (14)

Thus
\[
\Upsilon(n) \sim \frac{n}{\log(n)}
\] (15)

And subsequently, as by the Prime Number Theorem,
\[
\pi(n) \sim \frac{n}{\log(n)}
\] (16)

It can be stated that
\[
\Upsilon(n) \sim \pi(n)
\] (17)

Which can be expressed also stating that
\[
\lim_{n \to \infty} \frac{\Upsilon(n)}{\pi(n)} = 1
\] (18)
Theorem 2. Let the prime counting function until a given natural number be

$$\pi(n) = \# \{ p \in P \mid p \leq n \}$$  \hspace{1cm} (19)

Let the sum of the first \( \lfloor \sqrt{n} \rfloor \) consecutive prime numbers be

$$\Psi(n) = \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} p_k$$  \hspace{1cm} (20)

It can be stated that

$$\Psi(n) \left( \frac{1}{\log (\sqrt{n})} \right)^2 \sim \pi(n)$$  \hspace{1cm} (21)

Which can be expressed also stating that

$$\lim_{n \to \infty} \frac{\Psi(n)}{\pi(n) \log^2 (\sqrt{n})} = 1$$  \hspace{1cm} (22)

Proof

From the application of the Prime Number Theorem and the Abel’s summation we have

$$\Psi(n) \sim \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} k \log (k)$$  \hspace{1cm} (23)

$$\sum_{k=1}^{\lfloor \sqrt{n} \rfloor} k \log (k) = \frac{\lfloor \sqrt{n} \rfloor (\lfloor \sqrt{n} \rfloor + 1)}{2} \log (\lfloor \sqrt{n} \rfloor) - \frac{1}{2} \int_{1}^{\lfloor \sqrt{n} \rfloor} (t + 1) \, dt =$$  \hspace{1cm} (24)

$$= \frac{\lfloor \sqrt{n} \rfloor^2}{2} \log (\lfloor \sqrt{n} \rfloor) + \frac{\lfloor \sqrt{n} \rfloor}{2} \log (\lfloor \sqrt{n} \rfloor) - \frac{\lfloor \sqrt{n} \rfloor^2}{4} - \frac{\lfloor \sqrt{n} \rfloor}{2} + \frac{3}{4}$$  \hspace{1cm} (25)

Thus

$$\Psi(n) \sim \frac{\lfloor \sqrt{n} \rfloor^2}{2} \log (\lfloor \sqrt{n} \rfloor) \sim \frac{n}{2} \log (\sqrt{n})$$  \hspace{1cm} (26)

As

$$\frac{n}{2} \log (\sqrt{n}) = \frac{n}{\log (n)} \log^2 (\sqrt{n})$$  \hspace{1cm} (27)
Therefore
\[ \Psi(n) \sim \frac{n}{\log(n)} \log^2(\sqrt{n}) \] (28)

And subsequently, as by the Prime Number Theorem,
\[ \pi(n) \sim \frac{n}{\log(n)} \] (29)

It can be stated that
\[ \Psi(n) \left( \frac{1}{\log(\sqrt{n})} \right)^2 \sim \pi(n) \] (30)

Which can be expressed also stating that
\[ \lim_{n \to \infty} \frac{\Psi(n)}{\pi(n) \log^2(\sqrt{n})} = 1 \] (31)

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References


