Matrix-Representations of Tensors

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Abstract

The metric tensor of Minkowski space-time, the electromagnetic field tensor, etc., are usually represented by 4×4 matrices in many textbooks, but in this paper we will demonstrate that this form of matrix-representation is unreasonable. We will introduce more reasonable rules of matrix-form for representing any (p,q)-type tensor.

The metric tensor of Minkowski space-time, $g_{\mu\nu}$ or $g^{\mu\nu}$, is usually represented by a 4×4 matrix in many textbooks[1], as below, but it can be demonstrated that this form of representation is unreasonable.

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Similarly, it is unreasonable to represent the electromagnetic field tensor $F^{\mu\nu}$ by a 4×4 square matrix, e.g. as in many textbooks, because it can not give the clear and reasonable intuitive image of database for tensor, especially while operating the multiplications of tensors, it can not correspond with the rules of matrix-multiplications. Certainly, we may not represent tensors with matrix-form, as in the Dirac's book *General Theory of Relativity*[2], but if we do, we ought to do it better. In order to avoid confusion, we introduce a more reasonable method for representing any (p, q)-type tensor with matrix.

1. Rules of Matrix-Forms

The basic rule : every superscript must be corresponding to a column, and every subscript must be corresponding to a row. Thus:

the matrix form of a contravariant vector A^μ is a column matrix:

$$\begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

where the elements of the matrix is numbers.

The matrix representation of a covariant vector A_ν is a row matrix:

$$\begin{pmatrix} A_0 & A_1 & A_2 & A_3 \end{pmatrix}$$

The matrix representation of tensor $A^{\mu\nu}$ is (when more than two superscrips, the left one represents the ordinal-number of the larger column) :

$$A^{\mu\nu} = \begin{pmatrix} A^{0\nu} \\ A^{1\nu} \\ A^{2\nu} \\ A^{3\nu} \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} A^{00} \\ A^{01} \\ A^{02} \\ A^{03} \end{pmatrix} \\ \begin{pmatrix} A^{10} \\ A^{11} \\ A^{12} \\ A^{13} \end{pmatrix} \\ \begin{pmatrix} A^{20} \\ A^{21} \\ A^{22} \\ A^{23} \end{pmatrix} \\ \begin{pmatrix} A^{30} \\ A^{31} \\ A^{32} \\ A^{33} \end{pmatrix} \end{bmatrix}$$

The matrix representation of $A_{\mu\nu}$ is (when more than two subscrips, the left one represents the ordinal-number of the larger row):

$$[(A_{00}, A_{01}, A_{02}, A_{03}), (A_{10}, A_{11}, A_{12}, A_{13}), (A_{20}, A_{21}, A_{22}, A_{23}), (A_{30}, A_{31}, A_{32}, A_{33})]$$

The matrix representation of A_μ^ν is :

$$A_\mu^\nu = \begin{bmatrix} A_\mu^0 \\ A_\mu^1 \\ A_\mu^2 \\ A_\mu^3 \end{bmatrix} = [A_0^\nu, A_1^\nu, A_2^\nu, A_3^\nu]$$

$$= \begin{bmatrix} (A_0^0, A_1^0, A_2^0, A_3^0) \\ (A_0^1, A_1^1, A_2^1, A_3^1) \\ (A_0^2, A_1^2, A_2^2, A_3^3) \\ (A_0^3, A_1^3, A_2^3, A_3^3) \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} A_0^0 \\ A_0^1 \\ A_0^2 \\ A_0^3 \end{pmatrix} & \begin{pmatrix} A_1^0 \\ A_1^1 \\ A_1^2 \\ A_1^3 \end{pmatrix} & \begin{pmatrix} A_2^0 \\ A_2^1 \\ A_2^2 \\ A_2^3 \end{pmatrix} & \begin{pmatrix} A_3^0 \\ A_3^1 \\ A_3^2 \\ A_3^3 \end{pmatrix} \end{bmatrix}$$

$$= \begin{pmatrix} A_0^0 & A_1^0 & A_2^0 & A_3^0 \\ A_0^1 & A_1^1 & A_2^1 & A_3^1 \\ A_0^2 & A_1^2 & A_2^2 & A_3^2 \\ A_0^3 & A_1^3 & A_2^3 & A_3^3 \end{pmatrix}$$

Columns or rows matrices as above can be operated additions and two types of multiplications: contraction-product (including scalar-product) and direct product (or called tensor product) , as following:

1) Contraction-Product: Whenever one subscript attached to a capital letter which represents a matrix is equal to one superscript attached to another, it must be operated once by the usual matrix-multiplication.

$$A_\mu B^\mu = \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \end{pmatrix} \begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix} = A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3$$

,

2) Direct product:

$$A_\mu B^\nu = B^\nu A_\mu = \begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix} \otimes \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \end{pmatrix} = \begin{pmatrix} A_0 B^0 & A_1 B^0 & A_2 B^0 & A_3 B^0 \\ A_0 B^1 & A_1 B^1 & A_2 B^1 & A_3 B^1 \\ A_0 B^2 & A_1 B^2 & A_2 B^2 & A_3 B^2 \\ A_0 B^3 & A_1 B^3 & A_2 B^3 & A_3 B^3 \end{pmatrix},$$

and for another example,

$$A_\mu B_\nu = \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \end{pmatrix} \otimes \begin{pmatrix} B_0 & B_1 & B_2 & B_3 \end{pmatrix}$$

$$= \left[\begin{pmatrix} A_0 B_0 & A_0 B_1 & A_0 B_2 & A_0 B_3 \end{pmatrix}, (...), (...), \begin{pmatrix} A_3 B_0 & A_3 B_1 & A_3 B_2 & A_3 B_3 \end{pmatrix} \right],$$

etc. Notice that the elements of the rows or columns can also be rows (or columns).

In this way, now we can give a more reasonable matrix-form of $g_{\mu\nu}$, the metric tensor of Minkowski space-time:

$$g_{\mu\nu} = [\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \end{pmatrix}]$$

compare it with the matrix-form mentioned at the beginning of the article.

Its contraction-product is, for instance:

$$g_{\mu\nu} A^\nu = [\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \end{pmatrix}] \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} A^0 + \begin{pmatrix} 0 & -1 & 0 & 0 \end{pmatrix} A^1 + \begin{pmatrix} 0 & 0 & -1 & 0 \end{pmatrix} A^2 + \begin{pmatrix} 0 & 0 & 0 & -1 \end{pmatrix} A^3$$

$$= \begin{pmatrix} A^0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -A^1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -A^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -A^3 \end{pmatrix}$$

$$= \begin{pmatrix} A^0 & -A^1 & -A^2 & -A^3 \end{pmatrix} = \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \end{pmatrix} = A_\mu$$

2. Conditions of Coordinate Transformation for the Matrix-form of Tensor

As mentioned above, tensor can be represented by a column or row matrix which elements itself are also column (or row) matrices, just like the vector can be represented by a column or row which elements are numbers. But the matrix which can represent a tensor must obey the rule of coordinate transformation, that is:

While operating a coordinate transformation, the Jacobian Matrix of coordinate transformation must operate the left-contraction-product directly acted to each of the columns (including each element which itself is a column) at the same time, and the Jacobian Matrix of reverse coordinate transformation must operate a right-contraction-product directly acting from right to each of the rows (including each element which itself is a row) at the same time, besides which the effects of acting on the larger matrix will be handed on to the smaller matrices.

For example, in the case of 2D, when coordinate transformation is : $\mathbf{x} \to \mathbf{x}'$,

$$T_{i'}^{j'} = \begin{bmatrix} \begin{pmatrix} T'^1_1 & T'^1_2 \end{pmatrix} \\ \begin{pmatrix} T'^2_1 & T'^2_2 \end{pmatrix} \end{bmatrix} = \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} \\ \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} \end{pmatrix} \begin{bmatrix} \begin{pmatrix} T^1_1 & T^1_2 \end{pmatrix} \begin{pmatrix} \frac{\partial x^1}{\partial x^{1'}} & \frac{\partial x^1}{\partial x^{2'}} \\ \frac{\partial x^2}{\partial x^{1'}} & \frac{\partial x^2}{\partial x^{2'}} \end{pmatrix} \\ \begin{pmatrix} T^2_1 & T^2_2 \end{pmatrix} \begin{pmatrix} \frac{\partial x^1}{\partial x^{1'}} & \frac{\partial x^1}{\partial x^{2'}} \\ \frac{\partial x^2}{\partial x^{1'}} & \frac{\partial x^2}{\partial x^{2'}} \end{pmatrix} \end{bmatrix} = \frac{\partial x^{j'}}{\partial x^n} \frac{\partial x^m}{\partial x^{i'}} T^n_m$$

another example:

$$T^{i'j'} = \begin{bmatrix} \begin{pmatrix} T'^1_1 \\ T'^1_2 \end{pmatrix} \\ \begin{pmatrix} T'^2_1 \\ T'^2_2 \end{pmatrix} \end{bmatrix} = \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} \\ \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} \end{pmatrix} \begin{bmatrix} \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} \\ \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} \end{pmatrix} \begin{pmatrix} T^1_1 \\ T^1_2 \end{pmatrix} \\ \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} \\ \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} \end{pmatrix} \begin{pmatrix} T^2_1 \\ T^2_2 \end{pmatrix} \end{bmatrix} = \frac{\partial x^{i'}}{\partial x^m} \frac{\partial x^{j'}}{\partial x^n} T^{mn}$$

ect.

3. Summary About the Matrix-form of Representation of N-Dimensional (p,q)-Type Tensor

Suppose there are n smaller columns (or rows) in a larger column or row matrix, we say there is one class in the form, and so on, until there are n numbers in the smallest columns or rows, then, if totally there are p classes of columns and q classes of rows in the form, it could be a matrix-form of the representation of n dimensional (p, q)-type tensor. (And besides it must obey the rule of coordinate transformation.)

Appendix

1. Matrix-form of Levi-Civita Symbol:

ϵ_{ijk}=[[(0,0,0);(0,0,1);(0,-1,0)];[(0,0,-1);(0,0,0);(1,0,0)];[(0,1,0);(-1,0,0);(0,0,0)]]

Now let ϵ^{ij}_{k} or ϵ^{i}_{jk} ect. only represents the corresponding one element in the matrix, according to the rules of arraying the matrices, we can find:

$$\epsilon^{ij}_{k} = \epsilon_{kij} = \epsilon_{ijk} = \epsilon^{i}_{jk}, \quad ect.$$

(here ϵ^{ij}_{k} or other one temporarily does not represent a matrix-form of a tensor, but only represents an element or a component of a matrix or a tensor.)

2. Matrix-form of electromagnetic field tensor:

$$F^{\mu\nu} = \begin{bmatrix} \begin{pmatrix} F^{00} \\ F^{01} \\ F^{02} \\ F^{03} \end{pmatrix} \\ \begin{pmatrix} F^{10} \\ F^{11} \\ F^{12} \\ F^{13} \end{pmatrix} \\ \begin{pmatrix} F^{20} \\ F^{21} \\ F^{22} \\ F^{23} \end{pmatrix} \\ \begin{pmatrix} F^{30} \\ F^{31} \\ F^{32} \\ F^{33} \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} \partial^0 A^0 - \partial^0 A^0 \\ \partial^0 A^1 - \partial^1 A^0 \\ \partial^0 A^2 - \partial^2 A^0 \\ \partial^0 A^3 - \partial^3 A^0 \end{pmatrix} \\ \begin{pmatrix} \partial^1 A^0 - \partial^0 A^1 \\ \partial^1 A^1 - \partial^1 A^1 \\ \partial^1 A^2 - \partial^2 A^1 \\ \partial^1 A^3 - \partial^3 A^1 \end{pmatrix} \\ \begin{pmatrix} \partial^2 A^0 - \partial^0 A^2 \\ \partial^2 A^1 - \partial^1 A^2 \\ \partial^2 A^2 - \partial^2 A^2 \\ \partial^2 A^3 - \partial^3 A^2 \end{pmatrix} \\ \begin{pmatrix} \partial^3 A^0 - \partial^0 A^3 \\ \partial^3 A^1 - \partial^1 A^3 \\ \partial^3 A^2 - \partial^2 A^3 \\ \partial^3 A^3 - \partial^3 A^3 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 0 \\ E^1 \\ E^2 \\ E^3 \end{pmatrix} \\ \begin{pmatrix} -E^1 \\ 0 \\ B^3 \\ -B^2 \end{pmatrix} \\ \begin{pmatrix} -E^2 \\ -B^3 \\ 0 \\ B^1 \end{pmatrix} \\ \begin{pmatrix} -E^3 \\ B^2 \\ -B^1 \\ 0 \end{pmatrix} \end{bmatrix}$$

so,

$$\partial_\mu F^{\mu\nu} = \begin{pmatrix} \partial_0 & \partial_1 & \partial_2 & \partial_3 \end{pmatrix} \begin{bmatrix} \begin{pmatrix} 0 \\ E^1 \\ E^2 \\ E^3 \end{pmatrix} \\ \begin{pmatrix} -E^1 \\ 0 \\ B^3 \\ -B^2 \end{pmatrix} \\ \begin{pmatrix} -E^2 \\ -B^3 \\ 0 \\ B^1 \end{pmatrix} \\ \begin{pmatrix} -E^3 \\ B^2 \\ -B^1 \\ 0 \end{pmatrix} \end{bmatrix}$$

$$= \partial_0 \begin{pmatrix} 0 \\ E^1 \\ E^2 \\ E^3 \end{pmatrix} + \partial_1 \begin{pmatrix} -E^1 \\ 0 \\ B^3 \\ -B^2 \end{pmatrix} + \partial_2 \begin{pmatrix} -E^2 \\ -B^3 \\ 0 \\ B^1 \end{pmatrix} + \partial_3 \begin{pmatrix} -E^3 \\ B^2 \\ -B^1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\partial_1 E^1 - \partial_2 E^2 - \partial_3 E^3 \\ \partial_0 E^1 - \partial_2 B^3 + \partial_3 B^2 \\ \partial_0 E^2 + \partial_1 B^3 - \partial_3 B^1 \\ \partial_0 E^3 - \partial_1 B^2 + \partial_2 B^1 \end{pmatrix}$$

Reference:

[1] Lewis H.Ryder: *Quantum Field Theory,*
Michael E. Peskin and Daniel V.Schroeder: *A Introduction to Quantum Field Theory,*
Mark Srednicki: *Quantum Field Theory, etc.*
[2] P.A.M.Dirac: *General Theory of Relativity*

(g may be denoted as η, and, "+" and "−" may be exchanged.)

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