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More On P-Union and P-Intersection of Neutrosophic Soft Cubic Set

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Abstract: The P-union ,P-intersection, P-OR and P-AND of neutrosophic soft cubic sets are introduced and their related properties are investigated. We show that the P-union and the P-intersection of two internal neutrosophic soft cubic sets are also internal neutrosophic soft cubic sets. The conditions for the P-union (P-intersection) of two T-external (resp. I- external, F- external) neutrosophic soft cubic sets to be T-external (resp. I- external, F- external) neutrosophic soft cubic sets is also dealt with. We provide conditions for the P-union (P-intersection) of two T-external (resp. I- external, F- external) neutrosophic soft cubic sets to be T-internal (resp. I- internal,F- internal) neutrosophic soft cubic sets. Further the conditions for the P-union (resp. P-intersection) of two neutrosophic soft cubic sets to be both T-external (resp. I- external, F- external) neutrosophic soft cubic sets and T-external (resp. I- external, F- external) neutrosophic soft cubic sets are also framed.

Keywords: Cubic set, Neutrosophic cubic set, Neutrosophic soft cubic set, T-internal (resp. I- internal,F- internal) neutrosophic soft cubic sets , T-external (resp. I- external, F- external) neutrosophic soft cubic set.

1 Introduction

Florentine Smarandache[10,11] coined neutrosophic sets and neutrosophic logic which extends the concept of the classical sets, fuzzy sets and its extensions. In neutrosophic set, indeterminacy is quantified explicity and truth-membership, indeterminacy-membership and falsity – membership are independent. This assumption is very important in many applications such as information fusion in which we try to combine the data from different sensors. Pabita Kumar Majii[18] had combined the Neutrosophic set with soft sets and introduced a new mathematical model ‘ Nuetrosophic soft set’. Y. B. Jun et al[2]., introduced a new notion, called a cubic set by using a fuzzy set and an interval-valued fuzzy set, and investigated several properties. Jun et al. [19] extended the concept of cubic sets to the neutrosophic cubic sets. [1] introduced neutrosophic soft cubic set and the notion of truth-internal (indeterminacy-internal, falsity-internal) neutrosophic soft cubic sets and truth-external (indeterminacy-internal, falsity-internal) neutrosophic soft cubic sets

As a continuation of the paper [1]We show that the P-union and the P-intersection of T-internal (resp. I-internal,F-internal) neutrosophic soft cubic sets are also T-internal (resp. I-internal,F-internal) neutrosophic soft cubic sets. We also provide conditions for the P-union (P-intersection) of two T-external (resp. I- external,F-external) neutrosophic soft cubic sets to be T-external (resp. I- external,F- external) neutrosophic soft cubic sets.

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We provide conditions for the P-union (resp. P-intersection) of two NSCS to be both T-external (resp. I-external,F- external) neutrosophic soft cubic sets and T-external (resp. I- external,F- external) neutrosophic soft cubic sets.

2 Preliminaries

2.1 Definition: [5] Let E be a universe. Then a fuzzy set μ over E is defined by X = { μ_x (x) / x: x ϵ E }where μ_x is called membership function of X and defined by $\mu_x : E \to [0,1]$. For each x E, the value $\mu_x(x)$ represents the degree of x belonging to the fuzzy set X.

2.2 Definition: [2] Let X be a non-empty set. By a cubic set, we mean a structure $\Xi = \{\langle x, A(x), \mu(x)\rangle \,|\, x \in X\}$

in which A is an interval valued fuzzy set (IVF) and μ is a fuzzy set. It is denoted by $\langle A, \mu\rangle$.

2.3 Definition: [9]Let U be an initial universe set and E be a set of parameters. Consider A ⊂ E. Let P(U) denotes the set of all neutrosophic sets of U. The collection (F, A) is termed to be the soft neutrosophic set over U, where F is a mapping given by F : A → P(U).

2.4 Definition : [4] Let X be an universe. Then a neutrosophic (NS) set λ is an object having the form

$\lambda = \{< x : T(x), I(x), F(x) >: x \in X\}$

where the functions T, I, F : X → $]^{-}0, 1^{+}[$ defines respectively the degree of Truth, the degree of indeterminacy, and the degree of Falsehood of the element x ∈ X to the set λ with the condition.

$$^{-}0 \le T(x) + I(x) + F(x) \le 3^{+}$$

2.5 Definition : [7] Let X be a non-empty set. An interval neutrosophic set (INS) A in X is characterized by the truth-membership function A_T, the indeterminacy-membership function A_I and the falsity-membership function A_F. For each point x ∈ X, $A_T(x), A_I(x), A_F(x) \subseteq [0,1]$.
For two INS
$A = \{<x, [A_T^-(x), A_T^+(x)], [A_I^-(x), A_I^+(x)], [A_F^-(x), A_F^+(x)]>: x \in X\}$
and
$B = \{<x, [B_T^-(x), B_T^+(x)], [B_I^-(x), B_I^+(x)], [B_F^-(x), B_F^+(x)]>: x \in X\}$
Then,

1. $A \tilde{\subseteq} B$ if and only if
   $A_T^-(x) \le B_T^-(x), A_T^+(x) \le B_T^+(x)$
   $A_I^-(x) \ge B_I^-(x), A_I^+(x) \ge B_I^+(x)$
   $A_F^-(x) \ge B_F^-(x), A_F^+(x) \ge B_F^+(x)$ for all x∈ X.
2. $A = B$ if and only if
   $A_T^-(x) = B_T^-(x), A_T^+(x) = B_T^+(x)$
   $A_I^-(x) = B_I^-(x), A_I^+(x) = B_I^+(x)$
   $A_F^-(x) = B_F^-(x), A_F^+(x) = B_F^+(x)$ for all x ∈ X.
3. $A^{\tilde{C}} = \{<x, [A_F^-(x), A_F^+(x)], [A_I^-(x), A_I^+(x)], [A_T^-(x), A_T^+(x)]>: x \in X\}$
4. $A \tilde{\cap} B = \{<x, [\min\{A_T^-(x), B_T^-(x)\}, \min\{A_T^+(x), B_T^+(x)\}], [\max\{A_I^-(x), B_I^-(x)\}, \max\{A_I^+(x), B_I^+(x)\}], [\max\{A_F^-(x), B_F^-(x)\}, \max\{A_F^+(x), B_F^+(x)\}]>: x \in X\}$
5. $A \tilde{\bigcup} B = \{<x, [\max\{A_T^-(x), B_T^-(x)\}, \max\{A_T^+(x), B_T^+(x)\}], [\min\{A_I^-(x), B_I^-(x)\}, \min\{A_I^+(x), B_I^+(x)\}], [\min\{A_F^-(x), B_F^-(x)\}, \min\{A_F^+(x), B_F^+(x)\}]>: x \in X\}$

2.6 Definition: [1]
Let U be an initial universe set. Let NC(U) denote the set of all neutrosophic cubic sets and E be the set of parameters. Let $A \subset E$ then $(P, A) = \{P(e_i) = \{<x, A_{e_i}(x), \lambda_{e_i}(x)>: x \in U\} e_i \in A \subset E\}$ where $A_{e_i}(x) = \{<x, A_{e_i}^T(x), A_{e_i}^I(x), A_{e_i}^F(x)> / x \in U\}$ is an interval neutrosophic set, $\lambda_{e_i}(x) = \{<x, (\lambda_{e_i}^T(x), \lambda_{e_i}^I(x), \lambda_{e_i}^T(x)> / x \in U\}$ is a neutrosophic set. The pair (P, A) is termed to be the neutrosophic soft cubic set over U where P is a mapping given by $P: A \to NC(U)$.

2.7 Definition: [1]
Let X be an initial universe set. A neutrosophic soft cubic set (P, A) in X is said to be

- truth-internal (briefly, T-internal) if the following inequality is valid
  $(\forall x \in X, e_i \in E)\ (A_{e_i}^{-T}(x) \le \lambda_{e_i}^T(x) \le A_{e_i}^{+T}(x)),$ (2.1)
- indeterminacy-internal (briefly, I-internal) if the following inequality is valid
  $(\forall x \in X, e_i \in E)\ (A_{e_i}^{-I}(x) \le \lambda_{e_i}^I(x) \le A_{e_i}^{+I}(x)),$ (2.2)
- falsity-internal (briefly, F-internal) if the following inequality is valid
  $(\forall x \in X, e_i \in E)(A_{e_i}^{-F}(x) \le \lambda_{e_i}^F(x) \le A_{e_i}^{+F}(x)).$ (2.3)

If a neutrosophic soft cubic set in X satisfies (2.1), (2.2) and (2.3) we say that (P, A) is an internal neutrosophic soft cubic set in X.

2.8 Definition: [1]
Let X be an initial universe set. A neutrosophic soft cubic set (P, A) in X is said to be

- truth-external (briefly, T-external) if the following inequality is valid
  $(\forall x \in X, e_i \in E)\ (\lambda_{e_i}^T(x) \notin (A_{e_i}^{-T}(x), A_{e_i}^{+T}(x))),$ (2.4)
- indeterminacy-external (briefly, I-external) if the following inequality is valid
  $(\forall x \in X, e_i \in E)\ (\lambda_{e_i}^I(x) \notin (A_{e_i}^{-I}(x), A_{e_i}^{+I}(x))),$ (2.5)
- falsity-external (briefly, F-external) if the following inequality is valid
  $(\forall x \in X, e_i \in E)\ (\lambda_{e_i}^F(x) \notin (A_{e_i}^{-F}(x), A_{e_i}^{+F}(x))).$ (2.6)

If a neutrosophic soft cubic set (P, A)) in *X* satisfies (2.4), (2.5) and (2.6), we say that (P, A) is an external neutrosophic soft cubic set in *X*.

2.9 Definition [1]
Let
$(P, I) = \{P(e_i) = \{<x, A_{e_i}(x), \lambda_{e_i}(x)>: x \in X\}\ e_i \in I\}$
and
$(Q, J) = \{Q(e_i) = B_i = \{<x, B_{e_i}(x), \mu_{e_i}(x)>: x \in X\}\ e_i \in J\}$

be two neutrosophic soft cubic sets in X. Let I and J be any two subsets of E (set of parameters), then we have the following

1. $(P, I) = (Q, J)$ if and only if the following conditions are satisfied
   a) I = J and

b) $P(e_i) = Q(e_i)$ for all $e_i \in I$ if and only if $A_{e_i}(x) = B_{e_i}(x)$ and $\lambda_{e_i}(x) = \mu_{e_i}(x)$ for all $x \in X$ corresponding to each $e_i \in I$.

2. (P, I) and (Q, J) are two neutrosophic soft cubic set then we define and denote P-order as $(P, I) \subseteq_P (Q, J)$ if and only if the following conditions are satisfied
   c) $I \subseteq J$ and
   d) $P(e_i) \leq_P Q(e_i)$ for all $e_i \in I$ if and only if $A_{e_i}(x) \subseteq B_{e_i}(x)$ and $\lambda_{e_i}(x) \leq \mu_{e_i}(x)$ for all $x \in X$ corresponding to each $e_i \in I$.
3. (P, I) and (Q, J) are two neutrosophic soft cubic set then we define and denote P- order as $(P, I) \subseteq_R (Q, J)$ if and only if the following conditions are satisfied
   e) $I \subseteq J$ and
   f) $P(e_i) \leq_R Q(e_i)$ for all $e_i \in I$ if and only if $A_{e_i}(x) \subseteq B_{e_i}(x)$ and $\lambda_{e_i}(x) \geq \mu_{e_i}(x)$ for all $x \in X$ corresponding to each $e_i \in I$.

2.10 Definition: [1]
Let (F, I) and (G, J) be two neutrosophic soft cubic sets (NSCS) in X where I and J are any two subsets of the parameteric set E. Then we define P-union of neutrosophic soft cubic set as $(F, I) \cup_P (G, J) = (H, C)$ where $C = I \cup J$

$$H(e_i) = \begin{Bmatrix} F(e_i) & \text{if } e_i \in I - J \\ G(e_i) & \text{if } e_i \in J - I \\ F(e_i) \vee_P G(e_i) & \text{if } e_i \in I \cap J \end{Bmatrix}$$

where $F(e_i) \vee_P G(e_i)$ is defined as

$F(e_i) \vee_P G(e_i)$ =
$\{< x, \max\{A_{e_i}(x), B_{e_i}(x)\}, (\lambda_{e_i} \vee \mu_{e_i})(x) >: x \in X\}$ $e_i \in I \cap J$
where $A_{e_i}(x), B_{e_i}(x)$ represent interval neutrosophic sets.
Hence
$F^T(e_i) \vee_P G^T(e_i)$ =
$\{< x, \max\{A^T_{e_i}(x), B^T_{e_i}(x)\}, (\lambda^T_{e_i} \vee \mu^T_{e_i})(x) >: x \in X\}$ $e_i \in I \cap J$,
$F^I(e_i) \vee_P G^I(e_i)$ =
$\{< x, \max\{A^I_{e_i}(x), B^I_{e_i}(x)\}, (\lambda^I_{e_i} \vee \mu^I_{e_i})(x) >: x \in X\}$ $e_i \in I \cap J$,
$F^F(e_i) \vee_P G^F(e_i)$ =
$\{< x, \max\{A^F_{e_i}(x), B^F_{e_i}(x)\}, (\lambda^F_{e_i} \vee \mu^F_{e_i})(x) >: x \in X\}$ $e_i \in I \cap J$.

2.11 Definition: [1]
Let (F, I) and (G, J) be two neutrosophic soft cubic sets (NSCS) in X where I and J are any subsets of parameter's set E.
Then we define P-intersection of neutrosophic soft cubic set as $(F, I) \cap_p (G, J) = (H, C)$ where $C = I \cap J$,
$H(e_i) = F(e_i) \wedge_P G(e_i)$
$H(e_i) = F(e_i) \wedge_P G(e_i)$ and $e_i \in I \cap J$. Here $F(e_i) \wedge_P G(e_i)$ is defined as
$F(e_i) \wedge_P G(e_i) = H(e_i)$ =
$\{< x, \min\{A_{e_i}(x), B_{e_i}(x)\}, (\lambda_{e_i} \wedge \mu_{e_i})(x) >: x \in X\}$ $e_i \in I \cap J$
.
where $A_{e_i}(x), B_{e_i}(x)$ represent interval neutrosophic sets.
Hence
$F^T(e_i) \wedge_P G^T(e_i)$ =
$\{< x, \min\{A^T_{e_i}(x), B^T_{e_i}(x)\}, (\lambda^T_{e_i} \wedge \mu^T_{e_i})(x) >: x \in X\}$ $e_i \in I \cap J$,
$F^I(e_i) \wedge_P G^I(e_i)$ =
$\{< x, \min\{A^I_{e_i}(x), B^I_{e_i}(x)\}, (\lambda^I_{e_i} \wedge \mu^I_{e_i})(x) >: x \in X\}$ $e_i \in I \cap J$,

$F^F(e_i) \wedge_P G^F(e_i)$ =
$\{< x, \min\{A^F_{e_i}(x), B^F_{e_i}(x)\}, (\lambda^F_{e_i} \wedge \mu^F_{e_i})(x) >: x \in X\}$ $e_i \in I \cap J$

3 More On P-union And P-intersection Of Neutrosophic Soft Cubic Set

Defintion: 3.1
Let
$(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and
$(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ be neutrososphic soft cubic set (NSCS) in X. Then

[1] P-OR is denoted by $(F, I) \vee_p (G, J)$ and defined as $(F, I) \vee_p (G, J) = (H, I \times J)$ where $H(\alpha_i, \beta_i) = F(\alpha_i) \cup_P G(\beta_i)$ for all $(\alpha_i, \beta_i) \in I \times J$.

[2] P-AND is denoted by $(F, I) \wedge_p (G, J)$ and defined as $(F, I) \wedge_p (G, J) = (H, I \times J)$ where $H(\alpha_i, \beta_i) = F(\alpha_i) \cap_P G(\beta_i)$ for all $(\alpha_i, \beta_i) \in I \times J$.

Example: 3.2

Let X = {x_1, x_2,x_3} be initial universe and E = {e_1, e_2} parameter's set. Let (F, I) be a neutrosophic soft cubic set over X and defined as $(F, I) = \{ F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I \}$ and

X	F(e_1)		F(e_2)	
	<Ae_1(x),	λ e_1(x) >	<Ae_2(x),	λ e_2(x) >
x_1	[0.5,0.6][0.6,0.7][0.5,0.6]	[0.4,0.5,0.6]	[0.3,0.6][0.2,0.7][0.2,0.4]	[0.3,0.4,0.4]
x_2	[0.4,0.5][0.7,0.8][0.2,0.3]	[0.5,0.6,0.6]	[0.3,0.5][0.6,0.8][0.2,0.6]	[0.4,0.7,0.5]
x_3	[0.2,0.3][0.2,0.3][0.3,0.5]	[0.3,0.4,0.6]	[0.4,0.7][0.2,0.5][0.3,0.6]	[0.5,0.6,0.6]

$(G, J) = \{ G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J \}$

X	G(e_1)		G(e_2)	
	<Be_1(x),	μ e_1(x) >	< Ae_2(x),	μ e_2(x) >
x_1	[0.7,0.9][0.3,0.5][0.3,0.4]	[0.7,0.4,0.6]	[0.4,0.7][0.1,0.3][0.1,0.2]	[0.5,0.2,0.2]
x_2	[0.5,0.6][0.3,0.7][0.1,0.2]	[0.6,0.4,0.2]	[0.4,0.6][0.4,0.7][0.2,0.5]	[0.6,0.5,0.4]
x_3	[0.3,0.4][0.1,0.2][0.2,0.4]	[0.5,0.3,0.5]	[0.5,0.8][0.1,0.4][0.1,0.4]	[0.7,0.3,0.4]

P-OR is denoted by $(H, I \times J) = (F, I) \vee_P (G, J)$

where

$I \times J = \{(e_1, e_1), (e_1, e_2), (e_2, e_1), (e_2, e_2)\}$ is defined

X	H(e_1,e_1)		H(e_1,e_2)		H(e_2,e_1)		H(e_2,e_2)	
	F(e_1) ∪ G(e_1)		F(e_1) ∪ G(e_2)		F(e_2) ∪ G(e_1)		F(e_2) ∪ G(e_1)	
x_1	[0.7,0.9][0.6,0.7][0.5,0.6]	[0.7,0.5,0.6]	[0.5,0.6][0.6,0.7][0.5,0.6]	[0.5,0.5,0.6]	[0.7,0.9][0.3,0.5][0.3,0.4]	[0.7,0.4,0.5]	[0.4,00.7][0.2,0.7][0.2,0.4]	[0.5,0.4,0.4]
x_2	[0.5,0.6][0.7,0.8][0.2,0.3]	[0..6,0.6,0.6]	[0.4,0.6][0.7,0.8][0.2,0.5]	[[0.6,0.6,0.6]	[0.5,0.6][0.6,0.8][0.2,0.6]	[0..6,0.7,0.5]	[0.4,0.6][0.6,0.8][0.2,0.6]	[0.6,0.7,0.5]
x_3	[0.3,0.4][0.2,0.3][0.3,0.5]	[0.5,0.4,0.6]	[0.5,0.8][0.2,0.3][0.3,0.5]	[0.7,0.4,0.6]	[0.4,0.7][0.2,05][0.3,06]	[0.5,0.6,0.6]	[0.5,0.8][0.2,0.5][0.3,0.6]	[0.7,0.6,0.6]

Definition:3.3

The complement of a neutrosophic soft cubic set $(F, I) = \{ F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I \}$ is denoted by $(F, I)^C$ and defined as

$(F, I)^C = \{((F, I)^c = (F^C, \neg I)\}$, where $F^C : \neg I \to NC(X)$ and

$F^c(e_i) = (F(\neg e_i))^c$ *for all* $e_i \in \neg I$

$= (F(e_i))^c$ $(as\ \neg(\neg e_i) = e_i)$

$(F, I)^C = \{(F(e_i))^C = \{< x, A^C_{e_i}(x), \lambda^C_{e_i}(x) >: x \in X\}\ e_i \in I\}$.

$(F, I)^C =$

$\{< x, \left([1 - A^{+T}_{e_i}, 1 - A^{-T}_{e_i}], [1 - A^{+I}_{e_i}, 1 - A^{-I}_{e_i}], [1 - A^{+F}_{e_i}, 1 - A^{-F}_{e_i}]\right),$
$\left(1 - \lambda^T_{e_i}, 1 - \lambda^I_{e_i}, 1 - \lambda^F_{e_i}\right) > x \in X\}\ e_i \in I$.

Example:3.4

Let X = {x_1, x_2} be initial universe and E = {e_1, e_2} parameter's set. Let (F, I) be a neutrosophic soft cubic set over X and defined as $(F, I) = \{ F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I \}$

X	F(e_1)		F(e_2)	
	< Ae_1(x), λ e_1(x) >		< Ae_2(x), λ e_2(x) >	
x_1	[0.3,0.5][0.1,0.4][0.5,0.8]	[0.6,0.5.0.7]	[0.4,0.6][0.5,0.7][0.6,0.9]	[0.5,0.4,0.4]
x_2	[0.6,0.8][0.4,0.7][0.4,0.7]	[0.7,0.5,0.3]	[0.2,0.4][0.4,0.7][0.3,0.6]	[0.3,0.7,0.8]

Then

$(F, I)^C = \{(F(e_i))^C = \{< x, A^C_{e_i}(x), \lambda^C_{e_i}(x) >: x \in X\}\ e_i \in I\}$

is defined as.

X	F^c (e_1)		F^c (e_2)	
	< A^c e_1(x), λ^c e_1(x) >		< A^c e_1(x), λ^c e_1(x) >	
x_1	[0.5,0.7][0.6,0.9][0.2,0.5]	[0.4,0.5,0.3]	[0.4,0.6][0.3,0.5][0.1,0.4]	[0.5,0.6,0.6]
x_2	[0.2,0.4][0.3,0.6][[0.3,0.6]	[0.3,0.5,0.7]	[0.6,0.8][0.3,0.6][0.4.0.7]	[0.7,0.3,0.2]

Proposition :3.5

Let X be initial universe and I, J, L and S subsets of parametric set E. Then for any neutrosophic soft cubic sets $\mathcal{A} = (F, I)$, $\mathcal{B} = (G, J)$, $\mathcal{C} = (E, L)$, $\mathcal{D} = (T, S)$ the following properties hold

(1) if $\mathcal{A} \subseteq_P \mathcal{B}$ and $\mathcal{B} \subseteq_P \mathcal{C}$ then $A \subseteq_P \mathcal{C}$.
(2) if $\mathcal{A} \subseteq_P \mathcal{B}$ then $\mathcal{B}^c \subseteq_P \mathcal{A}^c$.
(3) if $\mathcal{A} \subseteq_P \mathcal{B}$ and $\mathcal{A} \subseteq_P C$ then $\mathcal{A} \subseteq_P \mathcal{B} \cap_P \mathcal{C}$.
(4) if $\mathcal{A} \subseteq_P \mathcal{B}$ and $\mathcal{C} \subseteq_P \mathcal{B}$ then $\mathcal{A} \cup_P \mathcal{C} \subseteq_P \mathcal{B}$.
(5) if $\mathcal{A} \subseteq_P \mathcal{B}$ and $\mathcal{C} \subseteq_P \mathcal{D}$ then $\mathcal{A} \cup_P \mathcal{C} \subseteq_P \mathcal{B} \cup_P \mathcal{D}$ and $\mathcal{A} \cap_P \mathcal{C} \subseteq_P \mathcal{B} \cap_P \mathcal{D}$.

Proof: Proof is straight forward

Theorem:3.6 Let (F, I) be a neutrosophic soft cubic set over X.

(1) If (F, I) is an internal neutrosophic soft cubic set, then $(F, I)^c$ is also an internal neutrosophic soft cubic set (INSCS).

(2) If (F, I) is an external neutrosophic soft cubic set, then $(F, I)^c$ is also an external Neutrosophic soft cubic set (ENSCS).

Proof.

(1) Given

$(F, I) = \{ F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\} \ e_i \in I \}$

is an INSCS this implies

$A_{e_i}^{-T}(x) \le \lambda_{e_i}^{T}(x) \le A_{e_i}^{+T}(x)$,

$A_{e_i}^{-I}(x) \le \lambda_{e_i}^{I}(x) \le A_{e_i}^{+I}(x)$,

$A_{e_i}^{-F}(x) \le \lambda_{e_i}^{F}(x) \le A_{e_i}^{+F}(x)$,

for all $e_i \in I$ and for all $x \in X$.

thisimplies

$1 - A_{e_i}^{+T}(x) \le 1 - \lambda_{e_i}^{T}(x) \le 1 - A_{e_i}^{-T}(x)$,

$1 - A_{e_i}^{+I}(x) \le 1 - \lambda_{e_i}^{I}(x) \le 1 - A_{e_i}^{-I}(x)$,

$1 - A_{e_i}^{+F}(x) \le 1 - \lambda_{e_i}^{F}(x) \le 1 - A_{e_i}^{-F}(x)$

for all $e_i \in I$ and for all $x \in X$.

Hence $(F, I)^c$ is an INSCS .

(2) Given

$(F, I) = \{ F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\} \ e_i \in I \}$ is an ENSCS this implies

$\lambda_{e_i}^{T}(x) \notin (A_{e_i}^{-T}(x), A_{e_i}^{+T}(x))$,

$\lambda_{e_i}^{I}(x) \notin (A_{e_i}^{-I}(x), A_{e_i}^{+I}(x))$

$\lambda_{e_i}^{F}(x) \notin (A_{e_i}^{-F}(x), A_{e_i}^{+F}(x))$

for all $e_i \in I$ and for all $x \in X$.

Since $\lambda_{e_i}^{T}(x) \notin (A_{e_i}^{-T}(x), A_{e_i}^{+T}(x))$ &

$0 \le A_{e_i}^{-T}(x) \le A_{e_i}^{+T}(x) \le 1$,

$\lambda_{e_i}^{I}(x) \notin (A_{e_i}^{-I}(x), A_{e_i}^{+I}(x)$ &

$0 \le A_{e_i}^{-I}(x) \le A_{e_i}^{+I}(x) \le 1$,

$\lambda_{e_i}^{F}(x) \notin (A_{e_i}^{-F}(x), A_{e_i}^{+F}(x))$ &

$0 \le A_{e_i}^{-F}(x) \le A_{e_i}^{+F}(x) \le 1$

So we have

$\lambda_{e_i}^{T}(x) \le A_{e_i}^{-T}(x)$ or $A_{e_i}^{+T}(x) \le \lambda_{e_i}^{T}(x)$,

$\lambda_{e_i}^{I}(x) \le A_{e_i}^{-I}(x)$ or $A_{e_i}^{+I}(x) \le \lambda_{e_i}^{I}(x)$,

$\lambda_{e_i}^{F}(x) \le A_{e_i}^{-F}(x)$ or $A_{e_i}^{+F}(x) \le \lambda_{e_i}^{F}(x)$

this implies

$1 - \lambda_{e_i}^{T}(x) \ge 1 - A_{e_i}^{-T}(x)$ or $1 - A_{e_i}^{+T}(x) \ge 1 - \lambda_{e_i}^{T}(x)$,

$1 - \lambda_{e_i}^{I}(x) \ge 1 - A_{e_i}^{-I}(x)$ or $1 - A_{e_i}^{+I}(x) \ge 1 - \lambda_{e_i}^{I}(x)$,

$1 - \lambda_{e_i}^{F}(x) \ge 1 - A_{e_i}^{-F}(x)$ or $1 - A_{e_i}^{+F}(x) \ge 1 - \lambda_{e_i}^{F}(x)$,

for all $e_i \in I$ and for all $x \in X$.

Thus $1 - \lambda_{e_i}^{T}(x) \notin (1 - A_{e_i}^{-T}(x), 1 - A_{e_i}^{+T}(x))$,

$1 - \lambda_{e_i}^{I}(x) \notin (1 - A_{e_i}^{-I}(x), 1 - A_{e_i}^{+I}(x))$,

$1 - \lambda_{e_i}^{F}(x) \notin (1 - A_{e_i}^{-F}(x), 1 - A_{e_i}^{+F}(x))$

Hence (F, I) is an ENSCS .

Theorem: 3.7

Let

$(F, I) = \{ F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\} \ e_i \in I \}$ and

$(G, J) = \{ G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\} \ e_i \in J \}$

be internal nuetrosophic cubic soft sets. Then,

(1) $(F, I) \cup_p (G, J)$ is an INSCS

(2) $(F, I) \cap_p (G, J)$ is an INSCS

Proof:

(1) Since (F, I) and (G, J) are internal neutrosophic soft cubic sets. So for (F, I) we have

$A_{e_i}^{-T}(x) \le \lambda_{e_i}^{T}(x) \le A_{e_i}^{+T}(x)$,

$A_{e_i}^{-I}(x) \le \lambda_{e_i}^{I}(x) \le A_{e_i}^{+I}(x)$, $A_{e_i}^{-F}(x) \le \lambda_{e_i}^{F}(x) \le A_{e_i}^{+F}(x)$

for all $e_i \in I$ and for all $x \in X$.

Also for (G, J) we $B_{e_i}^{-T}(x) \le \mu_{e_i}^{T}(x) \le B_{e_i}^{+T}(x)$,

$B_{e_i}^{-I}(x) \le \mu_{e_i}^{I}(x) \le B_{e_i}^{+I}(x)$, $B_{e_i}^{-F}(x) \le \mu_{e_i}^{F}(x) \le B_{e_i}^{+F}(x)$

for all $e_i \in J$ and for all $x \in X$. Then we have

$\max\{A_{e_i}^{-T}(x), B_{e_i}^{-T}(x)\} \le (\lambda_{e_i}^{T} \vee \mu_{e_i}^{T})(x) \le \max\{A_{e_i}^{+T}(x), B_{e_i}^{+T}(x)\}$,

$\max\{A_{e_i}^{-I}(x), B_{e_i}^{-I}(x)\} \le (\lambda_{e_i}^{I} \vee \mu_{e_i}^{I})(x) \le \max\{A_{e_i}^{+I}(x), B_{e_i}^{+I}(x)\}$,

$\max\{A_{e_i}^{-F}(x), B_{e_i}^{-F}(x)\} \le (\lambda_{e_i}^{F} \vee \mu_{e_i}^{F})(x) \le \max\{A_{e_i}^{+F}(x), B_{e_i}^{+F}(x)\}$,

for all $e_i \in I \cup J$ and for all $x \in X$. .

Now by definition of P-union of (F, I) and (G, J), we have $(F, I) \cup_p (G, J) = (H, C)$ where $I \cup J = C$ and

$$H(e_i) = \begin{cases} F(e_i) & if \ e \in I - J \\ G(e_i) & if \ e \in J - I \\ F(e_i) \vee_p \ G(e_i) & if \ e \in I \cap J \end{cases}$$

$if \ e_i \in I \cap J$, then $F(e_i) \vee_p G(e_i)$ is defined as

$F(e_i) \vee_p G(e_i) = H(e_i) =$

$\{< x, \max\{A_{e_i}(x), B_{e_i}(x)\}, (\lambda_{e_i} \vee \mu_{e_i})(x), x \in X, e_i \in I \cap J\}$.

where

$F^T(e_i) \vee_p G^T(e_i) =$
$\{< x, \max\{A^T_{e_i}(x), B^T_{e_i}(x)\}, (\lambda_{e_i} \vee \mu_{e_i})^T(x), x \in X, e_i \in I \cap J\},$

$F^I(e_i) \vee_p G^I(e_i) =$
$\{< x, \max\{A^I_{e_i}(x), B^I_{e_i}(x)\}, (\lambda_{e_i} \vee \mu_{e_i})^I(x), x \in X, e_i \in I \cap J\},$

$F^F(e_i) \vee_p G^F(e_i) =$
$\{< x, \max\{A^F_{e_i}(x), B^F_{e_i}(x)\}, (\lambda_{e_i} \vee \mu_{e_i})^F(x), x \in X, e_i \in I \cap J\}.$

Thus $(F, I) \cup_p (G, J)$ is an INSCS if $e_i \in I \cap J$. If $e_i \in I - J$ or $e_i \in J - I$ then the result is trivial. Hence $(F, I) \cup_p (G, J)$ is an INSCS in all cases.

(2) Since $(F, I) \cap_p (G, J) = (H, C)$ where $I \cap J = C$ and $H(e_i) = F(e_i) \wedge_p G(e_i)$. If $e_i \in I \cap J$ then $F(e_i) \wedge_p G(e_i)$ is defined as
$H(e_i) = F(e_i) \wedge_p G(e_i) =$
$\{< x, \min\{A_{e_i}(x), B_{e_i}(x)\}, (\lambda_{e_i} \wedge \mu_{e_i})(x) > x \in X, e \in I \cap J\}$. Also given that (F, I) and (G, J) are INSCS.

So far we have

$A^{-T}_{e_i}(x) \le \lambda^T_{e_i}(x) \le A^{+T}_{e_i}(x)$, $A^{-I}_{e_i}(x) \le \lambda^I_{e_i}(x) \le A^{+I}_{e_i}(x)$,
$A^{-F}_{e_i}(x) \le \lambda^F_{e_i}(x) \le A^{+F}_{e_i}(x)$

for all $e_i \in I$ and for all $x \in X$.

And for (G, J) we have $B^{-T}_{e_i}(x) \le \mu^T_{e_i}(x) \le B^{+T}_{e_i}(x)$,
$B^{-I}_{e_i}(x) \le \mu^I_{e_i}(x) \le B^{+I}_{e_i}(x)$, $B^{-F}_{e_i}(x) \le \mu^F_{e_i}(x) \le B^{+F}_{e_i}(x)$

for all $e_i \in J$ and for all $x \in X$.

$\min\{A^{-T}_{e_i}(x), B^{-T}_{e_i}(x)\} \le (\lambda^T_{e_i} \wedge \mu^T_{e_i})(x) \le \min\{A^{+T}_{e_i}(x), B^{+T}_{e_i}(x)\}$,
$\min\{A^{-I}_{e_i}(x), B^{-I}_{e_i}(x)\} \le (\lambda^I_{e_i} \wedge \mu^I_{e_i})(x) \le \min\{A^{+I}_{e_i}(x), B^{+I}_{e_i}(x)\}$
$\min\{A^{-F}_{e_i}(x), B^{-F}_{e_i}(x)\} \le (\lambda^F_{e_i} \wedge \mu^F_{e_i})(x) \le \min\{A^{+F}_{e_i}(x), B^{+F}_{e_i}(x)\}$

for all $e_i \in I \cap J$ and for all $x \in X$.

Hence $(F, I) \cap_p (G, J)$ is an INSCS .

Definition: 3.8

Given two neutrosophic soft cubic sets (NSCS) $(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and $(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$, if we interchange λ and μ,

Then the new neutrosophic soft cubic set (NSCS) are denoted and defined as
$(F, I)^* = \{F(e_i) = \{< x, A_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and
$(G, J)^* = \{G(e_i) = \{< x, B_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in J\}$ respectively.

Theorem 3.9

For two ENSCSs $(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and $(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ in X , if $(F, I)^*$ and $(G, J)^*$ are INSCS in X then $(F, I) \cup_P (G, J)$ is an INSCS in X.

Proof:

Since $(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and $(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ are ENSCS.

Then for (F, I) we have $\lambda^T_{e_i}(x) \notin (A^{-T}_{e_i}(x), A^{+T}_{e_i}(x))$,
$\lambda^I_{e_i}(x) \notin (A^{-I}_{e_i}(x), A^{+I}_{e_i}(x))$, $\lambda^F_{e_i}(x) \notin (A^{-F}_{e_i}(x), A^{+F}_{e_i}(x))$
for all $e_i \in I$ and for all $x \in X$ and (G, J) we have
$\mu^T_{e_i}(x) \notin (B^{-T}_{e_i}(x), B^{+T}_{e_i}(x))$, $\mu^I_{e_i}(x) \notin (B^{-I}_{e_i}(x), B^{+I}_{e_i}(x))$,
$\mu^F_{e_i}(x) \notin (B^{-F}_{e_i}(x), B^{+F}_{e_i}(x))$
for all $e_i \in J$ and for all $x \in X$. Also given that $(F, I)^* = \{F(e_i) = \{< x, A_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and $(G, J)^* = \{G(e_i) = \{< x, B_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in J\}$ are INSCS so this implies $A^{-T}_{e_i}(x) \le \mu^T_{e_i}(x) \le A^{+T}_{e_i}(x)$,
$A^{-I}_{e_i}(x) \le \mu^I_{e_i}(x) \le A^{+I}_{e_i}(x)$, $A^{-F}_{e_i}(x) \le \mu^F_{e_i}(x) \le A^{+F}_{e_i}(x)$
for all $e_i \in I$ and for all $x \in X$. And
$B^{-T}_{e_i}(x) \le \lambda^T_{e_i}(x) \le B^{+T}_{e_i}(x)$
$B^{-I}_{e_i}(x) \le \lambda^I_{e_i}(x) \le B^{+I}_{e_i}(x)$, $B^{-F}_{e_i}(x) \le \lambda^F_{e_i}(x) \le B^{+F}_{e_i}(x)$.
for all $e_i \in J$ and for all $x \in X$. Since (F, I) and (G, J) are ENSCS and $(F, I)^*$ and $(G, J)^*$ are INSCS. Thus by definition of ENSCS and INSCS all the possibilities are under

1) (a1) $\lambda^T_{e_i}(x) \le A^{-T}_{e_i}(x) \le \mu^T_{e_i}(x) \le A^{+T}_{e_i}(x)$

(a2) $\lambda^{+I}_{e_i}(x) \le A^{-I}_{e_i}(x) \le \mu^I_{e_i}(x) \le A^{+I}_{e_i}(x)$

(a3) $\lambda^F_{e_i}(x) \le A^{-F}_{e_i}(x) \le \mu^F_{e_i}(x) \le A^{+F}_{e_i}(x)$

(b1) $\mu^T_{e_i}(x) \le B^{-T}_{e_i}(x) \le \lambda^T_{e_i}(x) \le B^{+T}_{e_i}(x)$

(b2) $\mu_{e_i}^{I}(x)\le B_{e_i}^{-I}(x)\le \lambda_{e_i}^{I}(x)\le B_{e_i}^{+I}(x)$

(b3) $\mu_{e_i}^{F}(x)\le B_{e_i}^{-F}(x)\le \lambda_{e_i}^{F}(x)\le B_{e_i}^{+F}(x)$

2) (a1) $A_{e_i}^{-T}(x)\le \mu_{e_i}^{T}(x)\le A_{e_i}^{+T}(x)\le \lambda_{e_i}^{T}(x)$

(a2) $A_{e_i}^{-I}(x)\le \mu_{e_i}^{I}(x)\le A_{e_i}^{+I}(x)\le \lambda_{e_i}^{I}(x)$

(a3) $A_{e_i}^{-F}(x)\le \mu_{e_i}^{F}(x)\le A_{e_i}^{+F}(x)\le \lambda_{e_i}^{F}(x)$

(b1) $B_{e_i}^{-T}(x)\le \lambda_{e_i}^{T}(x)\le B_{e_i}^{+T}(x)\le \mu_{e_i}^{T}(x)$

(b2) $B_{e_i}^{-I}(x)\le \lambda_{e_i}^{I}(x)\le B_{e_i}^{+I}(x)\le \mu_{e_i}^{I}(x)$

(b3) $B_{e_i}^{-F}(x)\le \lambda_{e_i}^{F}(x)\le B_{e_i}^{+F}(x)\le \mu_{e_i}^{F}(x)$

3) (a1) $\lambda_{e_i}^{T}(x)\le A_{e_i}^{-T}(x)\le \mu_{e_i}^{T}(x)\le A_{e_i}^{+T}(x)$

(a2) $\lambda_{e_i}^{I}(x)\le A_{e_i}^{-I}(x)\le \mu_{e_i}^{I}(x)\le A_{e_i}^{+I}(x)$

(a3) $\lambda_{e_i}^{F}(x)\le A_{e_i}^{-F}(x)\le \mu_{e_i}^{F}(x)\le A_{e_i}^{+F}(x)$

(b1) $B_{e_i}^{-T}(x)\le \lambda_{e_i}^{T}(x)\le B_{e_i}^{+T}(x)\le \mu_{e_i}^{T}(x)$

(b2) $B_{e_i}^{-I}(x)\le \lambda_{e_i}^{I}(x)\le B_{e_i}^{+I}(x)\le \mu_{e_i}^{I}(x)$

(b3) $B_{e_i}^{-F}(x)\le \lambda_{e_i}^{F}(x)\le B_{e_i}^{+F}(x)\le \mu_{e_i}^{F}(x)$

4) (a2) $A_{e_i}^{-T}(x)\le \mu_{e_i}^{T}(x)\le A_{e_i}^{+T}(x)\le \lambda_{e_i}^{T}(x)$

(a2) $A_{e_i}^{-I}(x)\le \mu_{e_i}^{I}(x)\le A_{e_i}^{+I}(x)\le \lambda_{e_i}^{I}(x)$

(a2) $A_{e_i}^{-F}(x)\le \mu_{e_i}^{F}(x)\le A_{e_i}^{+F}(x)\le \lambda_{e_i}^{F}(x)$

(b1) $\mu_{e_i}^{T}(x)\le B_{e_i}^{-T}(x)\le \lambda_{e_i}^{T}(x)\le B_{e_i}^{+T}(x)$

(b2) $\mu_{e_i}^{I}(x)\le B_{e_i}^{-I}(x)\le \lambda_{e_i}^{I}(x)\le B_{e_i}^{+I}(x)$

(b2) $\mu_{e_i}^{F}(x)\le B_{e_i}^{-F}(x)\le \lambda_{e_i}^{F}(x)\le B_{e_i}^{+F}(x)$

Since P-union of (F, I) and (G, J) is denoted and defined as $(F, I) \cup_P (G, J) = (H, C)$ where $I \cup J = C$ and

$$H(e_i) = \begin{cases} F(e_i) & if\ e \in I - J \\ G(e_i) & if\ e \in J - I \\ F(e_i) \vee_P G(e_i) & if\ e \in I \cap J \end{cases}$$

$if\ e_i \in I \cap J$, then $F(e_i) \vee_R G(e_i)$ is defined as

$F(e_i) \vee_P G(e_i) = H(e_i) = \{< x, \max\{A_{e_i}(x), B_{e_i}(x)\}, (\lambda_{e_i} \vee \mu_{e_i})(x), x \in X, e_i \in I \cap J\}$

where

$F^T(e_i) \vee_P G^T(e_i) =$

$$\left\{< x, \max\{A^T_{e_i}(x), B^T_{e_i}(x)\}, (\lambda^T_{e_i} \vee \mu^T_{e_i})(x), x \in X, e_i \in I \cap J\right\}$$

$F^I(e_i) \vee_P G^I(e_i) =$

$$\left\{< x, \max\{A^I_{e_i}(x), B^I_{e_i}(x)\}, (\lambda^I_{e_i} \vee \mu^I_{e_i}(x), x \in X, e_i \in I \cap J\right\}$$

,

$F^F(e_i) \vee_P G^F(e_i) =$

$$\left\{< x, \max\{A^F_{e_i}(x), B^F_{e_i}(x)\}, (\lambda^F_{e_i} \vee \mu^F_{e_i}(x), x \in X, e_i \in I \cap J\right\}$$

for all $e_i \in I \cap J$ and for all $x \in X$.

.

Case: 1

If $H(e_i) = F(e_i)$ that is if $e_i \in I - J$ then from (1)(a1) and (2)(a1) , we have $\lambda_{e_i}^{T}(x) = A_{e_i}^{-T}(x)$ and $\lambda_{e_i}^{T}(x) = A_{e_i}^{+T}(x)$

for all $e_i \in I$ and for all $x \in X$.

Thus

$A_{e_i}^{-T}(x)\le \lambda_{e_i}^{T}(x)\le A_{e_i}^{+T}(x)$,

for all $e_i \in I - J$ and for all $x \in X$.

Similarly we can prove for (1)(a2) , (2)(a2) and (1)(a3) , (2)(a3).

Thus $A_{e_i}^{-I}(x)\le \lambda_{e_i}^{I}(x)\le A_{e_i}^{+I}(x)$ and

$A_{e_i}^{-F}(x)\le \lambda_{e_i}^{F}(x)\le A_{e_i}^{+F}(x)$,

for all $e_i \in I - J$ and for all $x \in X$.

Case: 2

If $H(e_i) = G(e_i)$ that is if $e_i \in J - I$ then from (1)(b1) and (2)(b1) , we have $\mu_{e_i}^{T}(x) = B_{e_i}^{-T}(x)$ *and* $\mu_{e_i}^{T}(x) = B_{e_i}^{+T}(x)$

for all $e_i \in I$ and for all $x \in X$. Thus

$B_{e_i}^{-T}(x)\le \mu_{e_i}^{T}(x)\le B_{e_i}^{+T}(x)$,

for all $e_i \in J - I$ and for all $x \in X$. Similarly we can prove for (1)(b2) and (2)(b2) and (1)(b3) and (2)(b3). Thus

$B_{e_i}^{-I}(x)\le \mu_{e_i}^{I}(x)\le B_{e_i}^{+I}(x)$ and

$B_{e_i}^{-F}(x)\le \mu_{e_i}^{F}(x)\le B_{e_i}^{+F}(x)$,

for all $e_i \in J - I$ and for all $x \in X$.

Case: 3

If $H(e_i) = F(e_i) \vee_P G(e_i)$ that is if $e_i \in I \cap J$, then from (1)(a1) and (1)(b1) , we have $A_{e_i}^{-T}(x)\le \lambda_{e_i}^{T}(x)\le A_{e_i}^{+T}(x)$ for all $e_i \in I$ and for all $x \in X$.

and

$B_{e_i}^{-T}(x)\le \mu_{e_i}^{T}(x)\le B_{e_i}^{+T}(x)$ for all $e_i \in J$ and for all $x \in X$.

Hence (i)
$e_i \in I \cap J$ then

$\max\{A^{-T}_{e_i}(x), B^{-T}_{e_i}(x)\} \leq \left(\lambda^T_{e_i} \vee \mu^T_{e_i}\right)(x) \leq \max\{A^{+T}_{e_i}(x), B^{+T}_{e_i}(x)\}$.

Similarly we can prove (1)(a2) , (1)(b2) and (1)(a3), (1)(b3) .
Thus

$\max\{A^{-I}_{e_i}(x), B^{-I}_{e_i}(x)\} \leq \left(\lambda^I_{e_i} \vee \mu^I_{e_i}\right)(x) \leq \max\{A^{+I}_{e_i}(x), B^{+I}_{e_i}(x)\}$

, $\max\{A^{-F}_{e_i}(x), B^{-F}_{e_i}(x)\} \leq \left(\lambda^F_{e_i} \vee \mu^F_{e_i}\right)(x) \leq \max\{A^{+F}_{e_i}(x), B^{+F}_{e_i}(x)\}$

.

Thus in all the three cases $(F, I) \cup_P (G, J)$ is an INSCS in X.

Theorem: 3.10
For two ENSCSs
$(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and
$(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ in X if
$(F, I)^* = \{F(e_i) = \{< x, A_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and
$(G, J)^* = \{G(e_i) = \{< x, B_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in J\}$ are INSCS in X then $(F, I) \cap_P (G, J)$ is an INSCS in X.

Proof: By similar way to Theorem 3.9 we can obtain the result.

Theorem: 3.11
Let
$(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and
$(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ be ENSCSs in X such that
$(F, I)^* = \{F(e_i) = \{< x, A_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in I\}$ *and*
$(G, J)^* = \{G(e_i) = \{< x, B_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in J\}$ be ENSCS in X. Then P-union of (F, I) and (G, J) is an ENSCS in X.

Proof:
Since (F, I), (G, J), $(F, I)^*$ and $(G, J)^*$ are ENSCS so by definition of an external soft cubic set for (F, I), (G, J), $(F, I)^*$ and $(G, J)^*$ we have
$\lambda^T_{e_i}(x) \notin (A^{-T}_{e_i}(x), A^{+T}_{e_i}(x))$, $\lambda^I_{e_i}(x) \notin (A^{-I}_{e_i}(x), A^{+I}_{e_i}(x))$,
$\lambda^F_{e_i}(x) \notin (A^{-F}_{e_i}(x), A^{+F}_{e_i}(x))$,
for all $e_i \in I$ and for all $x \in X$.
$\mu^T_{e_i}(x) \notin (B^{-T}_{e_i}(x), B^{+T}_{e_i}(x))$, $\mu^I_{e_i}(x) \notin (B^{-I}_{e_i}(x), B^{+I}_{e_i}(x))$,
$\mu^F_{e_i}(x) \notin (B^{-F}_{e_i}(x), B^{+F}_{e_i}(x))$ for all $e_i \in J$ and for all $x \in X$.
$\mu^T_{e_i}(x) \notin (A^{-T}_{e_i}(x), A^{+T}_{e_i}(x))$, $\mu^I_{e_i}(x) \notin (A^{-I}_{e_i}(x), A^{+I}_{e_i}(x))$,
$\mu^F_{e_i}(x) \notin (A^{-F}_{e_i}(x), A^{+F}_{e_i}(x))$
for all $e_i \in I$ and for all $x \in X$.
$\lambda^T_{e_i}(x) \notin (B^{-T}_{e_i}(x), B^{+T}_{e_i}(x))$, $\lambda^I_{e_i}(x) \notin (B^{-I}_{e_i}(x), B^{+I}_{e_i}(x))$,
$\lambda^F_{e_i}(x) \notin (B^{-F}_{e_i}(x), B^{+F}_{e_i}(x))$ for all $e_i \in J$ and for all $x \in X$ respectively.
Thus we have

$\left(\lambda^T_{e_i} \vee \mu^T_{e_i}\right)(x) \notin \{\max\{A^{-T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \max\{A^{+T}_{e_i}(x), B^{+T}_{e_i}(x)\}\}$,

$\left(\lambda^I_{e_i} \vee \mu^I_{e_i}\right)(x) \notin \{\max\{A^{-I}_{e_i}(x), B^{-I}_{e_i}(x)\}, \max\{A^{+I}_{e_i}(x), B^{+I}_{e_i}(x)\}\}$,

$\left(\lambda^F_{e_i} \vee \mu^F_{e_i}\right)(x) \notin \{\max\{A^{-F}_{e_i}(x), B^{-F}_{e_i}(x)\}, \max\{A^{+F}_{e_i}(x), B^{+F}_{e_i}(x)\}\}$

for all $e_i \in I \cap J$ and for all $x \in X$. Thus we have

$\left(\lambda_{e_i} \vee \mu_{e_i}\right)(x) \notin \max\{A_{e_i}(x), B_{e_i}(x)\}$

for all $e_i \in I \cap J$ and for all $x \in X$. Also since $(F, I) \cup_P (G, J) = (H, C)$ where $I \cup J = C$ and

$$H(e_i) = \begin{cases} F(e_i) & \text{if } e \in I - J \\ G(e_i) & \text{if } e \in J - I \\ F(e_i) \vee_p G(e_i) & \text{if } e \in I \cap J \end{cases}$$

if $e \in I \cap J$, then $F(e_i) \vee_p G(e_i)$ is defined as

$F(e_i) \vee_p G(e_i) = H(e_i) =$
$\{< x, \max\{A_{e_i}(x), B_{e_i}(x)\}, (\lambda_{e_i} \vee \mu_{e_i})(x), x \in X, e_i \in I \cap J\}$.
where

$F^T(e_i) \vee_p G^T(e_i) =$

$$\left\{< x, \max\{A^T_{e_i}(x), B^T_{e_i}(x)\}, \left(\lambda^T_{e_i} \vee \mu^T_{e_i}\right)(x), x \in X, e_i \in I \cap J\right\}$$

$F^I(e_i) \vee_p G^I(e_i) =$

$$\left\{< x, \max\{A^I_{e_i}(x), B^I_{e_i}(x)\}, \left(\lambda^I_{e_i} \vee \mu^I_{e_i}\right)(x), x \in X, e_i \in I \cap J\right\}$$

$F^F(e_i) \vee_p G^F(e_i) =$

$\left\{< x, \max\{A^F_{e_i}(x), B^F_{e_i}(x)\}, \left(\lambda^F_{e_i} \vee \mu^F_{e_i}\right)(x), x \in X, e_i \in I \cap J\right\}$

By definition of an external soft cubic set $(F, I) \cup_P (G, J)$ is an ENSCS in X.

Example: 3.12

Let (P, I) and (Q, J) be neutrosophic soft cubic sets in X where

$(P, I) = P(e_1)$
$= \{< x, ([0.3,0.5],[0.2,0.5],[0.5,0.7]), (0.8,0.3,0.4) > e_1 \in I\}$

$(Q, J) = Q(e_1)$
$= \{< x, ([0.7,0.9][0.6,0.8][0.4,0.7]), (0.4,0.7,0.3) > e_1 \in J\}$

for all $x \in X$

Then (P, I) and (Q, J) are T-external neutrosophic cubic sets in X and $(P, I) \cap_P (Q, J) = (P, I) \bigcap (Q, J) = P \cap Q(e_1)$
$= \{< x, ([0.3,0.5][0.2,0.5],[0.4,0.7]), (0.4,0.3,0.3) > e_1 \in I \cap J\}$
for all $x \in X$. $(P, I) \cap_P (Q, J)$ is not an T-external neutrosophic cubic set since $\left(\lambda^T_{e_1} \wedge \mu^T_{e_1}\right)(x) = 0.4 \in (0.3, 0.5) = \left(\left(A^T_{e_1} \cap B^T_{e_1}\right)^{-}(x), \left(A^T_{e_1} \cap B^T_{e_1}\right)^{+}(x)\right)$

From the above example it is clear that P-intersection of T-external neutrosophic soft cubic sets may not be an T-external neutrosophic soft cubic set. We provide a condition for the P-intersection of T-external (resp. I-external and F-external) neutrosophic soft cubic sets to be T-external (resp. I-external and F-external) neutrosophic soft cubic set.

Theorem: 3.13

Let

$(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) > : x \in X\}\ e_i \in I\}$ and

$(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) > : x \in X\}\ e_i \in J\}$ be

T- ENSCSs in X such that

$$\left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \in \left(\begin{array}{l} \max\left\{\{\min\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \min\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\}, \\ \min\left\{\max\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \max\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\} \end{array}\right) \quad (3.7)$$

for all $e_i \in I$ and for all $e_i \in J$ and for all $x \in X$.

Then $(F, I) \cap_P (G, J)$ is also an T- ENSCS.

Proof

Consider $(F, I) \cap_P (G, J) = (H, C)$ where $I \cap J = C$

where $H(e_i) = F(e_i) \wedge_p G(e_i)$ is defined as

$F(e_i) \wedge_p G(e_i) = H(e_i)$
$= \left\{< x, \min\{A_{e_i}(x), B_{e_i}(x)\}, (\lambda_{e_i} \wedge \mu_{e_i})(x), x \in X, e_i \in I \cap J\right\}$.

For each $e \in I \cap J$,

Take

$\alpha^T_{e_i} = \min\left\{\max\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \max\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\}$ and

$\beta^T_{e_i} = \max\left\{\{\min\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \min\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\}$

.

Then $\alpha^T_{e_i}$ is one of $A^{-T}_{e_i}(x)$, $B^{-T}_{e_i}(x)$, $A^{+T}_{e_i}(x)$, $B^{+T}_{e_i}(x)$.

Now we consider $\alpha^T_{e_i} = A^{-T}_{e_i}(x)$ or $A^{+T}_{e_i}(x)$ only, as the remaining cases are similar to this one.

If $\alpha^T_{e_i} = A^{-T}_{e_i}(x)$ then

$B^{-T}_{e_i}(x) \le B^{+T}_{e_i}(x) \le A^{-T}_{e_i}(x) \le A^{+T}_{e_i}(x)$ and so $\beta^T_{e_i} = B^{+T}_{e_i}(x)$

thus $B^{-T}_{e_i}(x) = (A^T_{e_i} \cap B^T_{e_i})^{-}(x) \le (A^T_{e_i} \cap B^T_{e_i})^{+}(x) = B^{+T}_{e_i}(x) = \beta^T_{e_i} < \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x)$.

Hence $\left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \notin \left((A^T_{e_i} \cap B^T_{e_i})^{-}(x), (A^T_{e_i} \cap B^T_{e_i})^{+}(x)\right)$

.

If $\alpha^T_{e_i} = A^{+T}_{e_i}(x)$, then $B^{-T}_{e_i}(x) \le A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$ and so $\beta^T_{e_i} = \max\{A^{-T}_{e_i}(x), B^{-T}_{e_i}(x)\}$.

Assume that $\beta^T_{e_i} = A^{-T}_{e_i}(x)$ then $B^{-T}_{e_i}(x) \le A^{-T}_{e_i}(x) < \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \le A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$.

So from this we can write $B^{-T}_{e_i}(x) \le A^{-T}_{e_i}(x) < \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) < A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$ or $B^{-T}_{e_i}(x) \le A^{-T}_{e_i}(x) < \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) = A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$.

For this case $B^{-T}_{e_i}(x) \le A^{-T}_{e_i}(x) < \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) < A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$ it is contradiction to the fact that (F, I) and (G, J) are T-ENSCS.

For the case $B^{-T}_{e_i}(x) \le A^{-T}_{e_i}(x) < \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) = A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$ we have $\left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \notin$

$\left((A_{e_i}^T \cap B_{e_i}^T)^-(x), A_{e_i}^T \cap B_{e_i}^T)^+(x)\right)$ because $\left(\lambda_{e_i}^T \wedge \mu_{e_i}^T\right)(x) = A_{e_i}^{+T}(x) = (A_{e_i}^T \cap B_{e_i}^T)^+(x)$. Again assume that $\beta_{e_i}^T = B_{e_i}^{-T}(x)$ then $A_{e_i}^{-T}(x) \leq B_{e_i}^{-T}(x) < \left(\lambda_{e_i}^T \wedge \mu_{e_i}^T\right)(x) \leq A_{e_i}^{+T}(x) \leq B_{e_i}^{+T}(x)$. From this we can write $A_{e_i}^{-T}(x) \leq B_{e_i}^{-T}(x) < \left(\lambda_{e_i}^T \wedge \mu_{e_i}^T\right)(x) < A_{e_i}^{+T}(x) \leq B_{e_i}^{+T}(x)$ or $A_{e_i}^{-T}(x) \leq B_{e_i}^{-T}(x) < \left(\lambda_{e_i}^T \wedge \mu_{e_i}^T\right)(x) = A_{e_i}^{+T}(x) \leq B_{e_i}^{+T}(x)$. For this case $A_{e_i}^{-T}(x) \leq B_{e_i}^{-T}(x) < \left(\lambda_{e_i}^T \wedge \mu_{e_i}^T\right)(x) < A_{e_i}^{+T}(x) \leq B_{e_i}^{+T}(x)$ it is contradiction to the fact that (F,I) and (G,J) are T-ENSCS. And if we take the case $A_{e_i}^{-T}(x) \leq B_{e_i}^{-T}(x) < \left(\lambda_{e_i}^T \wedge \mu_{e_i}^T\right)(x) = A_{e_i}^{+T}(x) \leq B_{e_i}^{+T}(x)$, we get have $\left(\lambda_{e_i}^T \wedge \mu_{e_i}^T\right)(x) \notin \left((A_{e_i}^T \cap B_{e_i}^T)^-(x), A_{e_i}^T \cap B_{e_i}^T)^+(x)\right)$ because $\left(\lambda_{e_i}^T \wedge \mu_{e_i}^T\right)(x) = A_{e_i}^{+T}(x) = (A_{e_i}^T \cap B_{e_i}^T)^+(x)$.

Hence in all the cases $(F,I) \cap_P (G,J)$ is an T-ENSCS in X.

Theorem: 3.14

Let
$(F,I) = \{ F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and
$(G,J) = \{ G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ be
I- ENSCSs in X such that

$$\left(\lambda_{e_i}^I \wedge \mu_{e_i}^I\right)(x) \in \begin{pmatrix} \min\left\{ \max\{A_{e_i}^{+I}(x), B_{e_i}^{-I}(x)\}, \max\{A_{e_i}^{-I}(x), B_{e_i}^{+I}(x)\}\right\}, \\ \max\left\{\{ \min\{A_{e_i}^{+I}(x), B_{e_i}^{-I}(x)\}, \min\{A_{e_i}^{-I}(x), B_{e_i}^{+I}(x)\}\right\} \end{pmatrix} \quad (3.8)$$

for all $e_i \in I$ and for all $e_i \in J$ and for all $x \in X$. Then $(F,I) \cap_P (G,J)$ is also an I – ENSCS

Proof:
By similar way to Theorem 3.13, we can obtain the result.

Theorem : 3.15

Let
$(F,I) = \{ F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ *and*
$(G,J) = \{ G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ be
F- ENSCSs in X such that

$$\left(\lambda_{e_i}^F \wedge \mu_{e_i}^F\right)(x) \in \begin{pmatrix} \min\left\{ \max\{A_{e_i}^{+F}(x), B_{e_i}^{-F}(x)\}, \max\{A_{e_i}^{-F}(x), B_{e_i}^{+F}(x)\}\right\}, \\ \max\left\{\{ \min\{A_{e_i}^{+F}(x), B_{e_i}^{-F}(x)\}, \min\{A_{e_i}^{-F}(x), B_{e_i}^{+F}(x)\}\right\} \end{pmatrix}$$

…………..(3.9)

for all $e_i \in I$ and for all $e_i \in J$ and for all $x \in X$. Then $(F,I) \cap_P (G,J)$ is also an F- ENSCS.

Proof : By similar way to Theorem 3.13, we can obtain the result

Corallary:3.16

Let
$(F,I) = \{ F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ *and*
$(G,J) = \{ G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ be
ENSCSs in X. Then P-intersection $(F,I) \cap_P (G,J)$ is also an ENSCS in X when the conditions (3.7), (3.8)and (3.9) are valid.

Theorem: 3.17

If neutrosophic soft cubic set
$(F,I) = \{ F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and
$(G,J) = \{ G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ in
X satisfy the following condition

$$\min\left\{ \max\{A_{e_i}^{+T}(x), B_{e_i}^{-T}(x)\}, \max\{A_{e_i}^{-T}(x), B_{e_i}^{+T}(x)\}\right\}$$
$$= \left(\lambda_{e_i}^T \wedge \mu_{e_i}^T\right)(x)$$
$$= \max\left\{\{ \min\{A_{e_i}^{+T}(x), B_{e_i}^{-T}(x)\}, \min\{A_{e_i}^{-T}(x), B_{e_i}^{+T}(x)\}\right\} \quad ….(11.1)$$

then the $(F,I) \cap_P (G,J)$ is both
an T-Internal Neutrosophic Soft Cubic Set and T-External Soft Neutrosophic Cubic Set
in X.

Proof: Consider $(F,I) \cap_P (G,J) = (H,C)$ where $I \cap J = C$ where $H(e_i) = F(e_i) \wedge_p G(e_i)$ is defined as
$F(e_i) \wedge_P G(e_i) = H(e_i) =$
$\{< x, \min\{ A_{e_i}(x), B_{e_i}(x)\}, (\lambda_{e_i} \wedge \mu_{e_i})(x) >: x \in X\}\ e_i \in I \cap J\}$
where

$F^T(e_i) \wedge_P G^T(e_i) =$
$\{< x, \min\{A^T_{e_i}(x), B^T_{e_i}(x)\}, (\lambda^T_{e_i} \wedge \mu^T_{e_i})(x) >: x \in X\}\ e_i \in I \cap J\}$ .For each $e_i \in I \cap J$ Take $\alpha^T_{e_i} = \min\left\{\max\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \max\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\}$ and $\beta^T_{e_i} = \max\left\{\{\min\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \min\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\}$. Then $\alpha^T_{e_i}$ is one of $A^{-T}_{e_i}(x)$, $B^{-T}_{e_i}(x)$, $A^{+T}_{e_i}(x)$, $B^{+T}_{e_i}(x)$. Now we consider $\alpha^T_{e_i} = A^{-T}_{e_i}(x)$, or $A^{+T}_{e_i}(x)$ only, as the remaining cases are similar to this one. If $\alpha^T_{e_i} = A^{-T}_{e_i}(x)$ then $B^{-T}_{e_i}(x) \le B^{+T}_{e_i}(x) \le A^{-T}_{e_i}(x), \le A^{+T}_{e_i}(x)$, and so $\beta^T_{e_i} = B^{+T}_{e_i}(x)$ this implies $A^{-T}_{e_i}(x) = \alpha^I_{e_i} = \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) = \beta^T_{e_i} = B^{+T}_{e_i}(x)$. Thus $B^{-T}_{e_i}(x) \le B^{+T}_{e_i}(x) = \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) = A^{-T}_{e_i}(x) \le A^{+T}_{e_i}(x)$, which implies that $\left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) = B^{+T}_{e_i}(x) = (A^T_{e_i} \cap B^T_{e_i})^+(x)$. Hence $\left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \notin \left((A^T_{e_i} \cap B^T_{e_i})^-(x), (A^T_{e_i} \cap B^T_{e_i})^+(x)\right)$ and $(A^T_{e_i} \cap B^T_{e_i})^-(x) \le \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \le (A^T_{e_i} \cap B^T_{e_i})^+(x)$. If $\alpha^T_{e_i} = A^{+T}_{e_i}(x)$ then $B^{-T}_{e_i}(x) \le A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$, and so $\left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) = A^{+T}_{e_i}(x) = (A^T_{e_i} \cap B^T_{e_i})^+(x)$. Hence $\left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \notin \left((A^T_{e_i} \cap B^T_{e_i})^-(x), (A^T_{e_i} \cap B^T_{e_i})^+(x)\right)$ and $(A^T_{e_i} \cap B^T_{e_i})^-(x) \le \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \le (A^T_{e_i} \cap B^T_{e_i})^+(x)$.
Consequently we note that $(F, I) \cap_P (G, J)$ is both T-internal neutrosophic soft cubic set and T-external soft neutrosophic cubic set in X.
Similarly we have the following theorems

Theorem 3.18
If neutrosophic soft cubic set $(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and $(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ in satisfy the following condition

$$\min\left\{\max\{A^{+I}_{e_i}(x), B^{-I}_{e_i}(x)\}, \max\{A^{-I}_{e_i}(x), B^{+I}_{e_i}(x)\}\right\}$$
$$= \left(\lambda^I_{e_i} \wedge \mu^I_{e_i}\right)(x)$$
$$= \max\left\{\{\min\{A^{+I}_{e_i}(x), B^{-I}_{e_i}(x)\}, \min\{A^{-I}_{e_i}(x), B^{+I}_{e_i}(x)\}\right\} \quad \text{.......(11.2)}$$

then the $(F, I) \cap_P (G, J)$ is both an I-internal neutrosophic soft cubic set and an I-external soft neutrosophic cubic set in X.

Theorem :3.19
If neutrosophic soft cubic set $(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and $(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ in X satisfy the following condition

$$\min\left\{\max\{A^{+F}_{e_i}(x), B^{-F}_{e_i}(x)\}, \max\{A^{-F}_{e_i}(x), B^{+F}_{e_i}(x)\}\right\}$$
$$= \left(\lambda^F_{e_i} \wedge \mu^F_{e_i}\right)(x)$$
$$= \max\left\{\{\min\{A^{+F}_{e_i}(x), B^{-F}_{e_i}(x)\}, \min\{A^{-F}_{e_i}(x), B^{+F}_{e_i}(x)\}\right\}$$

.........(11.3)then the $(F, I) \cap_P (G, J)$ is both an F-internal neutrosophic soft cubic set and an F-external soft neutrosophic cubic set in X.

Corollary:3.20
Let $(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and $(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in I\}$ be NSCSs in X. Then P-intersection $(F, I) \cap_P (G, J)$ is also an ENSCS and an INSCS in X when the conditions (11.1**)**, **(**11.2)and (11.3) are valid.

The following example shows that the P-union of T-external neutrosophic soft cubic sets may not be an T-external neutrosophic soft cubic set.

Example 3.21. Let (P, I) and (Q, J) be neutrosophic soft cubic sets in X where
$(P, I) = P(e_1) = \{< x, ([0.3, 0.5], [0.2, 0.5], [0.5, 0.7]), (0.8, 0.3, 0.4) > e_1 \in I\}$
, $(Q, J) = Q(e_1) = \{< x, ([0.7, 0.9][0.6, 0.8][0.4, 0.7]), (0.4, 0.7, 0.3) > e_1 \in J\}$

Then (P,I) and (Q,J) are T-external neutrosophic cubic sets in X and $(P,I)\cup(Q,J)=P\cup Q(e_1)$ $=\{<x,([0.7,0.9][0.6,0.8],[0.5,0.7]),(0.8,0.7,0.4)>\}$ $(P,I)\cup_p(Q,J)$ is not an T-external neutrosophic cubic set in X since $\left(\lambda_{e_1}^T\vee\mu_{e_2}^T\right)(x)=0.8\in(0.7,0.9)=\left(\left(A_{e_1}^T\cup B_{e_1}^T\right)^-(x),\left(A_{e_1}^T\cup B_{e_1}^T\right)^+(x)\right)$.

We consider a condition for the P-union of T-external (resp. I-external and F-external) neutrosophic soft cubic sets to be T-external (resp. I-external and F-external) neutrosophic soft cubic set.

Theorem 3.22

Let $(F,I)=\{F(e_i)=\{<x,A_{e_i}(x),\lambda_{e_i}(x)>:x\in X\}\ e_i\in I\}$ and $(G,J)=\{G(e_i)=\{<x,B_{e_i}(x),\mu_{e_i}(x)>:x\in X\}\ e_i\in J\}$ be T- ENSCSs in X such that

$$\left(\lambda_{e_i}^T\vee\mu_{e_i}^T\right)(x)\in\begin{pmatrix}\max\left\{\{\min\{A_{e_i}^{+T}(x),B_{e_i}^{-T}(x)\},\min\{A_{e_i}^{-T}(x),B_{e_i}^{+T}(x)\}\right\},\\ \min\left\{\max\{A_{e_i}^{+T}(x),B_{e_i}^{-T}(x)\},\max\{A_{e_i}^{-T}(x),B_{e_i}^{+T}(x)\}\right\}\end{pmatrix}$$

…………..(12.1)

for all $e_i\in I$ and for all $e_i\in J$ and for all $x\in X$. Then $(F,I)\cup_P(G,J)$ is also an T- ENSCS.

Proof:

Consider $(F,I)\cup_P(G,J)=(H,C)$ where $I\cup J=C$ and

$$H(e_i)=\begin{cases}F(e_i) & \text{if } e\in I-J\\ G(e_i) & \text{if } e\in J-I\\ F(e_i)\vee_p G(e_i) & \text{if } e\in I\cap J\end{cases}$$

where $H(e_i)=F(e_i)\vee_p G(e_i)$ is defined as

$F(e_i)\vee_p G(e_i)=H(e_i)=$

$\{<x,\max\{A_{e_i}(x),B_{e_i}(x)\},(\lambda_{e_i}\vee\mu_{e_i})(x),x\in X,e_i\in I\cap J\}$, where

$F^T(e_i)\vee_p G^T(e_i)=$

$\{<x,\max\{A_{e_i}^T(x),B_{e_i}^T(x)\},(\lambda_{e_i}^T\vee\mu_{e_i}^T)(x),x\in X,e_i\in I\cap J\}$,

If $e_i\in I\cap J$,

$$\alpha_{e_i}^T=\min\left\{\max\{A_{e_i}^{+T}(x),B_{e_i}^{-T}(x)\},\max\{A_{e_i}^{-T}(x),B_{e_i}^{+T}(x)\}\right\}$$

and

$$\beta_{e_i}^T=\max\left\{\{\min\{A_{e_i}^{+T}(x),B_{e_i}^{-T}(x)\},\min\{A_{e_i}^{-T}(x),B_{e_i}^{+T}(x)\}\right\}$$

Then $\alpha_{e_i}^T$ is one of $A_{e_i}^{-T}(x)$, $B_{e_i}^{-T}(x)$, $\alpha_{e_i}^T$ $A_{e_i}^{+T}(x)$, $B_{e_i}^{+T}(x)$. Now we consider $\alpha_{e_i}^T=A_{e_i}^{-T}(x)$ or $A_{e_i}^{+T}(x)$, only as the remaining cases are similar to this one.

If $\alpha_{e_i}^T=A_{e_i}^{-T}(x)$ then $B_{e_i}^{-T}(x)\le B_{e_i}^{+T}(x)\le A_{e_i}^{-T}(x)\le A_{e_i}^{+T}(x)$, and so $\beta_{e_i}^T=B_{e_i}^{+T}(x)$ .Thus $(A_{e_i}^T\cup B_{e_i}^T)^-(x)=A_{e_i}^{-T}(x)=\alpha_{e_i}^T>\left(\lambda_{e_i}^T\vee\mu_{e_i}^T\right)(x)$. Hence $\left(\lambda_{e_i}^T\vee\mu_{e_i}^T\right)(x)\notin\left((A_{e_i}^T\cup B_{e_i}^T)^-(x),(A_{e_i}^T\cup B_{e_i}^T)^+(x)\right)$. If $\alpha_{e_i}^T=A_{e_i}^{+T}(x)$, then $B_{e_i}^{-T}(x)\le A_{e_i}^{+T}(x)\le B_{e_i}^{+T}(x)$ and so $\beta_{e_i}^T=\max\{A_{e_i}^{-T}(x),B_{e_i}^{-T}(x)\}$. Assume that $\beta_{e_i}^T=A_{e_i}^{-T}(x)$ then $B_{e_i}^{-T}(x)\le A_{e_i}^{-T}(x)<\left(\lambda_{e_i}^T\vee\mu_{e_i}^T\right)(x)\le A_{e_i}^{+T}(x)\le B_{e_i}^{+T}(x)$. So from this we can write $B_{e_i}^{-T}(x)\le A_{e_i}^{-T}(x)<\left(\lambda_{e_i}^T\vee\mu_{e_i}^T\right)(x)<A_{e_i}^{+T}(x)\le B_{e_i}^{+T}(x)$ or $B_{e_i}^{-T}(x)\le A_{e_i}^{-T}(x)=\left(\lambda_{e_i}^T\vee\mu_{e_i}^T\right)(x)\le A_{e_i}^{+T}(x)\le B_{e_i}^{+T}(x)$.

For the case $B_{e_i}^{-T}(x)\le A_{e_i}^{-T}(x)<\left(\lambda_{e_i}^T\vee\mu_{e_i}^T\right)(x)<A_{e_i}^{+T}(x)\le B_{e_i}^{+T}(x)$ it is contradiction to the fact that (F,I) and (G,J) are T-ENSCS. For the case $B_{e_i}^{-T}(x)\le A_{e_i}^{-T}(x)=\left(\lambda_{e_i}^T\vee\mu_{e_i}^T\right)(x)\le A_{e_i}^{+T}(x)\le B_{e_i}^{+T}(x)$ we have $\left(\lambda_{e_i}^T\vee\mu_{e_i}^T\right)(x)\notin\left((A_{e_i}^T\cup B_{e_i}^T)^-(x),(A_{e_i}^T\cup B_{e_i}^T)^+(x)\right)$ because $(A_{e_i}^T\cup B_{e_i}^T)^-(x)=A_{e_i}^{-T}(x)=\left(\lambda_{e_i}^T\vee\mu_{e_i}^T\right)(x)$.

Again assume that $\beta_{e_i}^T=B_{e_i}^{-T}(x)$ then $A_{e_i}^{-T}(x)\le B_{e_i}^{-T}(x)\le\left(\lambda_{e_i}^T\vee\mu_{e_i}^T\right)(x)\le A_{e_i}^{+T}(x)\le B_{e_i}^{+T}(x)$, so from this we can write $A_{e_i}^{-T}(x) \le B_{e_i}^{-T}(x) < \left(\lambda_{e_i}^T \vee \mu_{e_i}^T\right)(x) < A_{e_i}^{+T}(x) \le B_{e_i}^{+T}(x)$ or $A_{e_i}^{-T}(x) \le B_{e_i}^{-T}(x) = \left(\lambda_{e_i}^T \vee \mu_{e_i}^T\right)(x) < A_{e_i}^{+T}(x) \le B_{e_i}^{+T}(x)$. For this case $A_{e_i}^{-T}(x) \le B_{e_i}^{-T}(x) < \left(\lambda_{e_i}^T \vee \mu_{e_i}^T\right)(x) < A_{e_i}^{+T}(x) \le B_{e_i}^{+T}(x)$ it is contradiction to the fact that (F,I) and (G,J) are T-ENSCS. And if we take the case $A_{e_i}^{-T}(x) \le B_{e_i}^{-T}(x) = \left(\lambda_{e_i}^T \vee \mu_{e_i}^T\right)(x) \le A_{e_i}^{+T}(x) \le B_{e_i}^{+T}(x)$, we get have $\left(\lambda_{e_i}^T \vee \mu_{e_i}^T\right)(x) \notin \left((A_{e_i}^T \cup B_{e_i}^T)^-(x), A_{e_i}^T \cup B_{e_i}^T)^+(x)\right)$ because $(A_{e_i}^T \cup B_{e_i}^T)^-(x) = B_{e_i}^{-T}(x) = \left(\lambda_{e_i}^T \vee \mu_{e_i}^T\right)(x)$. If $e_i \in I - J$ *or* $e_i \in J - I$,then we have trivial result. Hence $(F,I) \cup_P (G,J)$ is an T-ENSCS in X.

Similarly we have the following theorems

Theorem:3.23
Let
$(F,I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and
$(G,J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ be T- ENSCSs in X such that

$$\left(\lambda_{e_i}^T \vee \mu_{e_i}^T\right)(x) \in \begin{pmatrix} \max\left\{\{\min\{A_{e_i}^{+T}(x), B_{e_i}^{-T}(x)\}, \min\{A_{e_i}^{-T}(x), B_{e_i}^{+T}(x)\}\right\}, \\ \min\left\{\max\{A_{e_i}^{+T}(x), B_{e_i}^{-T}(x)\}, \max\{A_{e_i}^{-T}(x), B_{e_i}^{+T}(x)\}\right\} \end{pmatrix}$$

…………..(12.2)
for all $e_i \in I$ *and* for all $e_i \in J$ and for all $x \in X$. Then $(F,I) \cup_P (G,J)$ is also an T- ENSCS.

Theorem :3.24
Let
$(F,I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and
$(G,J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ be T- ENSCSs in X such that

$$\left(\lambda_{e_i}^T \vee \mu_{e_i}^T\right)(x) \in \begin{pmatrix} \max\left\{\{\min\{A_{e_i}^{+T}(x), B_{e_i}^{-T}(x)\}, \min\{A_{e_i}^{-T}(x), B_{e_i}^{+T}(x)\}\right\}, \\ \min\left\{\max\{A_{e_i}^{+T}(x), B_{e_i}^{-T}(x)\}, \max\{A_{e_i}^{-T}(x), B_{e_i}^{+T}(x)\}\right\} \end{pmatrix}$$

…………..(12.3)
for all $e_i \in I$ and for all $e_i \in J$ and for all $x \in X$. Then $(F,I) \cup_P (G,J)$ is also an T- ENSCS.

Corollary:3.25
Let
$(F,I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and
$(G,J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ be ENSCSs in X. Then $(F,I) \cup_P (G,J)$ is also an ENSCS in X when the conditions (12.1**),** (12.2)and (12.3) are valid.

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