How to Effect a Composite Rotation of a Vector via Geometric (Clifford) Algebra

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Abstract

We show how to express the representation of a composite rotation in terms that allow the rotation of a vector to be calculated conveniently via a spreadsheet that uses formulas developed, previously, for a single rotation. The work presented here (which includes a sample calculation) also shows how to determine the bivector angle that produces, in a single operation, the same rotation that is effected by the composite of two rotations.

“Rotation of the vector \( \mathbf{v} \) through the bivector angle \( \mathbf{M}_1 \mu_1 \), to produce the vector \( \mathbf{v}' \).”
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1 Introduction

Suppose that we rotate some vector \( v \) through the bivector angle \( M_1\mu_1 \) to produce the vector that we shall call \( v' \) (Fig. 1), and that we then rotate \( v' \) through the bivector angle \( M_2\mu_2 \) to produce the vector that we shall call \( v'' \). That sequence of rotations is called the composition of the two rotations. It is equal to the rotation through some bivector angle \( S\sigma \) ([1], pp. 89-91). Geometric Algebra (GA) is a convenient and efficient tool for manipulating rotations—single as well as composite—as abstract symbols, but what form does a numerical calculation of a rotation take in a concrete situation? And how can we calculate the bivector angle \( S\sigma \)?

Those are two of the questions that we will address in this document. Our procedure will make use of single-rotation formulas that were developed in [2]. We’ll begin with a review of how a given vector can be rotated via GA. In that review, we’ll discuss the important concept of the representation of a rotation, after which we’ll present an formula that can be implemented in Excel for to calculate single rotations of a given vector.

Having finished that review, we’ll see how to express the representation of a composite rotation in terms that can be substituted directly in the formula for single rotations. We’ll then work a sample problem in which we’ll calculate the results of successive rotations of a vector. We’ll also calculate the bivector angle that produces the same rotation in a single operation. The method used for calculating that bivector angle is presented in the Appendix.
Figure 1: Rotation of the vector $v$ through the bivector angle $M_2\mu_2$, to produce the vector $v'$.

Figure 2: Rotation of the vector $v'$ through the bivector angle $M_2\mu_2$, to produce the vector $v''$. 
2 A Brief Review of How a Rotation of a Given Vector Can be Effected via GA

References [3] (pp. 280-286) and [1] (pp. 89-91) derive and explain the following formula for finding the new vector, $w'$, that results from the rotation of a vector $w$ through the angle $\theta$ with respect to a plane that is parallel to the unit bivector $Q$:

$$w' = \cos(\theta)w + \sin(\theta)R_\theta(w)$$

Figure 4: Rotation of the vector $w$ through the bivector angle $Q_1$, to produce the vector $w'$. 
\[
\mathbf{w}' = \begin{bmatrix} e^{-\mathbf{Q}/2} & \mathbf{w} \\ \mathbf{w}^T & e^{\mathbf{Q}/2} \end{bmatrix}. \tag{2.1}
\]

For our convenience later in this document, we will follow Reference [1] (p. 89) in saying that the factor \(e^{-\mathbf{Q}/2}\) represents the rotation \(R_{\mathbf{Q}/2}\). That factor is a quaternion, but in GA terms it is a multivector:

\[
e^{-\mathbf{Q}/2} = \cos \frac{\theta}{2} - \mathbf{Q} \sin \frac{\theta}{2}. \tag{2.2}
\]

As further preparation for work that we’ll do later, we’ll mention that for any given right-handed reference system with orthonormal basis vectors \(\hat{a}, \hat{b}, \) and \(\hat{c},\) we may express the unit bivector \(\mathbf{Q}\) as a linear combination of the basis bivectors \(\hat{a}\hat{b}, \hat{b}\hat{c}, \) and \(\hat{a}\hat{c} :\)

\[
\mathbf{Q} = \hat{a}\hat{b}q_{ab} + \hat{b}\hat{c}q_{bc} + \hat{a}\hat{c}q_{ac},
\]
in which \(q_{ab}, q_{bc},\) and \(q_{ac}\) are scalars, and \(q_{ab}^2 + q_{bc}^2 + q_{ac}^2 = 1.\)

To present a convenient way of calculating rotations via Excel spreadsheets, Ref. [2] built upon that idea to write \(e^{-\mathbf{Q}/2}\) as

\[
e^{-\mathbf{Q}/2} = f_o - \left(\hat{a}\hat{b}f_{ab} + \hat{b}\hat{c}f_{bc} + \hat{a}\hat{c}f_{ac}\right), \tag{2.3}
\]

with \(f_o = \cos \frac{\theta}{2}; f_{ab} = q_{ab}\sin \frac{\theta}{2}; f_{bc} = q_{bc}\sin \frac{\theta}{2};\) and \(f_{ac} = q_{ac}\sin \frac{\theta}{2}.\) Similarly,

\[
e^{\mathbf{Q}/2} = f_o + \left(\hat{a}\hat{b}f_{ab} + \hat{b}\hat{c}f_{bc} + \hat{a}\hat{c}f_{ac}\right). \tag{2.4}
\]

Using these expressions for \(e^{-\mathbf{Q}/2}\) and \(e^{\mathbf{Q}/2}\), and writing \(\mathbf{w}\) as \(\mathbf{w} = \hat{a}w_a + \hat{b}w_b + \hat{c}w_c,\) Eq. (2.1) becomes

\[
\mathbf{w}' = \left[f_o - \hat{a}\hat{b}f_{ab} - \hat{b}\hat{c}f_{bc} - \hat{a}\hat{c}f_{ac}\right]\left[\hat{a}w_a + \hat{b}w_b + \hat{c}w_c\right]f_o + \hat{a}\hat{b}f_{ab} + \hat{b}\hat{c}f_{bc} + \hat{a}\hat{c}f_{ac}\right].
\]

By expanding and simplifying the right-hand side we obtain

\[
\mathbf{w}' = \hat{a}\left[w_a (f_o^2 - f_{ab}^2 + f_{bc}^2 - f_{ac}^2) + w_b (2f_o f_{ab} - 2f_{bc} f_{ac}) + w_c (2f_o f_{ac} + 2f_{bc} f_{ab})\right]
\]

\[
+ \hat{b}\left[w_a (2f_o f_{ab} - 2f_{bc} f_{ac}) + w_b (f_o^2 - f_{ab}^2 - f_{bc}^2 + f_{ac}^2) + w_c (2f_o f_{ac} - 2f_{bc} f_{ab})\right]
\]

\[
+ \hat{c}\left[w_a (2f_o f_{ac} + 2f_{bc} f_{ab}) + w_b (2f_o f_{bc} - 2f_{ac} f_{ab}) + w_c (f_o^2 + f_{ab}^2 - f_{bc}^2 - f_{ac}^2)\right]. \tag{2.5}
\]

Because this result can be implemented conveniently in (for example) a spreadsheet similar to Ref. [2], the sections that follow will show how to express the representation of a composite rotation in the form of Eq. (2.3).

3 Identifying the “Representation” of a Composite Rotation

Let’s begin by defining two unit bivectors, \(\mathbf{M}_1\) and \(\mathbf{M}_2:\)

\[
\mathbf{M}_1 = \hat{a}\hat{b}m_{1ab} + \hat{b}\hat{c}m_{1bc} + \hat{a}\hat{c}m_{1ac};
\]

\[
\mathbf{M}_2 = \hat{a}\hat{b}m_{2ab} + \hat{b}\hat{c}m_{2bc} + \hat{a}\hat{c}m_{2ac}.
\]
Now, write the rotation of a vector $v$ by the bivector angle $M_{1\mu_1}$ to produce the vector $v'$:

$$v' = \left[e^{-M_{1\mu_1}/2}\right]v\left[e^{M_{1\mu_1}/2}\right].$$

Next, we will rotate $v'$ by the bivector angle $M_{2\mu_2}$ to produce the vector $v''$:

$$v'' = \left[e^{-M_{2\mu_2}/2}\right]\left[e^{M_{2\mu_2}/2}\right]v'.$$

Combining those two equations,

$$v'' = \left[e^{-M_{2\mu_2}/2}\right]\left\{\left[e^{-M_{1\mu_1}/2}\right]v\left[e^{M_{1\mu_1}/2}\right]\right\}\left[e^{M_{2\mu_2}/2}\right].$$

The vector $v''$ was produced from $v$ via the composition of the rotations by the bivectors angles $M_{1\mu_1}$ and $M_{2\mu_2}$. The representation of that composition is the product $\left[e^{-M_{2\mu_2}/2}\right]\left[e^{-M_{1\mu_1}/2}\right]$. We’ll rewrite the previous equation to make that idea clearer:

$$v'' = \left\{\left[e^{-M_{2\mu_2}/2}\right]\left[e^{-M_{1\mu_1}/2}\right]\right\}\left\{\left[e^{M_{1\mu_1}/2}\right]\left[e^{M_{2\mu_2}/2}\right]\right\}v.$$

There exists an identifiable bivector angle —we’ll call it $S\sigma$— through which $v$ could have been rotated to produce $v''$ in a single operation rather than through the composition of rotations through $M_{1\mu_1}$ and $M_{2\mu_2}$. (See the Appendix.) But instead of going that route, let’s write $e^{-M_{1\mu_1}/2}$ and $e^{-M_{2\mu_2}/2}$ in a way that will enable us to use Eq. (2.3):

$$e^{-M_{1\mu_1}/2} = \sigma - \left(\hat{a}\hat{b}g_{ab} + \hat{b}\hat{c}g_{bc} + \hat{a}\hat{c}g_{ac}\right),$$

$$e^{-M_{2\mu_2}/2} = \sigma - \left(\hat{a}\hat{b}h_{ab} + \hat{b}\hat{c}h_{bc} + \hat{a}\hat{c}h_{ac}\right),$$

where $g_{ab} = \cos \left(\frac{\mu_1}{2}\right)g_{ab} = m_{1ab}\sin \left(\frac{\mu_1}{2}\right)$; $g_{bc} = m_{1bc}\sin \left(\frac{\mu_1}{2}\right)$; and $g_{ac} = m_{1ac}\sin \left(\frac{\mu_1}{2}\right)$, and $h_{ab} = \cos \left(\frac{\mu_2}{2}\right)h_{ab} = m_{2ab}\sin \left(\frac{\mu_2}{2}\right)h_{bc} = m_{2bc}\sin \left(\frac{\mu_2}{2}\right)$; and $h_{ac} = m_{2ac}\sin \left(\frac{\mu_2}{2}\right)$. Now, we write the representation of the the composition as

$$\left[e^{-M_{2\mu_2}/2}\right]\left[e^{-M_{1\mu_1}/2}\right]v = \left[e^{-M_{2\mu_2}/2}\right]\left[e^{-M_{1\mu_1}/2}\right]\left\{\left[e^{M_{1\mu_1}/2}\right]\left[e^{M_{2\mu_2}/2}\right]\right\}v.$$

After expanding that product and grouping like terms, the representation of the composite rotation can be written in a form identical to Eq. (2.3):

$$F_o - \left(\hat{a}\hat{b}F_{ab} + \hat{b}\hat{c}F_{bc} + \hat{a}\hat{c}F_{ac}\right),$$

with

$$F_o = \left(e^{-M_{2\mu_2}/2}e^{-M_{1\mu_1}/2}\right)_{0}$$

$$F_{ab} = h_o g_{ab} - h_{ab}g_{o} - h_{bc}g_{ac} - h_{ac}g_{bc},$$

$$F_{bc} = h_o g_{bc} + h_{bc}g_{o} - h_{ac}g_{ab} - h_{ac}g_{bc},$$

$$F_{ae} = h_o g_{ac} - h_{ab}g_{bc} + h_{bc}g_{ab} + h_{ac}g_{o}. $$

(3.2)
Therefore, with these definitions of \( \mathcal{F}_o, \mathcal{F}_{ab}, \mathcal{F}_{bc}, \) and \( \mathcal{F}_{ac} \), the vector \( \mathbf{v}' \) can be calculated from \( \mathbf{v} \) (written as \( \hat{a}v_a + \hat{b}v_b + \hat{c}v_c \)) via an equation that is analogous, term for term, with Eq. (2.5):

\[
\mathbf{v}' = \hat{a}[v_a (F_o^2 - F_{ab}^2 + F_{bc}^2 - F_{ac}^2) + v_b (2F_o F_{ab} - 2F_{bc} F_{ac}) + v_c (2F_o F_{ac} - 2F_{ab} F_{bc})] \\
+ \hat{b}[v_a (2F_o F_{ab} - 2F_{bc} F_{ac}) + v_b (F_o^2 - F_{ab}^2 - F_{bc}^2 + F_{ac}^2) + v_c (2F_o F_{bc} - 2F_{ab} F_{ac})] \\
+ \hat{c}[v_a (2F_o F_{ac} + 2F_{ab} F_{bc}) + v_b (2F_o F_{bc} - 2F_{ab} F_{ac}) + v_c (F_o^2 + F_{ab}^2 - F_{bc}^2 - F_{ac}^2)].
\]

(3.3)

At this point, you may (and should) be objecting that I’ve gotten ahead of myself. Please recall that Eq. (2.5) was derived starting from the “rotation” equation (2.1)

\[
\mathbf{w}' = [e^{-Q/2}] \mathbf{w} [e^{Q/2}].
\]

The quantities \( f_o, f_o, f_{ab}, f_{bc} \), and \( f_{ac} \) in Eq. (2.5), for which

\[
e^{-Q/2} = f_o - (\hat{a}\hat{b}f_{ab} + \hat{b}\hat{c}f_{bc} + \hat{a}\hat{c}f_{ac}),
\]

also meet the condition that

\[
e^{Q/2} = f_o + (\hat{a}\hat{b}f_{ab} + \hat{b}\hat{c}f_{bc} + \hat{a}\hat{c}f_{ac}).
\]

(3.5)

We are not justified in using \( \mathcal{F}_o, \mathcal{F}_{ab}, \mathcal{F}_{bc}, \) and \( \mathcal{F}_{ac} \) in Eq. (2.5) unless we first prove that these composite-rotation “\( \mathcal{F} \)'s”, for which

\[
\mathcal{F}_o - (\hat{a}\hat{b}\mathcal{F}_{ab} + \hat{b}\hat{c}\mathcal{F}_{bc} + \hat{a}\hat{c}\mathcal{F}_{ac}) = e^{-M_{1\mu}/2} e^{-M_{2\mu}/2},
\]

(3.6)

also meet the condition that

\[
\mathcal{F}_o + (\hat{a}\hat{b}\mathcal{F}_{ab} + \hat{b}\hat{c}\mathcal{F}_{bc} + \hat{a}\hat{c}\mathcal{F}_{ac}) = e^{M_{1\mu}/2} e^{-M_{2\mu}/2}.
\]

(3.7)

Although more-elegant proofs may well exist, “brute force and ignorance” gets the job done. We begin by writing \( e^{M_{1\mu}/2} e^{-M_{2\mu}/2} \) in a way that is analogous to that which was presented in the text that preceded Eq. (3.1):

\[
\begin{bmatrix}
g_o + (\hat{a}\hat{b}g_{ab} + \hat{b}\hat{c}g_{bc} + \hat{a}\hat{c}g_{ac}) \\
h_o + (\hat{a}\hat{b}h_{ab} + \hat{b}\hat{c}h_{bc} + \hat{a}\hat{c}h_{ac})
\end{bmatrix}
\]

\[
\begin{bmatrix}
e^{M_{1\mu}/2} \\
e^{M_{2\mu}/2}
\end{bmatrix}
\]

Expanding, simplifying, and regrouping, we find that \( e^{M_{1\mu}/2} e^{-M_{2\mu}/2} \) is indeed equal to \( \mathcal{F}_o + (\hat{a}\hat{b}\mathcal{F}_{ab} + \hat{b}\hat{c}\mathcal{F}_{bc} + \hat{a}\hat{c}\mathcal{F}_{ac}) \), as required.

4 A Sample Calculation

The vector \( \mathbf{v} = \frac{4}{3}\hat{a} - \frac{4}{3}\hat{b} + \frac{16}{3}\hat{c} \) is rotated through the bivector angle \( \hat{a}\hat{b}\pi/2 \) radians to produce a new vector, \( \mathbf{v}' \). That vector is then rotated through the bivector angle \( \left(\frac{\hat{a}\hat{b}}{\sqrt{3}} + \frac{\hat{b}\hat{c}}{\sqrt{3}} - \frac{\hat{a}\hat{c}}{\sqrt{3}}\right) \left(-\frac{2\pi}{3}\right) \) to produce vector \( \mathbf{v}'' \). Calculate
Figure 5: Rotation of \( \mathbf{v} \) through the bivector angle \( \hat{a}\hat{b}\pi/2 \), to produce the vector \( \mathbf{v}' \).

a) The vectors \( \mathbf{v}' \) and \( \mathbf{v}'' \), and

b) The bivector angle \( \mathbf{S}\sigma \) through which \( \mathbf{v} \) could have been rotated to produce \( \mathbf{v}'' \) in a single operation.

We begin by calculating vector \( \mathbf{v}' \). The rotation is diagrammed in Fig. 5.

As shown in Fig. 6, \( \mathbf{v}' = \frac{4}{3}\hat{a} + \frac{4}{3}\hat{b} + \frac{16}{3}\hat{c} \).

We'll calculate \( \mathbf{v}'' \) in two ways: as the rotation of \( \mathbf{v}' \) by the bivector angle \( \left( \frac{\hat{a}\hat{b}}{\sqrt{3}} + \frac{\hat{b}\hat{c}}{\sqrt{3}} - \frac{\hat{a}\hat{c}}{\sqrt{3}} \right) \left( -\frac{2\pi}{3} \right) \), and as the result of the rotation by the composite of the two individual rotations. The rotation of \( \mathbf{v}' \) by \( \left( \frac{\hat{a}\hat{b}}{\sqrt{3}} + \frac{\hat{b}\hat{c}}{\sqrt{3}} - \frac{\hat{a}\hat{c}}{\sqrt{3}} \right) \left( -\frac{2\pi}{3} \right) \) is diagrammed in Fig. 7. Fig. 8 shows that \( \mathbf{v}'' = \frac{4}{3}\hat{a} + \frac{16}{3}\hat{b} + \frac{4}{3}\hat{c} \).

As we can see from Fig. 9, that result agrees with that which was obtained by calculating \( \mathbf{v}'' \) in a single step, as the composition of the individual rotations. Fig. 9 also shows that the bivector angle \( \mathbf{S}\sigma \) is \( \hat{b}\hat{c} \left( -\pi/2 \right) \), which we can also write as \( \hat{c}\hat{b} \left( \pi/2 \right) \). That rotation is diagrammed in Fig. 10.

5 Summary

We have seen how to express the representation of a composite rotation in terms that allow the rotation of a vector to be calculated conveniently via a spreadsheet that used formulas developed in [2] for a single rotation. The work presented here also shows how to determine the bivector angle that produces, in a single operation, the same rotation that is effected by the composite of two rotations.
Figure 6: A spreadsheet (Ref. [5]) that uses Eq. (2.5) to calculate $\mathbf{v}'$ as the rotation of $\mathbf{v}$ through the bivector angle $\hat{\mathbf{a}} \hat{\mathbf{b}} \pi/2$.

Figure 7: Rotation of $\mathbf{v}'$. Note the significance of the negative sign of the scalar angle: the direction in which $\mathbf{v}'$ is to be rotated is contrary to the orientation of the bivector. That significance is clearer in Fig. 10.
Rotation of a Vector by a Given Bivector Angle

Derivation is part of the document that is available at https://www.slideshare.net/JamesSmith245/how-to-effect-a-desired-rotation-of-a-vector-about-a-given-axis-via-geometric-clifford-algebra

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<th>Gray fields are informational.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pink fields are checks.</td>
<td></td>
</tr>
</tbody>
</table>

The vector, $\mathbf{v}'$, to be rotated

Components $\mathbf{a}_\text{hat}, \mathbf{b}_\text{hat}, \mathbf{c}_\text{hat}$

<table>
<thead>
<tr>
<th>$\mathbf{a}_\text{hat}$</th>
<th>$\mathbf{b}_\text{hat}$</th>
<th>$\mathbf{c}_\text{hat}$</th>
</tr>
</thead>
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<tr>
<td>1.333333333</td>
<td>1.333333333</td>
<td>5.333333333</td>
</tr>
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</table>

Components of the unit bivector, $\mathbf{M}_2$

<table>
<thead>
<tr>
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<th>$\mathbf{b}<em>\text{hat}'\mathbf{c}</em>\text{hat}$</th>
<th>$\mathbf{a}<em>\text{hat}'\mathbf{c}</em>\text{hat}$</th>
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<tr>
<td>0.57735027</td>
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<td>-0.57735058</td>
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Angle of rotation $\mu_2$ in radians.

<table>
<thead>
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</thead>
<tbody>
<tr>
<td>-2.094395102</td>
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Calculated values of the factors $f$

<table>
<thead>
<tr>
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<th>$f_{ab}$</th>
<th>$f_{bc}$</th>
<th>$f_{abc}$</th>
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<tbody>
<tr>
<td>0.5</td>
<td>-0.5</td>
<td>-0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Check: Sum of the squares of the $f$'s $= 1$?

Result

The vector, $\mathbf{v}''$, that results from the rotation

Components $\mathbf{a}_\text{hat}, \mathbf{b}_\text{hat}, \mathbf{c}_\text{hat}$

<table>
<thead>
<tr>
<th>$\mathbf{a}_\text{hat}$</th>
<th>$\mathbf{b}_\text{hat}$</th>
<th>$\mathbf{c}_\text{hat}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.333333333</td>
<td>5.333333333</td>
<td>1.333333333</td>
</tr>
</tbody>
</table>

Check: $||\mathbf{v}''||=||\mathbf{v}'||$?

| $||\mathbf{v}'||$ | $||\mathbf{v}''||$ |
|------------------|------------------|
| 5.65685425       | 5.65685425       |

Figure 8: A spreadsheet (Ref. 5) that uses Eq. (2.5) to calculate $\mathbf{v}''$ as the rotation of $\mathbf{v}'$. 
Figure 9: A spreadsheet (Ref. [6]) that uses Eq. (3.2) to calculate \( v'' \) via the composite rotation of \( v \).
Figure 10: Rotation of \( \mathbf{v} \) by \( S\sigma \) to produce \( \mathbf{v}'' \) in a single operation. Note the significance of the negative sign of the scalar angle: the direction in which \( \mathbf{v}' \) rotated is contrary to the orientation of the bivector \( \hat{b}\hat{c} \), and contrary also to the direction of the rotation from \( \hat{b} \) to \( \hat{c} \).

6 Appendix: Identifying the Bivector Angle \( S\sigma \) through which the Vector \( \mathbf{v} \) can be Rotated to Produce \( \mathbf{v}'' \) in a Single Operation

Let \( \mathbf{v} \) be an arbitrary vector. We want to identify the bivector angle \( S\sigma \) through which the initial vector, \( \mathbf{v} \), can be rotated to produce the same vector \( \mathbf{v}'' \) that results from the rotation of \( \mathbf{v} \) through the composite rotation by \( M_1\mu_1 \), then by \( M_2\mu_2 \):

\[
\begin{bmatrix} e^{-M_2\mu_2/2} \end{bmatrix} \begin{bmatrix} e^{-M_1\mu_1/2} \end{bmatrix} \begin{bmatrix} \mathbf{v} \end{bmatrix} \begin{bmatrix} e^{M_1\mu_1/2} \end{bmatrix} \begin{bmatrix} e^{M_2\mu_2/2} \end{bmatrix} = \mathbf{v}' = \begin{bmatrix} e^{-S\sigma/2} \end{bmatrix} \begin{bmatrix} \mathbf{v} \end{bmatrix} \begin{bmatrix} e^{S\sigma/2} \end{bmatrix}. \tag{6.1}
\]

We want Eq. (6.1) to be true for all vectors \( \mathbf{v} \). Therefore, \( e^{S\sigma/2} \) must be equal to \( \begin{bmatrix} e^{M_1\mu_1/2} \end{bmatrix} \begin{bmatrix} e^{M_2\mu_2/2} \end{bmatrix} \), and \( e^{-S\sigma/2} \) must be equal to \( \begin{bmatrix} e^{-M_1\mu_1/2} \end{bmatrix} \begin{bmatrix} e^{M_2\mu_2/2} \end{bmatrix} \). The second of those conditions is the same as saying that the representations of the \( S\sigma \) rotation and the composite rotation must be equal. We'll write that condition using the \( F \)’s defined in Eq. (3.2), with \( S \) expressed in terms of the unit bivectors \( \hat{a}\hat{b} \), \( \hat{b}\hat{c} \), and \( \hat{a}\hat{c} \):

\[
\cos \frac{\sigma}{2} - \left( \hat{a}\hat{b}S_{ab} + \hat{b}\hat{c}S_{bc} + \hat{a}\hat{c}S_{ac} \right) \sin \frac{\sigma}{2} = F_o - \left( \hat{a}\hat{b}F_{ab} + \hat{b}\hat{c}F_{bc} + \hat{a}\hat{c}F_{ac} \right).
\]

Now, we want to identify \( \sigma \) and the coefficients of \( \hat{a}\hat{b} \), \( \hat{b}\hat{c} \), and \( \hat{a}\hat{c} \). First, we note that both sides of the previous equation are multivectors. According to the postulates of GA, two multivectors \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are equal if and only if for every grade \( k \), \( \langle \mathcal{A}_1 \rangle_k = \langle \mathcal{A}_2 \rangle_k \). Equating the scalar parts, we see that \( \cos \frac{\sigma}{2} = F_o \). Equating the bivector parts gives

\[
\left( \hat{a}\hat{b}S_{ab} + \hat{b}\hat{c}S_{bc} + \hat{a}\hat{c}S_{ac} \right) \sin \frac{\sigma}{2} = \hat{a}\hat{b}F_{ab} + \hat{b}\hat{c}F_{bc} + \hat{a}\hat{c}F_{ac}.
\]
\[ \mathbf{b}\mathbf{c}F_{bc} + \mathbf{a}\mathbf{c}F_{ac}. \]

Comparing like terms, \( S_{ab} = F_{ab}/\sin \frac{\sigma}{2} \), \( S_{bc} = F_{bc}/\sin \frac{\sigma}{2} \), and \( S_{ac} = F_{ac}/\sin \frac{\sigma}{2} \).

Next, we need to find \( \sin \frac{\sigma}{2} \). Although we could do so via \( \sin \frac{\sigma}{2} = \sqrt{1 - \cos^2 \frac{\sigma}{2}} \), for the purposes of this discussion we will use the fact that \( S \) is, by definition, a unit bivector. Therefore, \( \|S\| = 1 \), leading to

\[ \|\sin \frac{\sigma}{2}\| = \|\mathbf{a}\mathbf{b}F_{ab} + \mathbf{b}\mathbf{c}F_{bc} + \mathbf{a}\mathbf{c}F_{ac}\| = \sqrt{F^2_{ab} + F^2_{bc} + F^2_{ac}}. \]

Now, the question is whether we want to use \( \sin \frac{\sigma}{2} = +\sqrt{F^2_{ab} + F^2_{bc} + F^2_{ac}} \), or \( \sin \frac{\sigma}{2} = -\sqrt{F^2_{ab} + F^2_{bc} + F^2_{ac}} \). The truth is that we can use either: if we use 

\[ -\sqrt{F^2_{ab} + F^2_{bc} + F^2_{ac}} \]

instead of \( +\sqrt{F^2_{ab} + F^2_{bc} + F^2_{ac}} \), then the sign of \( S \) changes as well, leaving the product \( S\sin \frac{\sigma}{2} \) unaltered.

The choice having been made, we can find the scalar angle \( \sigma \) from the values of \( \sin \frac{\sigma}{2} \) and \( \cos \frac{\sigma}{2} \), thereby determining the bivector angle \( S\sigma \).

References


http://vixra.org/abs/1708.0462


https://drive.google.com/file/d/0B2C4TqxB32RRxk7tDkJmE8PwUE/view?usp=sharing.