Question 402: A Definite Integral

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abstract
This note presents a definite integral

I. Introduction: The Integral

1. $$I = \int_0^1 \sqrt[5]{\frac{x}{1+\sqrt{1-x^2}}}\, dx$$

2. $$I = 1 + \frac{\sqrt{5}}{20}\ln\left(7+3\sqrt{5}\right) - \frac{\pi}{10}\sqrt{2-\frac{2}{\sqrt{5}}} - \frac{\sqrt{5}\,\ln(2)}{20} - \frac{\ln(2)}{5}$$

II. Related Integrals

3. $$I = 5\int_0^1 \frac{x^5}{\sqrt[5]{1+\sqrt{1-x^{10}}}}\, dx$$

4. $$I = \int_0^{\pi/2} \sqrt[5]{\frac{\sin x}{1+\cos x}}\, \cos x \, dx$$

5. $$I=\int_0^{\pi/2} \sqrt[5]{\frac{\cos x}{1+\sin x}}\ \sin x\, dx$$

6. $$I=2\int_0^{\pi/4} \sqrt[5]{\tan x}\ \cos(2x)\ dx$$

7. $$I=\int_0^{\infty} \sqrt[5]{\frac{\sinh x}{1+\cosh x}}\ \frac{1}{(\cosh x)^2}\ dx$$

8. $$I=2\int_0^{\infty} \frac{\sqrt[5]{\tanh x}}{(\cosh(2x))^2}\ dx$$

9. $$I=\int_0^1 \frac{x}{\sqrt[5]{(1+x)\left(1-x^2\right)^2}}\ dx$$

10. $$I=\int_0^1 \frac{x}{\sqrt[5]{(1+x)^3(1-x)^2}}\ dx$$

11. $$I=\int_0^1 \frac{\left(1-x^5\right)^2}{1+x^{10}}\ dx$$

12. $$I=1-\frac{1}{5}\int_0^1 \frac{x^{-2/5}}{1+x}\ dx$$

III. Series

13. $$I=1-\sum_{n=0}^{\infty} 2^{-n-1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{5k+3}$$

14. $$I=\frac{25\sqrt[5]{4}}{2} \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)_n \frac{2^{-n}}{n!\,(5n+3)(5n+8)}$$

15. $$I=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{10n+1}-\frac{2}{10n+6}+\frac{1}{10n+11}\right)$$

16. $$I=1-\frac{1}{3}F\left(\left\{1,\frac{3}{5}\right\},\left\{\frac{8}{5}\right\},-1\right)=1-\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n(3/5)_n}{(8/5)_n}$$

17. $$I=1-\frac{1}{6}F\left(\{1,1\},\left\{\frac{8}{5}\right\},\frac{1}{2}\right)=1-\frac{1}{6}\sum_{n=0}^{\infty}\frac{2^{-n}\cdot n!}{(8/5)_n}$$

18. $$I=1-\frac{1}{3\sqrt[5]{8}}F\left(\left\{\frac{3}{5},\frac{3}{5}\right\},\left\{\frac{8}{5}\right\},\frac{1}{2}\right)=1-\frac{1}{3\sqrt[5]{8}}\sum_{n=0}^{\infty}\frac{2^{-n}(3/5)_n^2}{n!\,(8/5)_n}$$

Remarks:

- $\pi=4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=3.141592\ldots$
- $F(\{a,b\},\{c\},x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\,n!}x^n\ ,\ |x|<1$
- $(a)_n=a(a+1)(a+2)\ldots(a+n-1)\ ,\ (a)_0=1$

References:

[1] Boros, G., and Moll, V.H.: Irresistible Integrals , Cambridge University Press: UK, USA, 2004.
[2] Gradshteyn, I.S., and Ryzhik, I.M.: Table of Integrals, Series, and Products. 5th. ed., ed. Alan Jeffrey. Academic Press,1994.