The de-composition of the mechanical force

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Abstract

This paper is prepared to show the synthesis of the Newtonian mechanical force and its counter part the inertial force. It is shows that the Newtonian mechanical force splits into two counter forces when it is act upon a rigid body. The paper is also shows the derivation of the momentum and the kinetic energy that occur due to the presence of the inertial force.

I. Theoretical Background

The encyclopaedia Britannica defines the inertial force as "any force invoked by an observer to maintain the validity of Isaac Newton's second law of motion in a reference frame that is rotating or otherwise accelerating at a constant rate.[1]" The inertial force is the main force behind the occurrence of the inertial energy and inertial momentum that which we are going to show there existence in this paper.

II. Analysis

II.1. Curvilinear Motion

II.1.1. Rigid body in a planar rotational motion:

Referring to figure(1), a rigid body "A" of mass m is free to rotate relative to its center of mass CM as it is also can be simultaneously free to rotate relative to point O. Thus it is pivoted at these two points. If an external force $\mathbf{F}$ acts on its center of mass such that it causes it to angularly accelerate with angular acceleration $\alpha\hat{k} = (d\omega/dt)\hat{k}$ relative to the axis of rotation O, where $\hat{k}$ is a unit vector perpendicular to the plane of the motion and ω is the angular velocity of rotation of the center of mass of the rigid body relative to the axis of rotation O. Hence, the total torque, $\boldsymbol{\tau}$, will be obtained by (assuming both axis of rotation, O and CM, are parallel):

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{r}_o \times \boldsymbol{F} \ , \\ &= \boldsymbol{r}_o \times \left(m\frac{d\omega}{dt}\hat{k} \times \boldsymbol{r}_o \right) = \ \boldsymbol{r}_o \times (m\alpha\hat{k} \times \boldsymbol{r}_o) \ . \end{aligned} \tag{1}$$

where $\boldsymbol{r}_o$ is the vector position of the center of mass of the rigid body relative to the axis of rotation O. Therefore, an inertial force $\mathbf{F}_i$ will occur at any element mass m_i of the rigid body. This inertial force is known as Euler force[2]

$$\begin{aligned} \mathbf{F}_i = \mathbf{F}_{\scriptscriptstyle Euler} &= -m_i\frac{d\omega}{dt}\hat{k} \times \boldsymbol{r}_i \ , \\ &= -m_i\alpha\hat{k} \times \boldsymbol{r}_i \ . \end{aligned} \tag{2}$$

where $\omega\hat{k}$ is the same angular velocity of rotation of the center of mass of the rigid body relative to the axis of rotation O and $\boldsymbol{r}_i$ is the vector position of the point where the acceleration is measured relative to the axis of the rotation O. Thus we have

$$\boldsymbol{\tau_i} = \boldsymbol{\rho}_i \times \mathbf{F}_i \tag{3}$$

where $\boldsymbol{\tau_i}$ is the torque of the element mass m_i relative to the center of mass of the rigid body due to Euler force $\mathbf{F}_i$ and $\boldsymbol{\rho}_i$ is the vector position of the element mass relative to the center of mass of the same rigid body. Hence the total torque, $\boldsymbol{\tau}_{-}$, due to the Euler force

$$\begin{aligned}\boldsymbol{\tau}_{-} &= \sum_i \boldsymbol{\rho}_i \times \mathbf{F}_i \ , \\ &= \sum_i \boldsymbol{\rho}_i \times (-m_i \alpha \hat{k} \times \boldsymbol{r}_i) \ ,\end{aligned}$$

from figure(1), we have $\boldsymbol{r}_i = \boldsymbol{r}_O + \boldsymbol{\rho}_i$. Hence one can write

$$\begin{aligned}\boldsymbol{\tau}_{-} &= \sum_i \boldsymbol{\rho}_i \times (-m_i \alpha \hat{k} \times \boldsymbol{r}_O - m_i \alpha \hat{k} \times \boldsymbol{\rho}_i) \ , \\ &= -\sum_i m_i \boldsymbol{\rho}_i \times (\alpha \hat{k} \times \boldsymbol{r}_O) - \sum_i \boldsymbol{\rho}_i \times (m_i \alpha \hat{k} \times \boldsymbol{\rho}_i) \ ,\end{aligned}$$

from the properties of the center of mass we know that $\sum_i m_i \boldsymbol{\rho}_i = 0$. Therefore, we obtain

$$\boldsymbol{\tau}_{-} = \sum_i \boldsymbol{\rho}_i \times (m_i \alpha (-\hat{k}) \times \boldsymbol{\rho}_i) \ . \tag{4}$$

Equation(4) can be mathematically simplified by doing the cross product using the Cartesian coordinate system and converting the summation to integration over the whole body[3][4]. Therefore one will find

$$\boldsymbol{\tau}_{-} = I_{CM} \alpha (-\hat{k}) \ . \tag{5}$$

where I_{CM} is the moment of inertia of the rigid body relative to its center of mass and $\alpha(-\hat{k}) = (d\omega/dt)(-\hat{k})$ is the angular acceleration of the rigid body relative to its center of mass where it is equal in magnitude to the rigid body angular acceleration relative to the axis of rotation O —assuming both axis of rotation O and CM are parallel (planar motion). Therefore, an <u>inertial torque</u> $\boldsymbol{\tau}_{-}$ will act over the rigid body —where $\boldsymbol{\tau}_{-} \neq 0$— such that it will cause the rigid body to rotate relative to its center of mass[1] CM in a direction counter to the direction of the rotation of the center of mass of the rigid body relative to the axis of rotation O.

1.A) Derivation of inertial torque from total torque:

It is known that since the mass is constrained to a circle the tangential acceleration of the mass of the rigid body "A" is $\alpha \hat{k} \times \boldsymbol{r}_O$ and since $\mathbf{F} = m\boldsymbol{a}$ the torque equation is given by:

$$\begin{aligned}\boldsymbol{\tau} = \ & \boldsymbol{r}_O \times (m\alpha \hat{k} \times \boldsymbol{r}_O) \ , \\ &= (\boldsymbol{r}_i - \boldsymbol{\rho}_i) \times (m\alpha \hat{k} \times (\boldsymbol{r}_i - \boldsymbol{\rho}_i)) \ , \\ &= \boldsymbol{r}_i \times (m\alpha \hat{k} \times (\boldsymbol{r}_i - \boldsymbol{\rho}_i)) + \boldsymbol{\rho}_i \times (m\alpha(-\hat{k}) \times (\boldsymbol{r}_i - \boldsymbol{\rho}_i)) \ , \\ &= \boldsymbol{r}_i \times (m\alpha \hat{k} \times \boldsymbol{r}_i) \ - \ \boldsymbol{r}_i \times (m\alpha \hat{k} \times \boldsymbol{\rho}_i) \\ &\quad + \ \boldsymbol{\rho}_i \times (m\alpha(-\hat{k}) \times \boldsymbol{r}_i) \ - \ \boldsymbol{\rho}_i \times (m\alpha(-\hat{k}) \times \boldsymbol{\rho}_i) \ ,\end{aligned}$$

[1]The observation of this phenomenon can be obtain easily by rotating a metallic disk pivoted at its center or by rotating a vessel containing ice cubes floating on water and can be exercise using your hand.

since $m = \sum_i m_i$ thus we have

$$\begin{aligned}
\boldsymbol{\tau} &= \boldsymbol{r}_i \times \left(\sum_i m_i \alpha \hat{k} \times \boldsymbol{r}_i \right) \;+\; \boldsymbol{r}_i \times \left(\sum_i m_i \alpha (-\hat{k}) \times \boldsymbol{\rho}_i \right) \\
&\quad + \; \boldsymbol{\rho}_i \times \left(\sum_i m_i \alpha (-\hat{k}) \times \boldsymbol{r}_i \right) \;-\; \boldsymbol{\rho}_i \times \left(\sum_i m_i \alpha (-\hat{k}) \times \boldsymbol{\rho}_i \right) \;, \\
&= \boldsymbol{r}_i \times \left(\sum_i m_i \alpha \hat{k} \times \boldsymbol{r}_i \right) \;+\; (\boldsymbol{r}_O + \boldsymbol{\rho}_i) \times \left(\sum_i m_i \alpha (-\hat{k}) \times \boldsymbol{\rho}_i \right) \\
&\quad + \; \boldsymbol{\rho}_i \times \left(\sum_i m_i \alpha (-\hat{k}) \times (\boldsymbol{r}_O + \boldsymbol{\rho}_i) \right) \;-\; \boldsymbol{\rho}_i \times \left(\sum_i m_i \alpha (-\hat{k}) \times \boldsymbol{\rho}_i \right) \;, \\
&= \sum_i \boldsymbol{r}_i \times (m_i \alpha \hat{k} \times \boldsymbol{r}_i) \;+\; \boldsymbol{r}_O \times \left(\alpha(-\hat{k}) \times \sum_i m_i \boldsymbol{\rho}_i \right) \\
&\quad + \; \sum_i \boldsymbol{\rho}_i \times \left(m_i \alpha (-\hat{k}) \times \boldsymbol{\rho}_i \right) \;+\; \sum_i m_i \boldsymbol{\rho}_i \times (\alpha(-\hat{k}) \times \boldsymbol{r}_O) \\
&\quad + \; \sum_i \boldsymbol{\rho}_i \times (m_i \alpha (-\hat{k}) \times \boldsymbol{\rho}_i) \;-\; \sum_i \boldsymbol{\rho}_i \times (m_i \alpha (-\hat{k}) \times \boldsymbol{\rho}_i) \;,
\end{aligned}$$

Again from the properties of the center of mass we have $\sum_i m_i \boldsymbol{\rho}_i = 0$. Therefore, one finds

$$\boldsymbol{r}_O \times (m \alpha \hat{k} \times \boldsymbol{r}_O) = \sum_i \boldsymbol{r}_i \times (m_i \alpha \hat{k} \times \boldsymbol{r}_i) \;+\; \sum_i \boldsymbol{\rho}_i \times (m_i \alpha (-\hat{k}) \times \boldsymbol{\rho}_i) \quad (6)$$

The second term in the RHS is nothing other than equation(4), the inertial torque. Hence we have

$$\boldsymbol{r}_O \times (m \alpha \hat{k} \times \boldsymbol{r}_O) = I_O \alpha \hat{k} \;+\; I_{CM} \alpha (-\hat{k}) \quad (7)$$

where I_O is the moment of inertia relative to the axis of rotation O which is a perpendicular distance $\boldsymbol{r}_O$ from the centre of mass. We can write it as:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_+ + \boldsymbol{\tau}_- \quad (8)$$

where $\boldsymbol{\tau} = \boldsymbol{r}_O \times (m \alpha \hat{k} \times \boldsymbol{r}_O)$ is the total torque. $\boldsymbol{\tau}_+ = I_O \alpha \hat{k}$ is the active torque and $\boldsymbol{\tau}_- = I_{CM} \alpha (-\hat{k})$ is the inertial torque. Thus the total torque $\boldsymbol{\tau}$ is a synthesis of two torques; the active torque $\boldsymbol{\tau}_+$ and the inertial torque $\boldsymbol{\tau}_-$. There is another thing that has to be notice in equation(7) when we rearrange it:

$$I_O \alpha \hat{k} = \boldsymbol{r}_O \times (m \alpha \hat{k} \times \boldsymbol{r}_O) - I_{CM} \alpha (-\hat{k}) \;,$$

and using the identity $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \circ \boldsymbol{C})\, \boldsymbol{B} - (\boldsymbol{A} \circ \boldsymbol{B})\, \boldsymbol{C}$, we obtain

$$I_O \alpha \hat{k} = \left[(\boldsymbol{r}_O \circ \boldsymbol{r}_O)\, m \alpha \hat{k} - \left(\boldsymbol{r}_O \circ m \alpha \hat{k} \right) \boldsymbol{r}_O \right] - I_{CM} \alpha (-\hat{k}) \;,$$

Since motion is planar then $\boldsymbol{r}_O$ and $\hat{k}$ are mutually orthogonal, so that

$$\begin{aligned} I_O\alpha\hat{k} &= (\boldsymbol{r}_O \circ \boldsymbol{r}_O)\, m\alpha\hat{k} - I_{CM}\alpha(-\hat{k}) \ , \\ &= mr_O^2\ \alpha\hat{k} - I_{CM}\alpha(-\hat{k}) \ , \end{aligned}$$

Dividing by α and dotted with $\hat{k}$, one obtain

$$I_O = mr_O^2 + I_{CM} \tag{9}$$

which is nothing other than the parallel axis theorem.

1.B) Angular momentum of a rigid body in a planar rotational motion:

It is known that[5] the angular momentum of a rigid body similar to "A" in its rotational motion is given by:

$$\boldsymbol{L} = \boldsymbol{r}_O \times \boldsymbol{P} \ + \ \sum_i \boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i m_i \tag{10}$$

The first term is the angular momentum (relativ to O) of the motion of the center of mass. The second is the angular momentum of the motion relative to the center of mass. Thus we can re-express equation(10) to say

$$\boldsymbol{L} = \boldsymbol{L}_{\text{motion of CM}} + \boldsymbol{L}_{\text{motion relative to CM}}$$

It is believed that: "this division of the total angular momentum into its orbital and spin parts is especially useful because it is often true (at least to a good approximation) that the two parts are separately conserved."[6]

Since we have a rigid body that rotating relative to fixed axes then equation(10) becomes

$$\boldsymbol{L} = \boldsymbol{r}_O \times (m\omega\hat{k} \times \boldsymbol{r}_O) \ + \ \sum_i \boldsymbol{\rho}_i \times (\Omega(\pm\hat{k}) \times \boldsymbol{\rho}_i) m_i \tag{11}$$

where $\omega\hat{k}$ is the rotational angular velocity of the rigid body relative to the axis of rotation O and $\Omega(\pm\hat{k})$ is its spin angular velocity relative to its center of mass. Taking the first term in the RHS and using the fact that $\boldsymbol{r}_i = \boldsymbol{r}_O + \boldsymbol{\rho}_i$, one finds

$$\begin{aligned} \boldsymbol{r}_O \times (m\omega\hat{k} \times \boldsymbol{r}_O) &= (\boldsymbol{r}_i - \boldsymbol{\rho}_i) \times (m\omega\hat{k} \times (\boldsymbol{r}_i - \boldsymbol{\rho}_i)) \ , \\ &= (\boldsymbol{r}_i - \boldsymbol{\rho}_i) \times (m\omega\hat{k} \times \boldsymbol{r}_i - m\omega\hat{k} \times \boldsymbol{\rho}_i) \ , \\ &= (\boldsymbol{r}_i - \boldsymbol{\rho}_i) \times (m\omega\hat{k} \times \boldsymbol{r}_i) - (\boldsymbol{r}_i - \boldsymbol{\rho}_i) \times (m\omega\hat{k} \times \boldsymbol{\rho}_i) \ , \\ &= \boldsymbol{r}_i \times (m\omega\hat{k} \times \boldsymbol{r}_i) - \boldsymbol{\rho}_i \times (m\omega\hat{k} \times \boldsymbol{r}_i) - \boldsymbol{r}_i \times (m\omega\hat{k} \times \boldsymbol{\rho}_i) \\ &\quad + \boldsymbol{\rho}_i \times (m\omega\hat{k} \times \boldsymbol{\rho}_i) \ , \end{aligned}$$

since $m = \sum_i m_i$ therefore

$$\begin{aligned} \boldsymbol{r}_O \times (m\omega\hat{k} \times \boldsymbol{r}_O) &= I_O\omega\hat{k} + I_{CM}\omega\hat{k} - \boldsymbol{\rho}_i \times (m\omega\hat{k} \times \boldsymbol{r}_i) - \boldsymbol{r}_i \times (m\omega\hat{k} \times \boldsymbol{\rho}_i) \ , \\ &= I_O\omega\hat{k} + I_{CM}\omega\hat{k} - \Big[(\boldsymbol{\rho_i} \circ \boldsymbol{r_i})\, m\omega\hat{k} - \Big(\boldsymbol{\rho_i} \circ m\omega\hat{k}\Big)\, \boldsymbol{r_i}\Big] \\ &\quad - \Big[(\boldsymbol{r_i} \circ \boldsymbol{\rho_i})\, m\omega\hat{k} - \Big(\boldsymbol{r_i} \circ m\omega\hat{k}\Big)\, \boldsymbol{\rho_i}\Big] \ , \end{aligned}$$

since $\boldsymbol{\rho}_i$ and $\hat{k}$ are mutually orthogonal and so are $\boldsymbol{r}_i$ and $\hat{k}$ then we have

$$\begin{aligned}
\boldsymbol{r}_O \times (m\omega\hat{k} \times \boldsymbol{r}_O) &= I_O\omega\hat{k} + I_{CM}\omega\hat{k} - (\boldsymbol{\rho}_i \circ \boldsymbol{r}_i)\, m\omega\hat{k} - (\boldsymbol{r_i} \circ \boldsymbol{\rho}_i)\, m\omega\hat{k} \\
&= I_O\omega\hat{k} + I_{CM}\omega\hat{k} - 2\,(\boldsymbol{r_i} \circ \boldsymbol{\rho}_i)\, m\omega\hat{k} \\
&= I_O\omega\hat{k} + I_{CM}\omega\hat{k} - 2\,((\boldsymbol{r}_O + \boldsymbol{\rho}_i) \circ \boldsymbol{\rho}_i)\, m\omega\hat{k} \\
&= I_O\omega\hat{k} + I_{CM}\omega\hat{k} - 2\,(\boldsymbol{r}_O \circ \boldsymbol{\rho}_i)\, m\omega\hat{k} - 2\,(\boldsymbol{\rho}_i \circ \boldsymbol{\rho}_i)\, m\omega\hat{k} \\
&= I_O\omega\hat{k} + I_{CM}\omega\hat{k} - 2\sum_i m_i\rho_i\; r_O \cos\gamma\; \omega\hat{k} - 2\sum_i m_i\rho_i^2\; \omega\hat{k}
\end{aligned}$$

where γ is the angle between $\boldsymbol{r}_O$ and $\boldsymbol{\rho}_i$, $\sum_i m_i\rho_i = 0$ and $\sum_i m_i\rho_i^2 = I_{CM}$. Hence we have

$$\boldsymbol{r}_O \times (m\omega\hat{k} \times \boldsymbol{r}_O) = I_O\omega\hat{k} + I_{CM}\omega(-\hat{k}) \tag{12}$$

Thus the total angular momentum (equation(11)) becomes

$$\begin{aligned}
\boldsymbol{L} &= I_O\omega\hat{k} + I_{CM}\omega(-\hat{k}) + \sum_i \boldsymbol{\rho}_i \times (\Omega(\pm\hat{k}) \times \boldsymbol{\rho}_i)m_i \\
&= I_O\omega\hat{k} + I_{CM}\omega(-\hat{k}) + I_{CM}\Omega(\pm\hat{k})
\end{aligned} \tag{13}$$

where $\omega\hat{k}$ is the rotational angular velocity of the rigid body relative to the axis of rotation O, $\Omega(\pm\hat{k})$ is its spin angular velocity (arbitrary) relative to its center of mass due to the application of any arbitrary external force over any point of the rigid body other than the center of mass and $\omega(-\hat{k})$ is an additional spin velocity relative to the center of mass due to the rotation of the rigid body relative to the axis of rotation O.

Since Ω is arbitrary then it can be zero, hence we can write equation(13) as:

$$\boldsymbol{L} = \boldsymbol{L}_+ + \boldsymbol{L}_- \tag{14}$$

where $\boldsymbol{L} = \boldsymbol{r}_O \times (m\omega\hat{k} \times \boldsymbol{r}_O)$ the <u>total</u> angular momentum, $\boldsymbol{L}_+ = I_O\omega\hat{k}$ the <u>active</u> angular momentum and $\boldsymbol{L}_- = I_{CM}\omega(-\hat{k})$ the <u>inertial</u> angular momentum.

1.C) Kinetic energy of a rigid body in a planar rotational motion:

It is known that[7][8] the kinetic energy T of a rigid body similar to "A" in its rotational motion relative to the axis of rotation O is given by:

$$T = \tfrac{1}{2}m\left(\omega\hat{k} \times \boldsymbol{r}_O \circ \omega\hat{k} \times \boldsymbol{r}_O\right) + \tfrac{1}{2}I_{CM}\left(\Omega(\pm\hat{k}) \circ \Omega(\pm\hat{k})\right) \tag{15}$$

where $\omega\hat{k} \times \boldsymbol{r}_O$ is the tangential velocity of the center of mass of the rigid body relative to the axis of rotation O and $\Omega(\pm\hat{k})$ is an arbitrary rotational angular velocity relative to the center of mass of the rigid body due to the application of any arbitrary external force over any point on the rigid body other than the center of mass of it (can be initially equal to zero).

When one uses the identity $(\boldsymbol{A}\times\boldsymbol{B}\circ\boldsymbol{C}\times\boldsymbol{D}) = (\boldsymbol{A}\circ\boldsymbol{C})(\boldsymbol{B}\circ\boldsymbol{D}) - (\boldsymbol{A}\circ\boldsymbol{D})(\boldsymbol{B}\circ\boldsymbol{C})$, he will get

$$T = \tfrac{1}{2}m\left[\left(\omega\hat{k}\circ\omega\hat{k}\right)(\boldsymbol{r}_O\circ\boldsymbol{r}_O) - \left(\omega\hat{k}\circ\boldsymbol{r}_O\right)\left(\boldsymbol{r}_O\circ\omega\hat{k}\right)\right] + \tfrac{1}{2}I_{CM}\left(\Omega(\pm\hat{k})\circ\Omega(\pm\hat{k})\right) \ ,$$

Since $\hat{k}$ and $\boldsymbol{r}_O$ are mutually orthogonal, so that

$$\begin{aligned}
T &= \tfrac{1}{2}m\left(\omega\hat{k}\circ\omega\hat{k}\right)(\boldsymbol{r}_O\circ\boldsymbol{r}_O) + \tfrac{1}{2}I_{CM}\left(\Omega(\pm\hat{k})\circ\Omega(\pm\hat{k})\right) \ , \qquad (16)\\
&= \tfrac{1}{2}m\left(\omega\hat{k}\circ\omega\hat{k}\right)(\boldsymbol{r}_i-\boldsymbol{\rho}_i\circ\boldsymbol{r}_i-\boldsymbol{\rho}_i) + \tfrac{1}{2}I_{CM}\left(\Omega(\pm\hat{k})\circ\Omega(\pm\hat{k})\right) \ ,\\
&= \tfrac{1}{2}m\left(\omega\hat{k}\circ\omega\hat{k}\right)[(\boldsymbol{r}_i\circ\boldsymbol{r}_i) - 2(\boldsymbol{r}_i\circ\boldsymbol{\rho}_i) + (\boldsymbol{\rho}_i\circ\boldsymbol{\rho}_i)] + \tfrac{1}{2}I_{CM}\left(\Omega(\pm\hat{k})\circ\Omega(\pm\hat{k})\right) \ ,\\
&= \tfrac{1}{2}\sum_i m_i(\boldsymbol{r}_i\circ\boldsymbol{r}_i)\left(\omega\hat{k}\circ\omega\hat{k}\right) - \sum_i m_i(\boldsymbol{r}_i\circ\boldsymbol{\rho}_i)\left(\omega\hat{k}\circ\omega\hat{k}\right)\\
&\quad + \tfrac{1}{2}\sum_i m_i(\boldsymbol{\rho}_i\circ\boldsymbol{\rho}_i)\left(\omega\hat{k}\circ\omega\hat{k}\right) + \tfrac{1}{2}I_{CM}\left(\Omega(\pm\hat{k})\circ\Omega(\pm\hat{k})\right) \ ,\\
&= \tfrac{1}{2}I_O\left(\omega\hat{k}\circ\omega\hat{k}\right) - \sum_i m_i(\boldsymbol{r}_i\circ\boldsymbol{\rho}_i)\left(\omega\hat{k}\circ\omega\hat{k}\right) + \tfrac{1}{2}I_{CM}\left(\omega\hat{k}\circ\omega\hat{k}\right)\\
&\quad + \tfrac{1}{2}I_{CM}\left(\Omega(\pm\hat{k})\circ\omega(\pm\hat{k})\right) \ ,\\
&= \tfrac{1}{2}I_O\left(\omega\hat{k}\circ\omega\hat{k}\right) - \sum_i m_i(\boldsymbol{r}_O+\boldsymbol{\rho}_i\circ\boldsymbol{\rho}_i)\left(\omega\hat{k}\circ\omega\hat{k}\right) + \tfrac{1}{2}I_{CM}\left(\omega\hat{k}\circ\omega\hat{k}\right)\\
&\quad + \tfrac{1}{2}I_{CM}\left(\Omega(\pm\hat{k})\circ\Omega(\pm\hat{k})\right) \ ,\\
&= \tfrac{1}{2}I_O\left(\omega\hat{k}\circ\omega\hat{k}\right) - \sum_i m_i(\boldsymbol{\rho}_i\circ\boldsymbol{\rho}_i)\left(\omega\hat{k}\circ\omega\hat{k}\right) - \sum_i m_i(\boldsymbol{r}_O\circ\boldsymbol{\rho}_i)\left(\omega\hat{k}\circ\omega\hat{k}\right)\\
&\quad + \tfrac{1}{2}I_{CM}\left(\omega\hat{k}\circ\omega\hat{k}\right) + \tfrac{1}{2}I_{CM}\left(\Omega(\pm\hat{k})\circ\Omega(\pm\hat{k})\right) \ ,\\
&= \tfrac{1}{2}I_O\left(\omega\hat{k}\circ\omega\hat{k}\right) - \tfrac{1}{2}I_{CM}\left(\omega\hat{k}\circ\omega\hat{k}\right) - \left(\sum_i m_i\rho_i\right) r_O\cos\gamma\left(\omega\hat{k}\circ\omega\hat{k}\right)\\
&\quad + \tfrac{1}{2}I_{CM}\left(\Omega(\pm\hat{k})\circ\Omega(\pm\hat{k})\right) \ ,
\end{aligned}$$

where γ is the angle between $\boldsymbol{r}_O$ and $\boldsymbol{\rho}_i$. Again $\sum_i m_i\rho_i = 0$. Therefore, we obtain

$$T = \tfrac{1}{2}I_O\left(\omega\hat{k}\circ\omega\hat{k}\right) - \tfrac{1}{2}I_{CM}\left(\omega\hat{k}\circ\omega\hat{k}\right) + \tfrac{1}{2}I_{CM}\left(\Omega(\pm\hat{k})\circ\Omega(\pm\hat{k})\right) \ . \qquad (17)$$

Subtracting equation(16) from (17) again one obtain the parallel axis theorem and if we subtracted equation(15) from (17) we obtain

$$\tfrac{1}{2}m\left(\omega\hat{k}\times\boldsymbol{r}_{O}\circ\omega\hat{k}\times\boldsymbol{r}_{O}\right)=\tfrac{1}{2}I_{O}\left(\omega\hat{k}\circ\omega\hat{k}\right)-\tfrac{1}{2}I_{CM}\left(\omega\hat{k}\circ\omega\hat{k}\right)\ . \tag{18}$$

Since there is not a negative kinetic energy therefore one can understand that the negative sign is assign to direction. Thus, the correct kinetic energy formula

$$T=\tfrac{1}{2}I_{O}\left(\omega\hat{k}\circ\omega\hat{k}\right)+\tfrac{1}{2}I_{CM}\left(\omega(-\hat{k})\circ\omega(-\hat{k})\right)+\tfrac{1}{2}I_{CM}\left(\Omega(\pm\hat{k})\circ\Omega(\pm\hat{k})\right)\ . \tag{19}$$

Since Ω is arbitrary then it can be zero, hence we can write equation(19) as:

$$T=T_{+}+T_{-} \tag{20}$$

where $T=\frac{1}{2}m\left(\omega\hat{k}\times\boldsymbol{r}_{O}\circ\omega\hat{k}\times\boldsymbol{r}_{O}\right)$ the <u>total</u> kinetic energy, $T_{+}=\frac{1}{2}I_{O}\left(\omega\hat{k}\circ\omega\hat{k}\right)$ the <u>active</u> kinetic energy and $T_{-}=\frac{1}{2}I_{CM}\left(\omega(-\hat{k})\circ\omega(-\hat{k})\right)$ the <u>inertial</u> kinetic energy.

II.1.2. Rigid body in spin motion:

Referring to figure(2), a rigid body "B" of mass m is free to rotate relative to its center of mass CM. When an external force $\mathbf{F}$ acts on the rigid body at any point other than its center of mass, then this force will create a torque

$$\boldsymbol{\tau}=I_{CM}\alpha_{s}\hat{k}$$

where I_{CM} is the moment of inertia of the rigid body relative to its center of mass CM, $\alpha_{s}\hat{k}=(d\Omega/dt)\hat{k}$ is the angular acceleration of the rigid body relative to its center of mass, Ω is the angular velocity of rotation of the rigid body relative to its center of mass (it is the same arbitrary angular velocity mentioned in equation(15)) and $\hat{k}$ is a unit vector perpendicular to the plan of motion. Therefore, Euler force $\mathbf{F}_{i}$ will occur at any element mass m_{i} of that rigid body and causes an inertial torque $\boldsymbol{\tau}_{-}$ such that it will spin relative to its center of mass in a direction counter to the direction of the original motion. Hence from figure(2) we have

$$\begin{aligned}\boldsymbol{\tau}_{-}&=\sum_{i}\boldsymbol{\rho}_{i}\times\mathbf{F}_{i}\ ,\\&=\sum_{i}\boldsymbol{\rho}_{i}\times\left(-m_{i}\frac{d\Omega}{dt}\ \hat{k}\times\boldsymbol{\rho}_{i}\right)\ ,\end{aligned} \tag{21}$$

Therefore, we obtain

$$\boldsymbol{\tau}_{-}=I_{CM}\alpha_{s}(-\hat{k})\ . \tag{22}$$

where $\alpha_{s}(-\hat{k})=(d\Omega/dt)(-\hat{k})$ is the angular acceleration of the rigid body relative to its center of mass due to the inertial torque $\boldsymbol{\tau}_{-}$ and $-\hat{k}$ is a unit vector anti-parallel to the unit vector $\hat{k}$.

2.A) Derivation of inertial torque from total torque in spin motion:

Since the active and inertial torques are acting on the body at the same time an unified expression is needed. From figure(3) an element torque $\boldsymbol{\tau}_l$ about the point O is given by:

$$\boldsymbol{\tau}_l = \boldsymbol{r}_l \times (m_l \alpha_1 \hat{k} \times \boldsymbol{r}_l) \tag{23}$$

Therefore, the total torque:

$$\boldsymbol{\tau} = \sum_l \boldsymbol{r}_l \times (m_l \alpha_1 \hat{k} \times \boldsymbol{r}_l) \tag{24}$$

where α_1 is the angular acceleration of the element mass m_l relative to the point O which is perpendicular distance $\boldsymbol{r}_l$ from it. Since $\boldsymbol{r}_l = \boldsymbol{r}_j - \boldsymbol{\rho}_l$ and $\sum\limits_l m_l = \sum\limits_j m_j = m$. Hence one can write

$$\begin{aligned}
\boldsymbol{\tau} &= \sum_l (\boldsymbol{r}_j - \boldsymbol{\rho}_l) \times (m_l \alpha_1 \hat{k} \times (\boldsymbol{r}_j - \boldsymbol{\rho}_l)) \ , \\
&= \sum_j \boldsymbol{r}_j \times (m_j \alpha_1 \hat{k} \times \boldsymbol{r}_j) - \sum_l \boldsymbol{r}_j \times (m_l \alpha_1 \hat{k} \times \boldsymbol{\rho}_l) - \sum_l \boldsymbol{\rho}_l \times (m_l \alpha_1 \hat{k} \times \boldsymbol{r}_j) \\
&\quad + \sum_l \boldsymbol{\rho}_l \times (m_l \alpha_1 \hat{k} \times \boldsymbol{\rho}_l) \ ,
\end{aligned}$$

Since we can always choose $\boldsymbol{r}_j$ such that $\boldsymbol{r}_j = \boldsymbol{\rho}_l$, so that we obtain

$$\begin{aligned}
\boldsymbol{\tau} &= \sum_j \boldsymbol{r}_j \times (m_j \alpha_1 \hat{k} \times \boldsymbol{r}_j) - \sum_l \boldsymbol{\rho}_l \times (m_l \alpha_1 \hat{k} \times \boldsymbol{\rho}_l) - \sum_l \boldsymbol{\rho}_l \times (m_l \alpha_1 \hat{k} \times \boldsymbol{\rho}_l) \\
&\quad + \sum_l \boldsymbol{\rho}_l \times (m_l \alpha_1 \hat{k} \times \boldsymbol{\rho}_l) \ , \\
&= \sum_j \boldsymbol{r}_j \times (m_j \alpha_1 \hat{k} \times \boldsymbol{r}_j) + \sum_l \boldsymbol{\rho}_l \times (m_l \alpha_1 (-\hat{k}) \times \boldsymbol{\rho}_l) \ , \\
&= \quad \sum_j \left[(\boldsymbol{r}_j \circ \boldsymbol{r}_j) \, m_j \alpha_1 \hat{k} - \left(\boldsymbol{r}_j \circ m_j \alpha_1 \hat{k} \right) \boldsymbol{r}_j \right] \\
&\quad + \sum_l \left[(\boldsymbol{\rho}_l \circ \boldsymbol{\rho}_l) \, m_l \alpha_1 (-\hat{k}) - \left(\boldsymbol{\rho}_l \circ m_l \alpha_1 (-\hat{k}) \right) \boldsymbol{\rho}_l \right] \ ,
\end{aligned}$$

Since $\boldsymbol{r}_j$ and $\hat{k}$ are mutually orthogonal and so are $\boldsymbol{\rho}_l$ and $-\hat{k}$ therefore we obtain

$$\begin{aligned}
\boldsymbol{\tau} &= \sum_j (\boldsymbol{r}_j \circ \boldsymbol{r}_j) \, m_j \alpha_1 \hat{k} + \sum_l (\boldsymbol{\rho}_l \circ \boldsymbol{\rho}_l) \, m_l \alpha_1 (-\hat{k}) \ , \\
&= I_{CM} \alpha_s (+\hat{k}) + I_{CM} \alpha_s (-\hat{k}) \ .
\end{aligned} \tag{25}$$

where α_s is the angular acceleration of the rigid body relative to its center of mass (since it is the only motion allowed). The second term in the RHS is nothing other than equation(22), the inertial torque.

One have to notice that the inertial torque ($I_{CM}\alpha_s(-\hat{k})$) does not cancel the active torque ($I_{CM}\alpha_s(+\hat{k})$) instead it acts like a restoring force and that gives the mass its inertial property.

2.B) The concept of di-vector:

When we write equation(25) as $\boldsymbol{\tau} = I_{CM}\alpha_s\left[((+1)+(-1))\hat{k}\right] = I_{CM}\alpha_s\left[0\hat{k}\right]$ we find ourself forced to define a new algebraic rules for quantities between square bracket (algebraic of directions). We know that the change in magnitude of a vector can be positive (increasing) or negative (decreasing) and that can happen independently from the direction (e.g. a spinning top that spin clockwise can be angularly accelerate or decelerate). Since $I_{CM}\alpha_s \neq 0$ that implies $0\hat{k} \neq 0$ therefore we redefine $0\hat{k}$ to $1\check{k}$ or simply $\check{k}$ and name it unit di-vector such that $\pm\check{k} = (\pm\check{k}_+) + (\mp\check{k}_-)$ or generally, $\pm\check{e} = (\pm\check{e}_+) + (\mp\check{e}_-)$ where $\check{e}$ is an arbitrary unit di-vector, the $\pm$ is the sign of direction and the subscript is for type of vector; $\check{e}_+$ is for active unit vector and $\check{e}_-$ is for inertial unit vector. We can write equation(25) as:

$$\boldsymbol{\tau} = I_{CM}\alpha_s(\pm\check{k}_+) + I_{CM}\alpha_s(\mp\check{k}_-) \; .$$

$$= \boldsymbol{\tau}_+ + \boldsymbol{\tau}_- \qquad (26)$$

where $\boldsymbol{\tau} = \sum_l \boldsymbol{r}_l \times (m_l\alpha_s(\pm\check{k}) \times \boldsymbol{r}_l)$ is the total torque or di-torque, $\boldsymbol{\tau}_+ = I_{CM}\alpha_s(\pm\check{k}_+)$ is the active/positive torque and $\boldsymbol{\tau}_- = I_{CM}\alpha_s(\mp\check{k}_-)$ is the inertial/negative torque. Thus the total torque $\boldsymbol{\tau}$ is a synthesis of two torques; the active torque $\boldsymbol{\tau}_+$ and the inertial torque $\boldsymbol{\tau}_-$ or the positive and negative torques.

2.C) Angular momentum of a rigid body in a planar spin motion:

From figure(3) the angular momentum of this rigid body in its spin motion is given by:

$$\boldsymbol{L} = \sum_l \boldsymbol{r}_l \times (m_l\Omega_1\hat{k} \times \boldsymbol{r}_l) \; , \qquad (27)$$

where Ω_1 is the spin angular velocity of the element mass m_l relative to the point O which is perpendicular distance $\boldsymbol{r}_l$ from it. Since $\boldsymbol{r}_l = \boldsymbol{r}_j - \boldsymbol{\rho}_l$ and $\sum_l m_l = \sum_j m_j = m$. Hence one can write

$$\begin{aligned}\boldsymbol{L} &= \sum_l (\boldsymbol{r}_j - \boldsymbol{\rho}_l) \times (m_l\Omega_1\hat{k} \times (\boldsymbol{r}_j - \boldsymbol{\rho}_l)) \; , \\ &= \sum_j \boldsymbol{r}_j \times (m_j\Omega_1\hat{k} \times \boldsymbol{r}_j) - \sum_l \boldsymbol{r}_j \times (m_l\Omega_1\hat{k} \times \boldsymbol{\rho}_l) - \sum_l \boldsymbol{\rho}_l \times (m_l\Omega_1\hat{k} \times \boldsymbol{r}_j) \\ &\quad + \sum_l \boldsymbol{\rho}_l \times (m_l\Omega_1\hat{k} \times \boldsymbol{\rho}_l) \; ,\end{aligned}$$

Again we can always choose $\boldsymbol{r}_j$ such that $\boldsymbol{r}_j = \boldsymbol{\rho}_l$, so that we obtain

$$\boldsymbol{L} = \sum_j \boldsymbol{r}_j \times (m_j \Omega_1 \hat{k} \times \boldsymbol{r}_j) + \sum_l \boldsymbol{\rho}_l \times (m_l \Omega_1 (-\hat{k}) \times \boldsymbol{\rho}_l) - \sum_l \boldsymbol{\rho}_l \times (m_l \Omega_1 \hat{k} \times \boldsymbol{\rho}_l)$$
$$+ \sum_l \boldsymbol{\rho}_l \times (m_l \Omega_1 \hat{k} \times \boldsymbol{\rho}_l) \ ,$$
$$= \sum_j \left[(\boldsymbol{r}_j \circ \boldsymbol{r}_j)\, m_j \Omega_1 \hat{k} - \left(\boldsymbol{r}_j \circ m_j \Omega_1 \hat{k} \right) \boldsymbol{r}_j \right]$$
$$+ \sum_l \left[(\boldsymbol{\rho}_l \circ \boldsymbol{\rho}_l)\, m_l \Omega_1 (-\hat{k}) - \left(\boldsymbol{\rho}_l \circ m_l \Omega_1 (-\hat{k}) \right) \boldsymbol{\rho}_l \right] \ ,$$

Again since $\boldsymbol{r}_j$ and $\hat{k}$ are mutually orthogonal and so are $\boldsymbol{\rho}_l$ and $-\hat{k}$ therefore we obtain

$$\boldsymbol{L} = I_{CM} \Omega \hat{k} + I_{CM} \Omega (-\hat{k}) \ ,$$

where Ω is the angular velocity of the rigid body relative to its center of mass (since it is the only motion allowed). We can write it in di-vector notation:

$$\boldsymbol{L} = I_{CM} \Omega (\pm \check{k}_+) + I_{CM} \Omega (\mp \check{k}_-) \ ,$$

$$= \boldsymbol{L}_+ + \boldsymbol{L}_- \ . \tag{28}$$

where $\boldsymbol{L} = \sum_l \boldsymbol{r}_l \times (m_l \Omega (\pm \check{k}) \times \boldsymbol{r}_l)$ the total angular momentum or di-angular angular momentum, $\boldsymbol{L}_+ = I_{CM} \Omega (\pm \check{k}_+)$ the active/positive angular momentum and $\boldsymbol{L}_- = I_{CM} \Omega (\mp \check{k}_-)$ the inertial/negative angular momentum.

2.D) Kinetic energy of a rigid body in a planar spin motion:

From figure(3) the total kinetic energy of this rigid body in its spin motion is given by:

$$T = \tfrac{1}{2} \sum_l m_l \left(\Omega_1 \hat{k} \times \boldsymbol{r}_l \circ \Omega_1 \hat{k} \times \boldsymbol{r}_l \right) \tag{29}$$

where $\Omega_1 \hat{k} \times \boldsymbol{r}_l$ is the tangential velocity of the element mass m_l which is perpendicular distance $\boldsymbol{r}_l$ from m_j . When one uses the identity $(\boldsymbol{A} \times \boldsymbol{B} \circ \boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \circ \boldsymbol{C})\,(\boldsymbol{B} \circ \boldsymbol{D}) - (\boldsymbol{A} \circ \boldsymbol{D})\,(\boldsymbol{B} \circ \boldsymbol{C})$ he will get

$$T = \tfrac{1}{2} \sum_l m_l \left[\left(\Omega_1 \hat{k} \circ \Omega_1 \hat{k} \right) (\boldsymbol{r}_l \circ \boldsymbol{r}_l) - \left(\Omega_1 \hat{k} \circ \boldsymbol{r}_l \right) \left(\boldsymbol{r}_l \circ \Omega_1 \hat{k} \right) \right] \ ,$$

Since the motion is planar then $\hat{k}$ and $\boldsymbol{r}_l$ are mutually orthogonal and we have $\sum\limits_l m_l = \sum\limits_j m_j = m$, so that

$$
\begin{aligned}
T &= \tfrac{1}{2}\sum_l m_l \left(\Omega_1\hat{k}\circ\Omega_1\hat{k}\right)(\boldsymbol{r}_l\circ\boldsymbol{r}_l) \ , \\
&= \tfrac{1}{2}\sum_l m_l \left(\Omega_1\hat{k}\circ\Omega_1\hat{k}\right)(\boldsymbol{r}_j-\boldsymbol{\rho}_l\circ\boldsymbol{r}_j-\boldsymbol{\rho}_l) \ , \\
&= \tfrac{1}{2}\sum_l m_l \left(\Omega_1\hat{k}\circ\Omega_1\hat{k}\right)\left[(\boldsymbol{r}_j\circ\boldsymbol{r}_j)-(\boldsymbol{r}_j\circ\boldsymbol{\rho}_l)-(\boldsymbol{\rho}_l\circ\boldsymbol{r}_j)+(\boldsymbol{\rho}_l\circ\boldsymbol{\rho}_l)\right] , \\
&= \tfrac{1}{2}\sum_j m_j(\boldsymbol{r}_j\circ\boldsymbol{r}_j)\left(\Omega_1\hat{k}\circ\Omega_1\hat{k}\right)-\tfrac{1}{2}\sum_l m_l(\boldsymbol{r}_j\circ\boldsymbol{\rho}_l)\left(\Omega_1\hat{k}\circ\Omega_1\hat{k}\right) \\
&\quad -\tfrac{1}{2}\sum_l m_l(\boldsymbol{\rho}_l\circ\boldsymbol{r}_j)\left(\Omega_1\hat{k}\circ\Omega_1\hat{k}\right)+\tfrac{1}{2}\sum_l m_l(\boldsymbol{\rho}_l\circ\boldsymbol{\rho}_l)\left(\Omega_1\hat{k}\circ\Omega_1\hat{k}\right) \ ,
\end{aligned}
$$

Again since $\boldsymbol{r}_j$ and $\hat{k}$ are mutually orthogonal and so are $\boldsymbol{\rho}_l$ and $-\hat{k}$ therefore we obtain

$$
\begin{aligned}
T &= \tfrac{1}{2}\sum_j m_j(\boldsymbol{r}_j\circ\boldsymbol{r}_j)\left(\Omega_1\hat{k}\circ\Omega_1\hat{k}\right)-\tfrac{1}{2}\sum_l m_l(\boldsymbol{\rho}_l\circ\boldsymbol{\rho}_l)\left(\Omega_1\hat{k}\circ\Omega_1\hat{k}\right) \\
&\quad -\tfrac{1}{2}\sum_l m_l(\boldsymbol{\rho}_l\circ\boldsymbol{\rho}_l)\left(\Omega_1\hat{k}\circ\Omega_1\hat{k}\right)+\tfrac{1}{2}\sum_l m_l(\boldsymbol{\rho}_l\circ\boldsymbol{\rho}_l)\left(\Omega_1\hat{k}\circ\Omega_1\hat{k}\right) \ , \\
&= \tfrac{1}{2}I_{CM}\left(\Omega_1\hat{k}\circ\Omega_1\hat{k}\right)-\tfrac{1}{2}I_{CM}\left(\Omega_1\hat{k}\circ\Omega_1\hat{k}\right) \ ,
\end{aligned}
$$

We have again the problem of the negative kinetic energy and we have the same solution mentioned earlier. Thus we have

$$T = \tfrac{1}{2}I_{CM}\left(\Omega(+\hat{k})\circ\Omega(+\hat{k})\right)+\tfrac{1}{2}I_{CM}\left(\Omega(-\hat{k})\circ\Omega(-\hat{k})\right) \ . \tag{30}$$

we can write it in di-vector notation:

$$
\begin{aligned}
T &= \tfrac{1}{2}I_{CM}\left((\pm\check{k}_+)\circ\Omega(\pm\check{k}_+)\right)+\tfrac{1}{2}I_{CM}\left(\Omega(\mp\check{k}_-)\circ\Omega(\mp\check{k}_-)\right) \ , \\
&= T_+ + T_-
\end{aligned} \tag{31}
$$

where $T=\frac{1}{2}\sum\limits_l m_l\left(\Omega(\pm\check{k})\times\boldsymbol{r}_l\circ\Omega(\pm\check{k})\times\boldsymbol{r}_l\right)$ the <u>total</u> kinetic energy or <u>di-kinetic energy</u>, $T_+ = \frac{1}{2}I_{CM}\left(\Omega(\pm\check{k}_+)\circ\Omega(\pm\check{k}_+)\right)$ the <u>active/positive</u> kinetic energy and
$T_- = \frac{1}{2}I_{CM}\left(\Omega(\mp\check{k}_-)\circ\Omega(\mp\check{k}_-)\right)$ the <u>inertial/negative</u> kinetic energy.

It is worth to mention that the inertial/negative kinetic energy occurs as potential energy when the force is applied.

2.E) The complete equations of motion for a rigid body in planar rotational motion:

At this point we can write the complete equations of motion for a rigid body in planar rotational motion by adding equations(28) and (31) to equations(13) and (19) consequently. Therefore, the total angular momentum for a rigid body in planar rotational motion:

$$\boldsymbol{L} = I_O\omega(\pm\check{k}_+) + I_{CM}\omega(\mp\check{k}_-) + I_{CM}\Omega(\pm\check{k}_+) + I_{CM}\Omega(\mp\check{k}_-) \tag{32}$$

The total kinetic energy for a rigid body in planar rotational motion:

$$\begin{aligned} T = \; & \tfrac{1}{2}I_O\left(\omega(\pm\check{k}_+)\circ\omega(\pm\check{k}_+)\right) + \tfrac{1}{2}I_{CM}\left(\omega(\mp\check{k}_-)\circ\omega(\mp\check{k}_-)\right) \\ & + \tfrac{1}{2}I_{CM}\left(\Omega(\pm\check{k}_+)\circ\Omega(\pm\check{k}_+)\right) + \tfrac{1}{2}I_{CM}\left(\Omega(\mp\check{k}_-)\circ\Omega(\mp\check{k}_-)\right) \end{aligned} \tag{33}$$

II.2. Rectilinear Motion

II.2.1. Rigid body in a planar translational motion:

Referring to figure(4), a rigid body "C" of mass m is free to translational move relative to the frame of reference S. When an external force $\mathbf{F}$ acts on the rigid body at its center of mass CM, then the body will move in straight line in the direction of the active force $\mathbf{F}$ with uniform acceleration $a\hat{e} = (dv/dt)\hat{e}$ relative to the S-frame; where "a" is the magnitude of the translational acceleration of the rigid body. $\hat{e}$ is a unit vector parallel to the plane of the motion. v is the translational velocity of the rigid body. Therefore, the total force acts on the body

$$\mathbf{F} = ma\hat{e}$$

Since the motion of the rigid body under influence of the active force $\mathbf{F}$ with respect to S-frame is non-inertial then an inertial force $\mathbf{F}_i$ will occur at every element mass m_i of the rigid body. Hence, the total inertial torque $\boldsymbol{\tau}_-$ due to this inertial force will be

$$\begin{aligned} \boldsymbol{\tau}_- &= \sum_i \boldsymbol{\rho}_i \times \mathbf{F}_i \ , \\ &= \sum_i \boldsymbol{\rho}_i \times \left(m_i\frac{dv}{dt}(-\hat{e})\right) \ , \\ &= \sum_i m_i\boldsymbol{\rho}_i \times \frac{dv}{dt}(-\hat{e}) \ , \end{aligned} \tag{34}$$

But from the properties of the center of mass we know that $\sum_i m_i\boldsymbol{\rho}_i = 0$. Therefore, we obtain

$$\boldsymbol{\tau}_- = 0 \ . \tag{35}$$

Thus, the inertial torque does *not* present in the rigid body translational motion. To find the total inertial force $\mathbf{F}_{inertial}$, one have

$$\mathbf{F}_{inertial} = \sum_i m_i \frac{dv}{dt}(-\hat{e}) \; = \; m\frac{dv}{dt}(-\hat{e}) \; ,$$

$$= \; ma(-\hat{e}) \; . \tag{36}$$

where $a(-\hat{e}) = (dv/dt)(-\hat{e})$ is the translational acceleration of the rigid body relative to the S-frame due to the inertial force.

3.A) Derivation of inertial force from total force:

Since there is no rotation or spin (see equation(35)) therefore all motion vectors occur parallel to the line of motion. Referring to figure(5) we can write the total force act on the rigid body "C":

$$\begin{aligned} \mathbf{F} &= \sum_i \mathbf{F}_j \\ &= \sum_i \mathbf{\Gamma}_j - \sum_i \mathbf{F}_i \\ &= \sum_j m_j a\hat{e} + \sum_i m_i a(-\hat{e}) \end{aligned}$$

Using the concept of di-vector that mentioned earlier to rewrite this result, one obtains

$$\mathbf{F} = ma(+\check{e}_+) + ma(-\check{e}_-) \tag{37}$$

and we can write it as:

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- \tag{38}$$

where $\mathbf{F} = ma(\pm\check{e})$ the total force or the di-force, $\mathbf{F}_+ = ma(\pm\check{e}_+)$ the active/positive force and $\mathbf{F}_- = ma(\mp\check{e}_-)$ the inertial/negative force.

3.B) Linear momentum of a rigid body in a planar translational motion:

Integrating equation(37) with respect to time yields

$$mv\hat{e} = mv(+\check{e}_+) + mv(-\check{e}_-) \tag{39}$$

Or simply:

$$\mathbf{p} = \mathbf{p}_+ + \mathbf{p}_- \tag{40}$$

where $\mathbf{p} = mv(\pm\check{e})$ the total momentum or the di-momentum, $\mathbf{p}_+ = mv(\pm\check{e}_+)$ the active/positive momentum and $\mathbf{p}_- = mv(\mp\check{e}_-)$ the inertial/negative momentum.

3.C) Kinetic energy of a rigid body in a planar translational motion:

Integrating equation(39) with respect to time yields

$$\tfrac{1}{2}m\,(v\check{e} \circ v\check{e}) = \tfrac{1}{2}m\,(v(+\check{e}_+) \circ v(+\check{e}_+)) + \tfrac{1}{2}m\,(v(-\check{e}_-) \circ v(-\check{e}_-)) \tag{41}$$

Or simply:

$$\mathbf{T} = \mathbf{T}_{+} + \mathbf{T}_{-} \tag{42}$$

where $\mathbf{T} = \frac{1}{2}m\,(v(\pm\check{e}) \circ v(\pm\check{e}))$ the total kinetic energy or the di-kinetic energy,
$\mathbf{T}_{+} = \frac{1}{2}m\,(v(\pm\check{e}_+) \circ v(\pm\check{e}_+))$ the active/positive kinetic energy and
$\mathbf{T}_{-} = \frac{1}{2}m\,(v(\mp\check{e}_-) \circ v(\mp\check{e}_-))$ the inertial/negative kinetic energy.

III. Conclusion

The Newtonian mechanical force when it acts on a rigid body it de-synthesizes into two counter forces which are the active force and the inertial force.

Both the active force and the inertial (fictitious) force are real forces and have one real origin.

The inertial force supplies the rigid body with additional kinetic energy and momentum and that independently from the kinetic energy and momentum which supplied by the active force.

One can states also that for any action there is a reaction and both synthesis to one action.

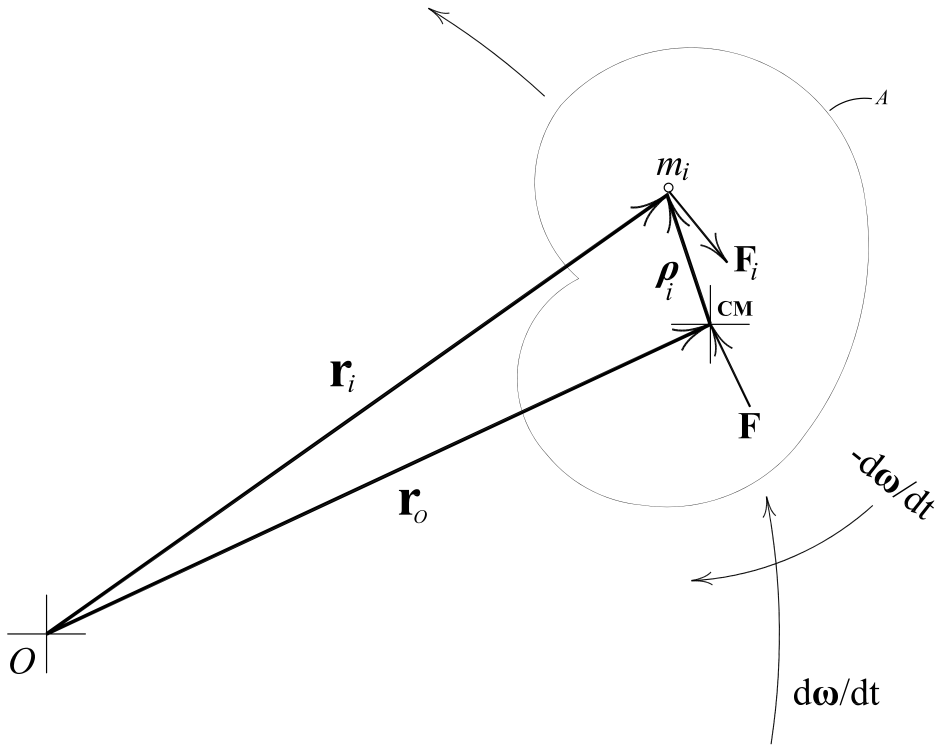

A
m_i
$\mathbf{F}_i$
$\boldsymbol{\rho}_i$
CM
$\mathbf{r}_i$
F
$\mathbf{r}_o$
-d**ω**/dt
O
d**ω**/dt

Figure 1

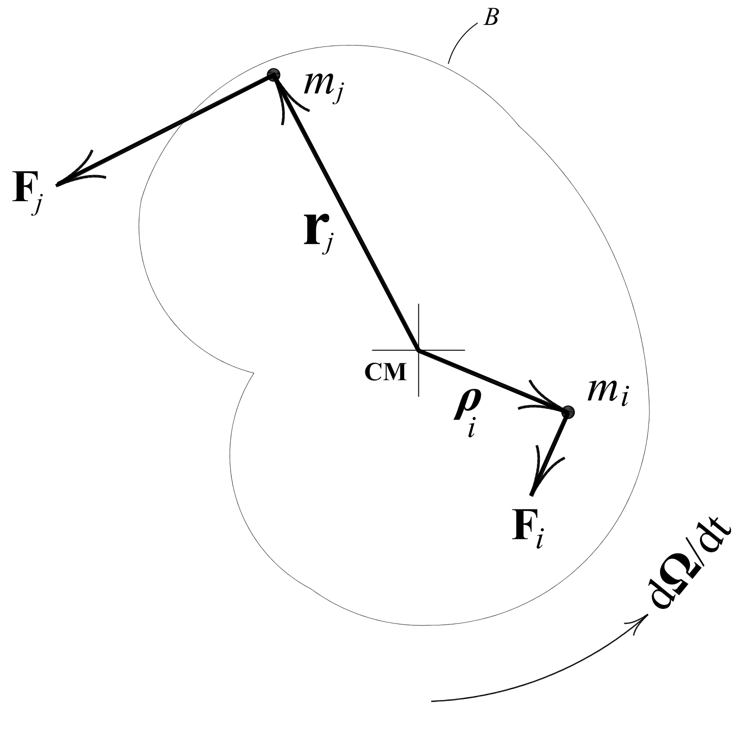

B
m_j
$\mathbf{F}_j$
$\mathbf{r}_j$
CM
$\boldsymbol{\rho}_i$
m_i
$\mathbf{F}_i$
d$\boldsymbol{\Omega}$/dt

Figure 2

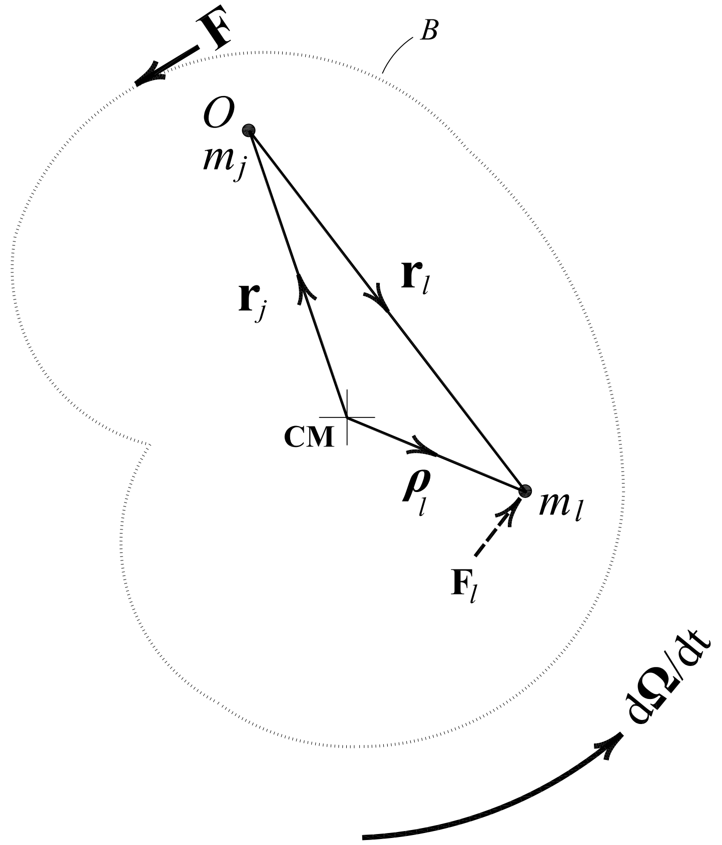

F
B
O
m_j
$\mathbf{r}_l$
$\mathbf{r}_j$
CM
$\boldsymbol{\rho}_l$
m_l
$\mathbf{F}_l$
d$\boldsymbol{\Omega}$/dt

Figure 3

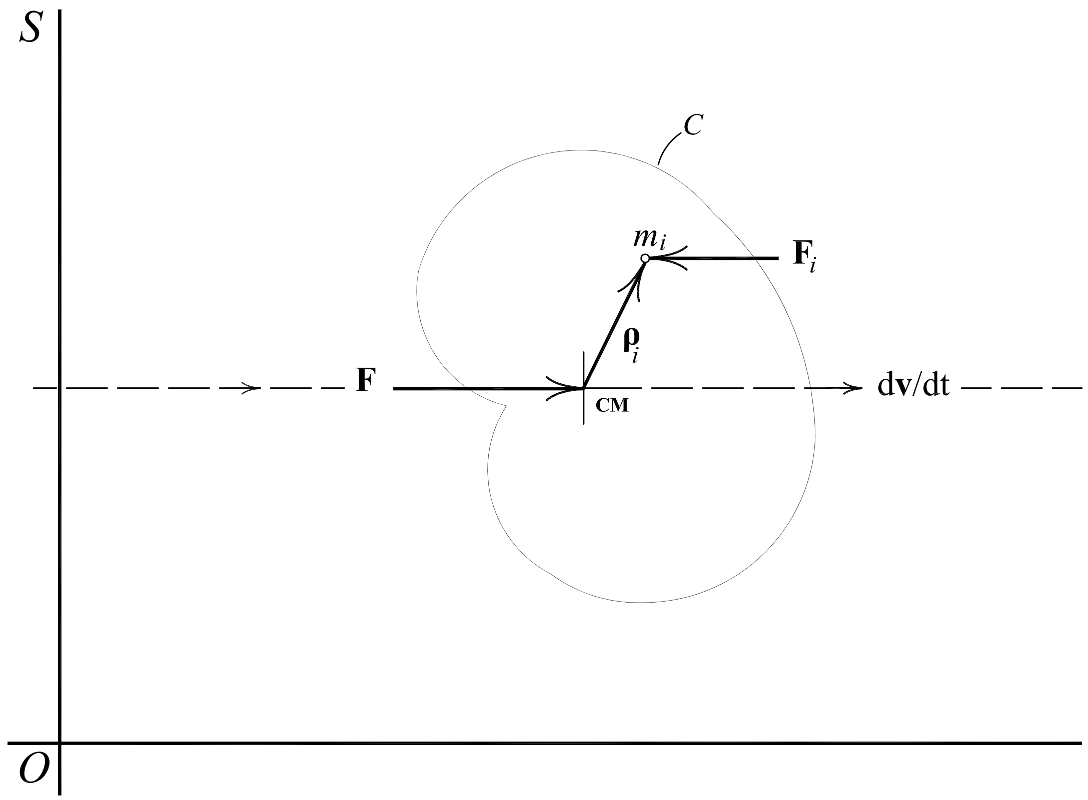
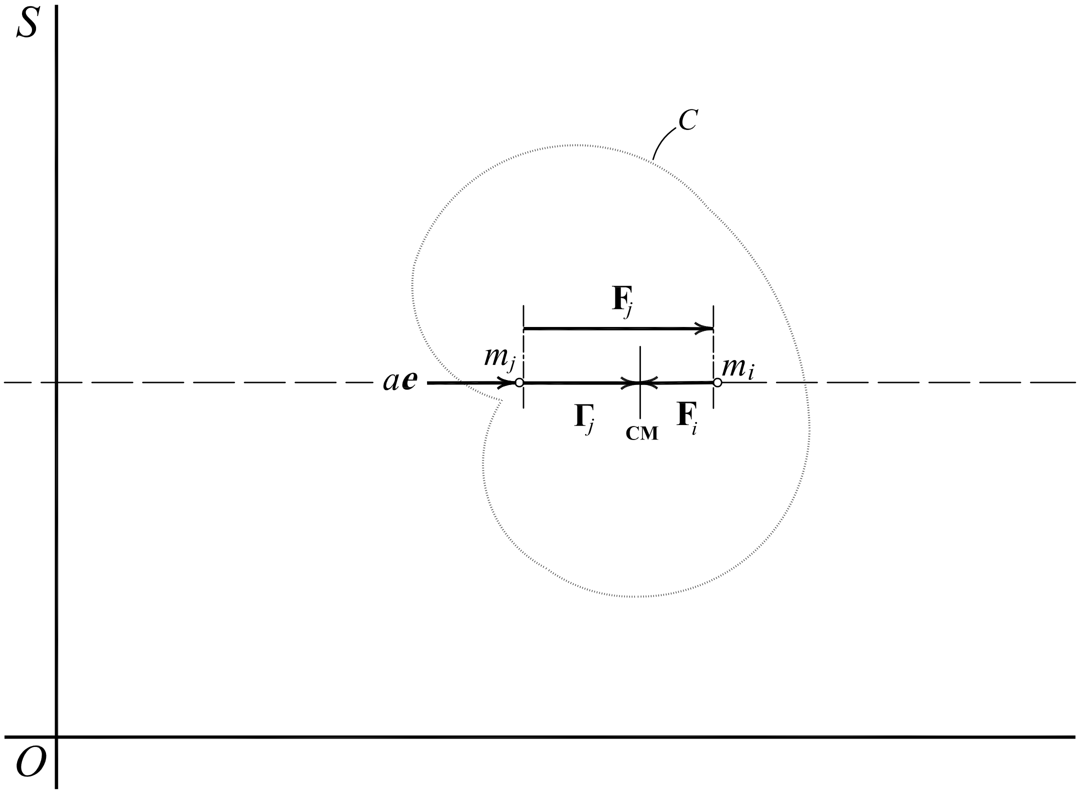


Figure 4



S
C
$\mathbf{F}_j$
m_j
m_i
$a\boldsymbol{e}$
$\boldsymbol{\Gamma}_j$
CM
$\mathbf{F}_i$
O

Figure 5

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