Bell's inequality is violated in classical systems as well as quantum systems

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Abstract

Bell's inequality is usually considered to belong to mathematics and not quantum mechanics. We think that this makes it difficult to understand Bell's theory. Thus in this paper, contrary to Bell's spirit (which inherits Einstein's spirit), we try to discuss Bell's inequality in the framework of quantum theory with the linguistic Copenhagen interpretation. And we clarify that whether or not Bell's inequality holds does not depend on whether classical systems or quantum systems, but depend on whether a kind of simultaneous measurements exist or not. And further we assert that our argument (based on the linguistic Copenhagen interpretation) should be regarded as a scientific representation of Bell's philosophical argument (based on Einstein's spirit).

Key phrases: Bohr-Einstein debates, Bell's inequality, Combined observable, Linguistic Copenhagen interpretation, Quantum Language

1 Review: Quantum language (= Measurement theory (=MT))

1.1 Introduction

Following refs. [6, 7, 8, 9, 10, 11], we shall review quantum language (i.e., quantum theory with the linguistic Copenhagen interpretation, or measurement theory), which has the following form:

$$\underset{\text{(= measurement theory)}}{\boxed{\text{Quantum language}}} = \underset{\text{(Axiom 1)}}{\boxed{\text{Measurement}}} + \underset{\text{(Axiom 2)}}{\boxed{\text{Causality}}} + \underset{\text{(how to use Axioms 1 and 2)}}{\boxed{\text{Linguistic (Copenhagen) interpretation}}} \tag{1}$$

We think that the location of quantum language in the history of world-description is as follows.

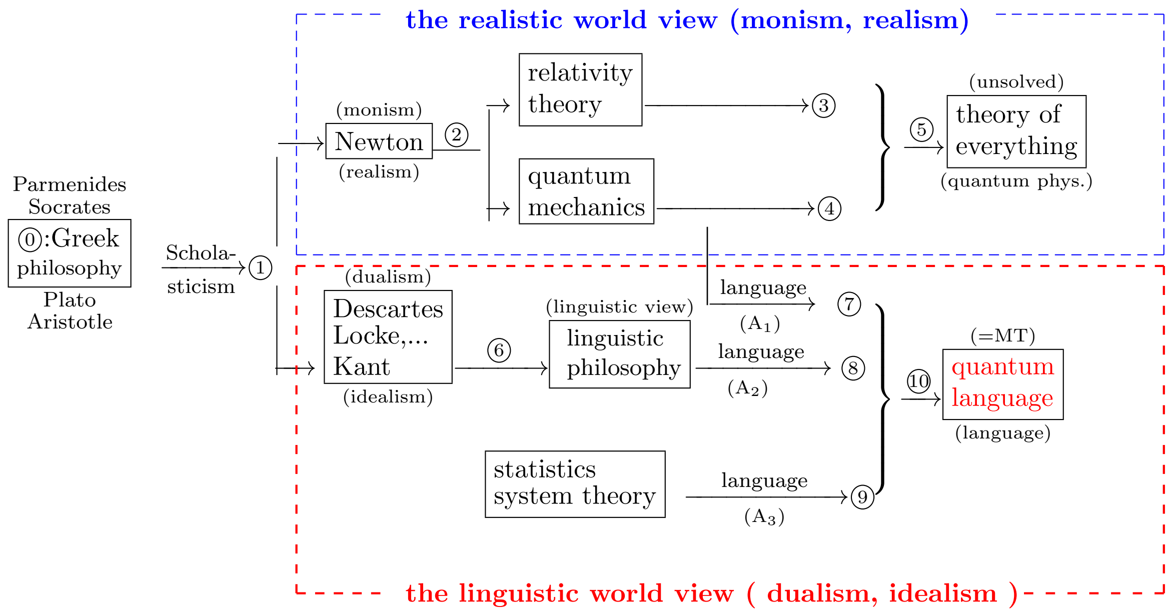

Figure 1: The history of the world-description

And in Figure 1, we think that the following four are equivalent (*cf.* ref. [10]):

(A_0) to propose quantum language (*cf.* ⑩ in Figure 1)

(A_1) to clarify the Copenhagen interpretation of quantum mechanics (*cf.* ⑦ in Figure 1)

(A_2) to clarify the final goal of the dualistic idealism (*cf.* ⑧ in Figure 1, see ref. [11])

(A_3) to reconstruct statistics in the dualistic idealism (*cf.* ⑨ in Figure 1)

In Bohr-Einstein debates (refs. [2, 5]), Einstein's standing-point is on the side of the realistic view in Figure 1. On the other hand, we think that Bohr's standing point is on the side of the linguistic view in Figure 1 (though Bohr might believe that the Copenhagen interpretation (proposed by his school) belongs to physics). In this paper, contrary to Bell's spirit (which inherits Einstein's spirit), we try to discuss Bell's inequality (refs. [1, 13]) in quantum language (i.e., quantum theory with the linguistic Copenhagen interpretation). And we clarify that whether or not Bell's inequality holds does not depend on whether classical systems or quantum systems (in Section 3), but depend on whether a kind of simultaneous measurements exist or not (in Section 2). And further we assert that our argument (based on the linguistic Copenhagen interpretation) should be regarded as a scientific representation of Bell's philosophical argument (based on Einstein's spirit).

1.2 Quantum language: Mathematical preparations

Now we briefly introduce quantum language as follows. Consider an operator algebra $B(H)$ (i.e., an operator algebra composed of all bounded linear operators on a Hilbert space H with the norm $\|F\|_{B(H)} = \sup_{\|u\|_H=1} \|Fu\|_H$), and consider the pair $[\mathcal{A}, \mathcal{N}]_{B(H)}$, called a *basic structure*. Here, $\mathcal{A}(\subseteq B(H))$ is a C^*-algebra, and $\mathcal{N}$ ($\mathcal{A} \subseteq \mathcal{N} \subseteq B(H)$) is a particular C^*-algebra (called a W^*-algebra) such that $\mathcal{N}$ is the weak closure of $\mathcal{A}$ in $B(H)$.
The measurement theory (=quantum language= the linguistic interpretation) is classified as follows.

$$\text{(B)} \qquad \text{measurement theory } = \begin{cases} (\mathrm{B}_1)\text{: quantum system theory} \quad (\text{when } \mathcal{A} = \mathcal{C}(H)) \\ \\ (\mathrm{B}_2)\text{: classical system theory} \quad (\text{when } \mathcal{A} = C_0(\Omega)) \end{cases}$$

That is, when $\mathcal{A} = \mathcal{C}(H)$, the C^*-algebra composed of all compact operators on a Hilbert space H, the (B_1) is called quantum measurement theory (or, quantum system theory), which can be regarded as the linguistic aspect of quantum mechanics. Also, when $\mathcal{A}$ is commutative (that is, when $\mathcal{A}$ is characterized by $C_0(\Omega)$, the C^*-algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space Ω (*cf.* [12, 14])), the (B_2) is called classical measurement theory.
Also, note (*cf.* [12]) that, when $\mathcal{A} = \mathcal{C}(H)$,

(i) $\mathcal{A}^* = \mathrm{Tr}(H)$ (=trace class), $\mathcal{N} = B(H)$, $\mathcal{N}_* = \mathrm{Tr}(H)$ (i.e., pre-dual space), thus, ${}_{\mathrm{Tr}(H)}\big(\rho, T\big)_{B(H)} = \mathrm{Tr}_H(\rho T)$ ($\rho \in \mathrm{Tr}(H), T \in B(H)$).

Also, when $\mathcal{A} = C_0(\Omega)$,

(ii) $\mathcal{A}^* =$"the space of all signed measures on Ω", $\mathcal{N} = L^\infty(\Omega, \nu)(\subseteq B(L^2(\Omega, \nu)))$, $\mathcal{N}_* = L^1(\Omega, \nu)$, where ν is some measure on Ω, thus, ${}_{L^1(\Omega,\nu)}\big(\rho, T\big)_{L^\infty(\Omega,\nu)} = \int_\Omega \rho(\omega)T(\omega)\nu(d\omega)$ ($\rho \in L^1(\Omega, \nu), T \in L^\infty(\Omega, \nu)$) (*cf.* [12]). The measure ν is usually omitted in this paper, that is, $L^\infty(\Omega, \nu)$ and $L^1(\Omega, \nu)$ is respectively written by $L^\infty(\Omega)$ and $L^1(\Omega)$.

Let $[\mathcal{A}, \mathcal{N}]_{B(H)}$ be a basic structure. Let $\mathcal{A}(\subseteq B(H))$ be a C^*-algebra, and let $\mathcal{A}^*$ be the dual Banach space of $\mathcal{A}$. That is, $\mathcal{A}^* = \{\rho \mid \rho \text{ is a continuous linear functional on } \mathcal{A}\}$, and the norm $\|\rho\|_{\mathcal{A}^*}$ is defined by $\sup\{|\rho(F)| \mid F \in \mathcal{A} \text{ such that } \|F\|_{\mathcal{A}}(= \|F\|_{B(H)}) \le 1\}$. Define the *mixed state* ρ ($\in \mathcal{A}^*$) such that $\|\rho\|_{\mathcal{A}^*} = 1$ and $\rho(F) \ge 0$ for all $F \in \mathcal{A}$ such that $F \ge 0$. And define the mixed state space $\mathfrak{S}^m(\mathcal{A}^*)$ such that

$$\mathfrak{S}^m(\mathcal{A}^*) = \{\rho \in \mathcal{A}^* \mid \rho \text{ is a mixed state}\}.$$

A mixed state $\rho(\in \mathfrak{S}^m(\mathcal{A}^*))$ is called a *pure state* if it satisfies that "$\rho = \theta\rho_1 + (1-\theta)\rho_2$ for some $\rho_1, \rho_2 \in \mathfrak{S}^m(\mathcal{A}^*)$ and $0 < \theta < 1$" implies "$\rho = \rho_1 = \rho_2$". Put

$$\mathfrak{S}^p(\mathcal{A}^*) = \{\rho \in \mathfrak{S}^m(\mathcal{A}^*) \mid \rho \text{ is a pure state}\},$$

which is called a *state space.* It is well known (*cf.* [12]) that $\mathfrak{S}^p(\mathcal{C}(H)^*) = \{|u\rangle\langle u| \text{ (i.e., the Dirac notation)} \mid \|u\|_H = 1\}$, and $\mathfrak{S}^p(C_0(\Omega)^*) = \{\delta_{\omega_0} \mid \delta_{\omega_0} \text{ is a point measure at } \omega_0 \in \Omega\}$, where $\int_\Omega f(\omega)\delta_{\omega_0}(d\omega) = f(\omega_0)$ $(\forall f \in C_0(\Omega))$. The latter implies that $\mathfrak{S}^p(C_0(\Omega)^*)$ can be also identified with Ω (called a *spectrum space* or simply *spectrum*) such as

$$\underset{\text{(state space)}}{\mathfrak{S}^p(C_0(\Omega)^*)} \ni \delta_\omega \leftrightarrow \omega \in \underset{\text{(spectrum)}}{\Omega}$$

For instance, in the above (ii) we must clarify the meaning of the "value" of $F(\omega_0)$ for $F \in L^\infty(\Omega, \nu)$ and $\omega_0 \in \Omega$. An element $F(\in \mathcal{N})$ is said to be *essentially continuous at* $\rho_0(\in \mathfrak{S}^p(\mathcal{A}^*))$, if there uniquely exists a complex number α such that

(C) if ρ $(\in \mathcal{N}_*, \|\rho\|_{\mathcal{N}_*} = 1)$ converges to $\rho_0(\in \mathfrak{S}^p(\mathcal{A}^*))$ in the sense of weak* topology of $\mathcal{A}^*$, that is,

$$\rho(G) \longrightarrow \rho_0(G) \ \ (\forall G \in \mathcal{A}(\subseteq \mathcal{N})),$$

then $\rho(F)$ converges to α.

And the value of $\rho_0(F)$ is defined by the α.

According to the noted idea (cf. [4]), an *observable* $\mathsf{O} := (X, \mathcal{F}, F)$ in $\mathcal{N}$ is defined as follows:

(i) [σ-field] X is a set, $\mathcal{F}(\subseteq 2^X (\equiv \mathcal{P}(X))$, the power set of X) is a σ-field of X, that is, "$\Xi_1, \Xi_2, ... \in \mathcal{F} \Rightarrow \cup_{n=1}^\infty \Xi_n \in \mathcal{F}$", "$\Xi \in \mathcal{F} \Rightarrow X \setminus \Xi \in \mathcal{F}$".

(ii) [Countable additivity] F is a mapping from $\mathcal{F}$ to $\mathcal{N}$ satisfying: (a): for every $\Xi \in \mathcal{F}$, $F(\Xi)$ is a non-negative element in $\mathcal{N}$ such that $0 \le F(\Xi) \le I$, (b): $F(\emptyset) = 0$ and $F(X) = I$, where 0 and I is the 0-element and the identity in $\mathcal{N}$ respectively. (c): for any countable decomposition $\{\Xi_1, \Xi_2, \ldots, \Xi_n, ...\}$ of Ξ $\big($i.e., $\Xi, \Xi_n \in \mathcal{F}$ $(n = 1, 2, 3, ...)$, $\cup_{n=1}^\infty \Xi_n = \Xi$, $\Xi_i \cap \Xi_j = \emptyset$ $(i \ne j)\big)$, it holds that $F(\Xi) = \sum_{n=1}^\infty F(\Xi_n)$ in the sense of weak* topology in $\mathcal{N}$.

1.3 Axiom 1 [Measurement] and Axiom 2 [Causality]

Measurement theory (B) is composed of two axioms (i.e., Axioms 1 and 2) as follows. With any *system* S, a basic structure $[\mathcal{A}, \mathcal{N}]_{B(H)}$ can be associated in which the measurement theory (B) of that system can be formulated. A *state* of the system S is represented by an element $\rho(\in \mathfrak{S}^p(\mathcal{A}^*))$ and an *observable* is represented by an observable $\mathsf{O} := (X, \mathcal{F}, F)$ in $\mathcal{N}$. Also, the *measurement of the observable* O *for the system* S *with the state* ρ is denoted by $\mathsf{M}_{\mathcal{N}}(\mathsf{O}, S_{[\rho]})$ $\big($ or more precisely, $\mathsf{M}_{\mathcal{N}}(\mathsf{O} := (X, \mathcal{F}, F), S_{[\rho]})$ $\big)$. An observer can obtain a measured value x $(\in X)$ by the measurement $\mathsf{M}_{\mathcal{N}}(\mathsf{O}, S_{[\rho]})$.

The Axiom 1 presented below is a kind of mathematical generalization of Born's probabilistic interpretation of quantum mechanics. And thus, it is a statement without reality.

Now we can present Axiom 1 in the W^*-algebraic formulation as follows.

Axiom 1 [Measurement]. *The probability that a measured value* x $(\in X)$ *obtained by the measurement* $\mathsf{M}_{\mathcal{N}}(\mathsf{O} := (X, \mathcal{F}, F), S_{[\rho]})$ *belongs to a set* $\Xi(\in \mathcal{F})$ *is given by* $\rho(F(\Xi))$ *if* $F(\Xi)$ *is essentially continuous at* $\rho(\in \mathfrak{S}^p(\mathcal{A}^*))$.

Next, we explain Axiom 2. Let $[\mathcal{A}_1, \mathcal{N}_1]_{B(H_1)}$ and $[\mathcal{A}_2, \mathcal{N}_2]_{B(H_2)}$ be basic structures. A continuous linear operator $\Phi_{1,2} : \mathcal{N}_2$ (with weak* topology) $\to \mathcal{N}_1$(with weak* topology) is called a *Markov operator,* if it satisfies that (i): $\Phi_{1,2}(F_2) \ge 0$ for any non-negative element F_2 in $\mathcal{N}_2$, (ii): $\Phi_{1,2}(I_2) = I_1$, where I_k is the identity in $\mathcal{N}_k$, $(k = 1, 2)$. In addition to the above (i) and (ii), in this paper we assume that $\Phi_{1,2}(\mathcal{A}_2) \subseteq \mathcal{A}_1$ and $\sup\{\|\Phi_{1,2}(F_2)\|_{\mathcal{A}_1} \mid F_2 \in \mathcal{A}_2 \text{ such that } \|F_2\|_{\mathcal{A}_2} \le 1\} = 1$.

It is clear that the dual operator $\Phi_{1,2}^* : \mathcal{A}_1^* \to \mathcal{A}_2^*$ satisfies that $\Phi_{1,2}^*(\mathfrak{S}^m(\mathcal{A}_1^*)) \subseteq \mathfrak{S}^m(\mathcal{A}_2^*)$. If it holds that $\Phi_{1,2}^*(\mathfrak{S}^p(\mathcal{A}_1^*)) \subseteq \mathfrak{S}^p(\mathcal{A}_2^*)$, the $\Phi_{1,2}$ is said to be deterministic. If it is not deterministic, it is said to be non-deterministic or decoherence. Also, note that, for any observable $\mathsf{O}_2 :=(X, \mathcal{F}, F_2)$ in $\mathcal{N}_2$, the $(X, \mathcal{F}, \Phi_{1,2}F_2)$ is an observable in $\mathcal{N}_1$.

Now Axiom 2 is presented as follows:
Axiom 2 [Causality]. *Let $t_1 \le t_2$. The causality is represented by a Markov operator $\Phi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}$.*

1.4 The linguistic interpretation (= the manual to use Axioms 1 and 2)

In the above, Axioms 1 and 2 are kinds of spells, (i.e., incantation, magic words, metaphysical statements), and thus, it is nonsense to verify them experimentally. Therefore, what we should do is not "to understand" but "to use". After learning Axioms 1 and 2 by rote, we have to improve how to use them through trial and error.

We can do well even if we do not know the linguistic interpretation. However, it is better to know the linguistic interpretation (= the manual to use Axioms 1 and 2), if we would like to make progress quantum language early.

The essence of the manual is as follows:

(D) *Only one measurement is permitted.* And thus, the state after a measurement is meaningless since it can not be measured any longer. Thus, the collapse of the wavefunction is prohibited (*cf.* [9]). We are not concerned with anything after measurement. That is, any statement including the phrase "**after the measurement**" is wrong. Also, the causality should be assumed only in the side of system, however, a state never moves. Thus, the Heisenberg picture should be adopted, and thus, the Schrödinger picture should be prohibited.

and so on. For details, see [10].

1.5 Simultaneous measurement, parallel measurement

Definition 1. (i): *Let $[\mathcal{A}, \mathcal{N}]_{B(H)}$ be a basic structure. Consider observables $\mathsf{O}_k = (X_k, \mathcal{F}_k, F_k)$ $(k = 1, 2, ..., K)$ in $\mathcal{N}$. Let $(\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k)$ be the product measurable space. An observable $\mathsf{O} = \times_{k=1}^K \mathsf{O}_k = (\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k, F)$ in $\mathcal{N}$ is called the simultaneous observable of O_k $(k = 1, 2, ..., K)$, if it holds that*

$$\mathop{\times}_{k=1}^{K} F_k(\Xi_k) = F(\mathop{\times}_{k=1}^{K} \Xi_k) \qquad (\forall \Xi_k \in \mathcal{F}_k) \tag{2}$$

Also, the measurement $\mathsf{M}_{\mathcal{N}}(\mathsf{O}, S_{[\rho_0]})$ is called a simultaneous measurement of measurements $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_k, S_{[\rho_0]})$ $(k = 1, 2, ..., K)$. Note that a simultaneous observable $\mathsf{O} = (\times_{k=1}^K X_k, \boxtimes_{k=1}^K, F)$ in $\mathcal{N}$ always exists if observables $\mathsf{O} = (\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k, F)$ commute, i.e.,

$$F_k(\Xi_k)F_l(\Xi_l) = F_l(\Xi_l)F_k(\Xi_k) \qquad (\forall \Xi_k \in \mathcal{F}_l, \forall \Xi_l \in \mathcal{F}_k, k \neq l) \tag{3}$$

(ii): *Let $[\mathcal{A}_k, \mathcal{N}_k]_{B(H_k)}$ $(k = 1, 2, ..., K)$ be basic structures, and let $[\bigotimes_{k=1}^K \mathcal{A}_k, \bigotimes_{k=1}^K \mathcal{N}_k]_{\bigotimes_{k=1}^K B(H_k)}$ be the tensor basic structure (cf. [10]). Consider measurements $\mathsf{M}_{\mathcal{N}_k}(\mathsf{O}_k = (X_k, \mathcal{F}_k, F_k), S_{[\rho_k]})$ $(k = 1, 2, ..., K)$ in $\mathcal{N}_k$. Let $\bigotimes_{k=1}^K \mathsf{O}_k = (\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k, \bigotimes_{k=1}^K F_k)$ be the parallel observable in a tensor W^*-algebra $\bigotimes_{k=1}^K \mathcal{N}_k$. And let $\bigotimes_{k=1}^K \rho_k \in \mathfrak{S}^p((\bigotimes_{k=1}^K \mathcal{A}_k)^*))$. Then, the measurement $\mathsf{M}_{\bigotimes_{k=1}^K \mathcal{N}_k}(\bigotimes_{k=1}^K \mathsf{O}_k = (\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k, \bigotimes_{k=1}^K F_k), S_{[\bigotimes_{k=1}^K \rho_k]})$ is called a parallel measurement of $\mathsf{M}_{\mathcal{N}_k}(\mathsf{O}_k = (X_k, \mathcal{F}_k, F_k), S_{[\rho_k]})$ $(k = 1, 2, ..., K)$. Note that the parallel measurement always exists uniquely.*

2 Bell's inequality in quantum language

2.1 Our view about Bell's inequality

In this paper, I assert that Bell's inequality should be studied in the framework of quantum theory (i.e., quantum theory with the linguistic Copenhagen interpretation). Let us start from the following definition, which is a slight modification of the simultaneous observable in Definition 1.

Definition 2. [Combined observable] *Let* $[\mathcal{A},\mathcal{N}]_{B(H)}$ *be a basic structure. Put* $X = \{-1,1\}$*. Consider four observables:* $\mathsf{O}_{13} = (X^2, \mathcal{P}(X^2), F_{13})$*,* $\mathsf{O}_{14} = (X^2, \mathcal{P}(X^2), F_{14})$*,* $\mathsf{O}_{23} = (X^2, \mathcal{P}(X^2), F_{23})$*,* $\mathsf{O}_{24} = (X^2, \mathcal{P}(X^2), F_{24})$ *in* $\mathcal{N}$*. The four observables are said to be combinable if there exists an observable* $\mathsf{O} = (X^4, \mathcal{P}(X^4), F)$ *in* $\mathcal{N}$ *such that*

$$
\begin{aligned}
&F_{13}(\{(x_1,x_3)\}) = F(\{x_1\} \times X \times \{x_3\} \times X), \quad F_{14}(\{(x_1,x_4)\}) = F(\{x_1\} \times X \times X \times \{x_4\})\\
&F_{23}(\{(x_2,x_3)\}) = F(X \times \{x_2\} \times \{x_3\} \times X), \quad F_{24}(\{(x_2,x_4)\}) = F(X \times \{x_2\} \times X \times \{x_4\})
\end{aligned}
\tag{4}
$$

for any $(x_1,x_2,x_3,x_4) \in X^4$*. The observable* O *is said to be a combined observable of* O_{ij} $(i = 1,2, j = 3,4)$*. Note that the* O *is regarded as a kind of simultaneous observable of* O_{ij} $(i = 1,2, j = 3,4)$*. Also, the measurement* $\mathsf{M}_{\mathcal{N}}(\mathsf{O} = (X^4, \mathcal{P}(X^4), F), S_{[\rho_0]})$ *is called the conbined measurement of* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{13}, S_{[\rho_0]})$*,* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{14}, S_{[\rho_0]})$*,* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{23}, S_{[\rho_0]})$ *and* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{24}, S_{[\rho_0]})$*.*

The following theorem is all of our insistence concerning Bell's inequality.

Theorem 3. [Bell's inequality in quantum language] *Let* $[\mathcal{A},\mathcal{N}]_{B(H)}$ *be a basic structure. Put* $X = \{-1,1\}$*. Fix the pure state* $\rho_0\big(\in \mathfrak{S}^p(\mathcal{A}^*)\big)$*. And consider the four measurements* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{13} = (X^2, \mathcal{P}(X^2), F_{13}), S_{[\rho_0]})$*,* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{14} = (X^2, \mathcal{P}(X^2), F_{14}), S_{[\rho_0]})$*,* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{23} = (X^2, \mathcal{P}(X^2), F_{23}), S_{[\rho_0]})$ *and* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{24} = (X^2, \mathcal{P}(X^2), F_{24}), S_{[\rho_0]})$*. Or equivalently, consider the parallel measurement* $\otimes_{i=1,2,j=3,4}\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{ij} = (X^2, \mathcal{P}(X^2), F_{ij}), S_{[\rho_0]})$*. Define four correlation functions* $(i = 1,2, j = 3,4)$*,*

$$
R_{ij} = \sum_{(u,v)\in X\times X} u \cdot v \ \ \rho_0(F_{ij}(\{(u,v)\}))
$$

Assume that four observables $\mathsf{O}_{13} = (X^2, \mathcal{P}(X^2), F_{13})$*,* $\mathsf{O}_{14} = (X^2, \mathcal{P}(X^2), F_{14})$*,* $\mathsf{O}_{23} = (X^2, \mathcal{P}(X^2), F_{23})$ *and* $\mathsf{O}_{24} = (X^2, \mathcal{P}(X^2), F_{24})$ *are combinable, that is, we have the observable* $\mathsf{O} = (X^4, \mathcal{P}(X^4), F)$ *in* $\mathcal{N}$ *such that it satisfies (4). Then we have the combined measurement* $\mathsf{M}_{\mathcal{N}}(\mathsf{O} = (X^4, \mathcal{P}(X^4), F), S_{[\rho_0]})$ *of* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{13}, S_{[\rho_0]})$*,* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{14}, S_{[\rho_0]})$*,* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{23}, S_{[\rho_0]})$ *and* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{24}, S_{[\rho_0]})$*. And further, we have Bell's inequality in quantum language as follows.*

$$
|R_{13} - R_{14}| + |R_{23} + R_{24}| \leqq 2 \tag{5}
$$

Proof. Clearly we see, $i = 1,2, j = 3,4$,

$$
R_{ij} = \sum_{(x_1,x_2,x_3,x_4)\in X\times X\times X\times X} x_i \cdot x_j \ \ \rho_0(F(\{(x_1,x_2,x_3,x_4)\})) \tag{6}
$$

$\big($ for example, $R_{13} = \sum_{(x_1,x_2,x_3,x_4)\in X\times X\times X\times X} x_1 \cdot x_3 \ \ \rho_0(F(\{(x_1,x_2,x_3,x_4)\}))$ $\big)$. Therefore, we see that

$$
\begin{aligned}
&|R_{13} - R_{14}| + |R_{23} + R_{24}|\\
=&\sum_{(x_1,x_2,x_3,x_4)\in X\times X\times X\times X} \Big[|x_1 \cdot x_3 - x_1 \cdot x_4| + |x_2 \cdot x_3 + x_2 \cdot x_4|\Big]\rho_0(F(\{(x_1,x_2,x_3,x_4)\}))\\
=&\sum_{(x_1,x_2,x_3,x_4)\in X\times X\times X\times X} \Big[|x_3 - x_4| + |x_3 + x_4|\Big]\rho_0(F(\{(x_1,x_2,x_3,x_4)\})) \leq 2
\end{aligned}
$$

This completes the proof. □

As the corollary of this theorem, we have the followings:

Corollary 4. *Consider the parallel measurement* $\otimes_{i=1,2,j=3,4} \mathsf{M}_{\mathcal{N}}(\mathsf{O}_{ij} = (X^2, \mathcal{P}(X^2), F_{ij}), S_{[\rho_0]})$ *as in Theorem 3. Let*

$$x = \big((x_{13}^1, x_{13}^2), (x_{14}^1, x_{14}^2), (x_{23}^1, x_{23}^2), (x_{24}^1, x_{24}^2)\big)$$
$$\in \bigtimes_{i,j=1,2} X^2 (\equiv \{-1,1\}^8)$$

be a measured value of the parallel measurement $\otimes_{i=1,2,j=3,4} \mathsf{M}_{\mathcal{N}}(\mathsf{O}_{ij} = (X^2, \mathcal{P}(X^2), F_{ij}), S_{[\rho_0]})$*. Let* N *be sufficiently large natural number. Consider* N*-parallel measurement* $\bigotimes_{n=1}^N [\otimes_{i=1,2,j=2,3} \mathsf{M}_{\mathcal{N}}(\mathsf{O}_{ij} := (X^2, \mathcal{P}(X^2), F_{ij}), S_{[\rho_0]})]$*. Let* $\{x^n\}_{n=1}^N$ *be the measured value. That is,*

$$\{x^n\}_{n=1}^N = \begin{bmatrix} \big((x_{13}^{1,1}, x_{13}^{2,1}), (x_{14}^{1,1}, x_{14}^{2,1}), (x_{23}^{1,1}, x_{23}^{2,1}), (x_{24}^{1,1}, x_{24}^{2,1})\big) \\ \big((x_{13}^{1,2}, x_{13}^{2,2}), (x_{14}^{1,2}, x_{14}^{2,2}), (x_{23}^{1,2}, x_{23}^{2,2}), (x_{24}^{1,2}, x_{24}^{2,2})\big) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \big((x_{13}^{1,N}, x_{13}^{2,N}), (x_{14}^{1,N}, x_{14}^{2,N}), (x_{23}^{1,N}, x_{23}^{2,N}), (x_{24}^{1,N}, x_{24}^{2,N})\big) \end{bmatrix} \in (X^8)^N$$

Here, note that the law of large numbers says:

$$R_{ij} \approx \frac{1}{N} \sum_{n=1}^N x_{ij}^{1,n} x_{ij}^{2,n} \qquad (i = 1,2, j = 3,4)$$

Then, it holds, by the formula (5), that

$$|\sum_{n=1}^N \frac{x_{13}^{1,n} x_{13}^{2,n}}{N} - \sum_{n=1}^N \frac{x_{14}^{1,n} x_{14}^{2,n}}{N}| + |\sum_{n=1}^N \frac{x_{23}^{1,n} x_{23}^{2,n}}{N} + \sum_{n=1}^N \frac{x_{24}^{1,n} x_{24}^{2,n}}{N}| \le 2, \tag{7}$$

which is also called Bell's inequality in quantum language.

Remark 5. [The conventional Bell's inequality (*cf.* [13])] The mathematical Bell's inequality is as follows: Let $(\Theta, \mathcal{B}, P)$ be a probability space. Let $(f_1, f_2, f_3, f_4) : \Theta \to X^4 (\equiv \{-1,1\}^4)$ be a measurable functions. Define the correlation functions $\widehat{R}_{ij} (i = 1,2, j = 3,4)$ by $\int_\Theta f_i(\theta) f_j(\theta) P(d\theta)$. Then, the following mathematical Bell's inequality holds:

$$|\widehat{R}_{13} - \widehat{R}_{14}| + |\widehat{R}_{23} - \widehat{R}_{24}| \le 2 \tag{8}$$

This is easily proved as follows.

$$\text{"the left-hand side of the above (8)"} \le \int_\Theta |f_3(\theta) + f_4(\theta)| P(d\theta) + \int_\Theta |f_3(\theta) - f_4(\theta)| P(d\theta) \le 2$$

□

3 Bell's inequality is violated in classical systems as well as quantum systems

In the previous section, we show that

(E_1) Under the combinable condition (*cf.* Definition 1), Bell's inequality (5) (or, (7)) holds in both classical systems and quantum systems.

Or, equivalently,

(E_2) If Bell's inequality (5) (or (7)) is violated, then the combined observable does not exist, and thus, we cannot obtain the measured value (by the measurement of the combined observable).

This makes us expect that

(F) Bell's inequality (5) (or (7)) is violated in classical systems as well as quantum systems without the combined condition.

This (F) was already shown in my previous paper [7]. However, I got a lot of questions concerning (F) from the readers. Thus, in this section, we again explain the (F) precisely.

For this, three steps ([Step:I] ∼[Step:III]) are prepared in what follows.

[Step: I]. Put $X = \{-1, 1\}$. Define complex numbers $a_k(= \alpha_k + \beta_k\sqrt{-1})$ $(k = 1, 2, 3, 4)$ such that $|a_k| = 1$. Define the probability space $(X^2, \mathcal{P}(X^2), \nu_{a_i a_j})$ such that $(i = 1, 2, j = 3, 4)$

$$
\begin{aligned}
&\nu_{a_i a_j}(\{(1,1)\}) = \nu_{a_i a_j}(\{(-1,-1)\}) = (1 - \alpha_i\alpha_j - \beta_i\beta_j)/4 \\
&\nu_{a_i a_j}(\{(-1,1)\}) = \nu_{a_i a_j}(\{(1,-1)\}) = (1 + \alpha_i\alpha_j + \beta_i\beta_j)/4
\end{aligned}
\tag{9}
$$

The correlation $R(a_i, a_j)$ $(i = 1, 2, j = 3, 4)$ is defined as follows:

$$
R(a_i, a_j) \equiv \sum_{(x_1,x_2)\in X\times X} x_1 \cdot x_2 \nu_{a_i a_j}(\{(x_1, x_2)\}) = -\alpha_i\alpha_j - \beta_i\beta_j \tag{10}
$$

Now we have the following problem:

(G) Find a measurement $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{a_i a_j} := (X^2, \mathcal{P}(X^2), F_{a_i a_j}), S_{[\rho_0]})$ $(i = 1, 2, j = 3, 4)$ in a basic structure $[\mathcal{A}, \mathcal{N}]_{B(H)}$ such that

$$
\nu_{a_i a_j}(\Xi) = \rho_0(F_{a_i a_j}(\Xi)) \qquad (\forall \Xi \in \mathcal{P}(X^2)) \tag{11}
$$

and

$$
\begin{aligned}
&F_{a_1 a_3}(\{x_1\} \times X) = F_{a_1 a_4}(\{x_1\} \times X) \quad F_{a_1 a_3}(X \times \{x_3\}) = F_{a_2 a_3}(X \times \{x_3\}) \\
&F_{a_2 a_3}(\{x_2\} \times X) = F_{a_2 a_4}(\{x_2\} \times X) \quad F_{a_1 a_4}(X \times \{x_4\}) = F_{a_2 a_4}(X \times \{x_4\}) \\
&\qquad (\forall x_k \in X(\equiv \{-1, 1\}), k = 1, 2, 3, 4)
\end{aligned}
$$

[Step: II].

Let us answer this problem (G) in the two cases (i.e., classical case and quantum case), that is,

$$
\bullet \begin{cases} \text{(i):the case of quantum systems: } [\mathcal{A} = B(\mathbb{C}^2 \otimes \mathbb{C}^2)] \\ \text{(ii):the case of classical systems: } [\mathcal{A} = C_0(\Omega \times \Omega)] \end{cases}
$$

(i):the case of quantum system: $[\mathcal{A} = B(\mathbb{C}^2) \otimes B(\mathbb{C}^2) = B(\mathbb{C}^2 \otimes \mathbb{C}^2)]$

Put

$$
e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (\in \mathbb{C}^2).
$$

For each a_k $(k=1,2,3,4)$, define the observable $\mathsf{O}_{a_k} \equiv \big(X, \mathcal{P}(X), G_{a_k}\big)$ in $B(\mathbb{C}^2)$ such that

$$G_{a_k}(\{1\}) = \frac{1}{2}\begin{bmatrix} 1 & \bar{a}_k \\ a_k & 1 \end{bmatrix}, \quad G_{a_k}(\{-1\}) = \frac{1}{2}\begin{bmatrix} 1 & -\bar{a}_k \\ -a_k & 1 \end{bmatrix}.$$

Then, we have four observable:

$$\widehat{\mathsf{O}}_{a_i} = (X, \mathcal{P}(X), G_{a_i} \otimes I), \quad \widehat{\mathsf{O}}_{a_j} = (X, \mathcal{P}(X), I \otimes G_{a_j}) \quad (i=1,2, j=3,4) \tag{12}$$

and further,

$$\mathsf{O}_{a_i a_j} = (X^2, \mathcal{P}(X^2), F_{a_i a_j} := G_{a_i} \otimes G_{a_j}) \qquad (i=1,2, j=3,4) \tag{13}$$

in $B(\mathbb{C}^2 \otimes \mathbb{C}^2)$, where it should be noted that $F_{a_i a_j}$ is separated by G_{a_i} and G_{a_j}.
Further define the singlet state $\rho_0 = |\psi_s\rangle\langle\psi_s| \ \big(\in \mathfrak{S}^p(B(\mathbb{C}^2 \otimes \mathbb{C}^2)^*)\big)$, where

$$\psi_s = (e_1 \otimes e_2 - e_2 \otimes e_1)/\sqrt{2}$$

Thus we have the measurement $\mathsf{M}_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(\mathsf{O}_{a_i a_j}, S_{[\rho_0]})$ in $B(\mathbb{C}^2 \otimes \mathbb{C}^2)$ $(i=1,2, j=3,4)$. The followings are clear: for each $(x_1, x_2) \in X^2 (\equiv \{-1,1\}^2)$,

$$\rho_0(F_{a_i a_j}(\{(x_1,x_2)\})) = \langle \psi_s, (G_{a_i}(\{x_1\}) \otimes G_{a_j}(\{x_2\}))\psi_s \rangle = \nu_{a_i a_j}(\{(x_1,x_2)\}) \quad (i=1,2, j=3,4) \tag{14}$$

For example, we easily see:

$$\begin{aligned}
&\rho_0(F_{a_i b_j}(\{(1,1)\})) = \langle \psi_s, (G_{a_i}(\{1\}) \otimes G_{a_j}(\{1\}))\psi_s \rangle \\
=&\frac{1}{8}\langle (e_1 \otimes e_2 - e_2 \otimes e_1), (\begin{bmatrix} 1 & \bar{a}_i \\ a_i & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \bar{a}_j \\ a_j & 1 \end{bmatrix})(e_1 \otimes e_2 - e_2 \otimes e_1)\rangle \\
=&\tfrac{1}{8}\langle (\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}), (\begin{bmatrix} 1 & \bar{a}_i \\ a_i & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \bar{a}_j \\ a_j & 1 \end{bmatrix})(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix})\rangle \\
=&\frac{1}{8}\langle (\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}), (\begin{bmatrix} 1 \\ a_i \end{bmatrix} \otimes \begin{bmatrix} \bar{a}_j \\ 1 \end{bmatrix} - \begin{bmatrix} \bar{a}_i \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ a_j \end{bmatrix})\rangle \\
=&\frac{1}{8}(2 - a\bar{a}_j - \bar{a}_i a_j) = (1 - \alpha_i \alpha_j - \beta_i \beta_j)/4 = \nu_{a_i a_j}(\{(1,1)\}).
\end{aligned}$$

Therefore, the measurement $\mathsf{M}_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(\mathsf{O}_{a_i a_j}, S_{[\rho_0]})$ satisfies the condition (G).

(ii):the case of classical systems: $[\mathcal{A} = C_0(\Omega) \otimes C_0(\Omega) = C_0(\Omega \times \Omega)]$

Put $\omega_0 (= (\omega_0', \omega_0'')) \in \Omega \times \Omega$、$\rho_0 = \delta_{\omega_0}$ $(\in \mathfrak{S}^p(C_0(\Omega \times \Omega)^*)$, i.e., the point measure at $\omega_0)$). Define the observable $\mathsf{O}_{a_i a_j} := (X^2, \mathcal{P}(X^2), F_{a_i a_j})$ in $L^\infty(\Omega \times \Omega)$ such that

$$[F_{a_i a_j}(\{(x_1,x_2)\})](\omega) = \nu_{a_i a_j}(\{(x_1,x_2)\}) \quad (\forall (x_1,x_2) \in X^2, i=1,2, j=3,4, \forall \omega \in \Omega \times \Omega) \tag{15}$$

Thus, we have four observables

$$\mathsf{O}_{a_i a_j} = (X^2, \mathcal{P}(X^2), F_{a_i a_j}) \qquad (i=1,2, j=3,4) \tag{16}$$

in $L^\infty(\Omega \times \Omega)$ (though the variables are not separable (*cf.* the formula (13)). Then, it is clear that the measurement $\mathsf{M}_{L^\infty(\Omega \times \Omega)}(\mathsf{O}_{a_i a_j}, S_{[\delta_{\omega_0}]})$ satisfies the condition (G).

[Step: III].

As defined by (9), consider four complex numbers $a_k(=\alpha_k+\beta_k\sqrt{-1}; k=1,2,3,4)$ such that $|a_k|=1$. Thus we have four observables

$$\mathsf{O}_{a_1a_3}:=(X^2,\mathcal{P}(X^2),F_{a_1a_3}),\quad \mathsf{O}_{a_1a_4}:=(X^2,\mathcal{P}(X^2),F_{a_1a_4}),$$
$$\mathsf{O}_{a_2a_3}:=(X^2,\mathcal{P}(X^2),F_{a_2a_3}),\quad \mathsf{O}_{a_2a_4}:=(X^2,\mathcal{P}(X^2),F_{a_2a_4}),$$

in the W^*-algebra $\mathcal{N}$. Thus, we have the parallel measurement $\otimes_{i=1,2,j=3,4}\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{a_ia_j}:=(X^2,\mathcal{P}(X^2),F_{a_ia_j}), S_{[\rho_0]})$ in $\otimes_{i=1,2,j=3,4}\mathcal{N}$.
Thus, putting

$$a_1=\sqrt{-1},\ a_2=1,\ a_3=\frac{1+\sqrt{-1}}{\sqrt{2}},\ a_4=\frac{1-\sqrt{-1}}{\sqrt{2}},$$

we see, by (10), that

$$|R(a_1,a_3)-R(a_1,a_4)|\ +\ |R(a_2,a_3)+R(a_2,a_4)|=2\sqrt{2} \tag{17}$$

Further, assume that the measured value is $x(\in X^8)$. That is,

$$x=\big((x_{13}^1,x_{13}^2),(x_{14}^1,x_{14}^2),(x_{23}^1,x_{23}^2),(x_{24}^1,x_{24}^2)\big)\in \mathop{\times}_{i,j=1,2} X^2(\equiv\{-1,1\}^8)$$

Let N be sufficiently large natural number. Consider N-parallel measurement $\bigotimes_{n=1}^N[\otimes_{i=1,2,j=3,4}\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{a_ia_j}:=(X^2,\mathcal{P}(X^2),F_{a_ia_j}), S_{[\rho_0]})]$. Assume that its measured value is $\{x^n\}_{n=1}^N$. That is,

$$\{x^n\}_{n=1}^N=\begin{bmatrix} \big((x_{13}^{1,1},x_{13}^{2,1}),(x_{14}^{1,1},x_{14}^{2,1}),(x_{23}^{1,1},x_{23}^{2,1}),(x_{24}^{1,1},x_{24}^{2,1})\big) \\ \big((x_{13}^{1,2},x_{13}^{2,2}),(x_{14}^{1,2},x_{14}^{2,2}),(x_{23}^{1,2},x_{23}^{2,2}),(x_{24}^{1,2},x_{24}^{2,2})\big) \\ \vdots\qquad\qquad\vdots\qquad\qquad\vdots \\ \Big((x_{13}^{1,N},x_{13}^{2,N}),(x_{14}^{1,N},x_{14}^{2,N}),(x_{23}^{1,N},x_{23}^{2,N}),(x_{24}^{1,N},x_{24}^{2,N})\Big)\end{bmatrix}\in\big(\mathop{\times}_{i=1,2,j=3,4}X^2\big)^N(\equiv\{-1,1\}^{8N})$$

Then, the law of large numbers says that

$$R(a_i,a_j)\approx\frac{1}{N}\sum_{n=1}^N x_{ij}^{1,n}x_{ij}^{2,n}\quad(i=1,2,j=3,4)$$

This and the formula (17) say that

$$|\sum_{n=1}^N\frac{x_{13}^{1,n}x_{13}^{2,n}}{N}-\sum_{n=1}^N\frac{x_{14}^{1,n}x_{14}^{2,n}}{N}|+|\sum_{n=1}^N\frac{x_{23}^{1,n}x_{23}^{2,n}}{N}+\sum_{n=1}^N\frac{x_{24}^{1,n}x_{24}^{2,n}}{N}|\approx 2\sqrt{2} \tag{18}$$

Therefore, Bell's inequality (5) (or (7)) is violated in classical systems as well as quantum systems.

4 Conclusions

In this paper, contrary to Bell's spirit (which inherits Einstein's spirit), we try to discuss Bell's inequality in the framework of quantum theory with the linguistic Copenhagen interpretation of quantum mechanics. And we show Theorem 3 (Bell's inequality in quantum language), which says the statement (E_2), that is,

(H) ($\equiv$ (E_2)): If Bell's inequality (5) (or (7)) is violated, then the combined observable does not exist, and thus, we cannot obtain the measured value.

Also, recall that Bell's original argument (based on Einstein's spirit) says, roughly speaking, that

(I) : If the mathematical Bell's inequality (8) is violated, then hidden variables do not exist.

which is rather philosophical. It should be note that the (I) is a statement in Einstein's spirit, on the other hand, the (H) is a statement in scientific theory (i.e., quantum theory with the linguistic Copenhagen interpretation). Therefore, we assert that our (H) is a scientific representation of the philosophical (I). If so, we can, for the first time, understand Bell's inequality in science.

We hope that our proposal will be examined from various points of view[1].

References

[1] Bell, J.S. *On the Einstein-Podolosky-Rosen Paradox,* Physics 1, 195–200 (1966)

[2] Bohr, N. *Can quantum-mechanical description of physical reality be considered complete ?,* Phys. Rev. (48) 696-702 (1935)

[3] Born, M. *Zur Quantenmechanik der Stoßprozesse (Vorläufige Mitteilung),* Z. Phys. (37) 863–867 1926

[4] Davies, E. B. *Quantum Theory of Open Systems,* Academic Press, 1976

[5] Einstein, A., Podolosky, B. and Rosen, N. *Can quantum-mechanical description of reality be considered completely ?* Phys. Rev. (47) 777–780 (1935)

[6] Ishikawa,S., *Mathematical Foundations of Measurement Theory,* Keio University Press Inc. 335pages, 2006. (`http://www.keio-up.co.jp/kup/mfomt/`)

[7] Ishikawa,S., *A New Interpretation of Quantum Mechanics,* Journal of quantum information science, Vol. 1, No. 2, 35-42, 2011 doi: 10.4236/jqis.2011.12005 (`http://www.scirp.org/journal/PaperInformation.aspx?paperID=7610`)

[8] Ishikawa,S., *Quantum Mechanics and the Philosophy of Language: Reconsideration of traditional philosophies,* Journal of quantum information science, Vol. 2, No. 1, 2-9, 2012 doi: 10.4236/jqis.2012.21002 (`http://www.scirp.org/journal/PaperInformation.aspx?paperID=18194`)

[9] Ishikawa,S., *Linguistic interpretation of quantum mechanics ; Projection Postulate,* Journal of quantum information science, Vol. 5, No.4 , 150-155, 2015, DOI: 10.4236/jqis.2015.54017 (`http://www.scirp.org/Journal/PaperInformation.aspx?PaperID=62464%20`)

[10] Ishikawa,S., *Linguistic interpretation of quantum mechanics: Quantum language* Version 2, Research Report (Department of mathematics, Keio university), (KSTS-RR-16/001, 2016, 426 pages) (`http://www.math.keio.ac.jp/academic/research_pdf/report/2016/16001.pdf`)
This preprint is a draft of my book *"Linguistic Interpretation of Quantum Mechanics –Towards World-Description in Quantum Language -"* Shiho-Shuppan Publisher,(2016)

[11] Ishikawa,S., *Final solution to mind-body problem by quantum language,* Journal of quantum information science, Vol. 7, No.2 , 48-56, 2017, DOI: 10.4236/jqis.2017.72005 (`http://www.scirp.org/Journal/PaperInformation.aspx?PaperID=76391`)

[12] Sakai, S., *C^*-algebras and W^*-algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete (Band 60), Springer-Verlag, (1971)

[13] Selleri, F. *Die Debatte um die Quantentheorie,* Friedr. Vieweg&Sohn Verlagsgesellscvhaft MBH, Braunschweig (1983)

[14] Yosida, K., *Functional Analysis,* Springer-Verlag, 6th edition, 1980.

[1] For the further information of quantum language, see my home page: http://www.math.keio.ac.jp/~ishikawa/indexe.html