Infinite Derivative Gravity and Exact Solutions of the Newton-Schroedinger Equation

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Abstract

Exact solutions to the stationary spherically symmetric Newton-Schroedinger equation are proposed in terms of integrals involving *generalized* Gaussians. The energy eigenvalues are also obtained in terms of these integrals which agree with the numerical results in the literature. A discussion of infinite-derivative-gravity follows which allows to generalize the Newton-Schroedinger equation by *replacing* the ordinary Poisson equation with a *modified* non-local Poisson equation associated with infinite-derivative gravity. Finally, we argue how to replace the nonlinear Newton-Schroedinger equation for a non-linear quantum-like Bohm-Poisson equation involving Bohm's quantum potential, and where the fundamental quantity is *no* longer the wave-function Ψ but the real-valued probability density ρ.

Keywords: Quantum Mechanics, Newton-Schroedinger Equation, Infinite Derivative Gravity, de Broglie-Bohm theory.

1 The Newton-Schroedinger Equation

Various arguments have been put forward from time to time to support the view that quantum state reduction is a phenomenon that occurs objectively, because of some gravitational influence [3], [2], [4], [6], [7]. According to a particular argument put forward by Penrose [3], a superposition of two quantum states, each of which would be stationary on its own, but for which there is a significant mass displacement between the two states, ought to be unstable and reduce to one state or the other within a certain characteristic average timescale T_G. This argument is motivated by a conflict between the basic principles of quantum mechanics and those of general relativity. It is accordingly proposed that T_G can be calculated in situations for which velocities and gravitational potentials are small in relativistic units, so that a Newtonian approximation is appropriate, and T_G is the reciprocal, in Planckian units, of the gravitational self-energy E_G of the difference between the mass distributions of the two states.

There is of course a substantial literature on the problem of wavefunction collapse and the related measurement problem. See, for example, [6], [7] and references therein. For a different idea about gravitationally-induced wave-function collapse see [4]. It has been pointed out by [2], [3] that one can regard the basic stationary states, into which a superposition of such states is to decay into (on a timescale of order $\hbar/E_G$), as stationary solutions of the Schroedinger equation where there is an additional term provided by a certain gravitational potential. The appropriate gravitational potential is the one which arises from the mass density given by the expectation value of the mass distribution in the state determined by the wave-function. In the practical situations under consideration in [3] (such as with the proposed class of experiments put forward there), it would be sufficient to consider Newtonian gravity. This leads us to consider what Penrose termed the Schroedinger-Newton equation.

This equation has had a long long history since the 1950's [1], [5]. It is the name given to the system coupling the Schroedinger equation to the Poisson equation. In the case of a single particle, this coupling is effected as follows: for the potential energy term in the Schroedinger equation take the gravitational potential energy determined by the Poisson equation from a matter density proportional to the probability density obtained from the wavefunction. For a single particle of mass m the system consists of the following pair of partial differential equations:

The Newton-Schroedinger equation is nonlinear and nonlocal modification of the Schroedinger equation given by

$$i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} \;=\; -\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r},t) \;+\; V(\vec{r},t)\;\Psi(\vec{r},t) \;+\; mV_G(\vec{r},t)\;\Psi(\vec{r},t) \qquad (1)$$

where $V(\vec{r},t)$ is the external potential acting on the particle and $mV_G(\vec{r},t)$ is the self-gravitational potential energy arising due a mass density obtained from the wave function of the particle itself. Given the Poisson equation sourced by a mass density $\rho = m|\Psi(\vec{r},t)|^2$

$$\nabla^2 V_G(\vec{r},t) \;=\; 4\pi\; Gm\;|\Psi(\vec{r},t)|^2 \qquad (2)$$

it leads to a self-gravitational potential

$$V_G(\vec{r},t) \;=\; -\,G\int\frac{m|\Psi(\vec{r'},t)|^2}{|\vec{r}-\vec{r'}|}\;d^3r' \qquad (3)$$

Inserting eq-(3) into eq-(1) leads to the integro-differential form of the nonlinear and nonlocal Newton-Schroedinger equation

$$i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} \;=\; -\;\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r},t) \;+\; V(\vec{r},t)\;\Psi(\vec{r},t) \;-\; \left(\;Gm^2\int\frac{|\Psi(\vec{r'},t)|^2}{|\vec{r}-\vec{r'}|}\;d^3r'\right)\;\Psi(\vec{r},t) \qquad (4)$$

This equation is based on the assumption that the point-particle is *smeared* over space such that its mass is distributed according to its wave function. Namely, there is a mass cloud over space whose net mass $m = \int m|\Psi(\vec{r}', t)|^2 d^3r'$ coincides with the mass of the point-particle. The mass cloud is self-gravitating and experiences a gravitational potential energy given by eq-(3). [1]

Let us set the external potential to zero and look for stationary solutions $\Psi(\vec{r}, t) = e^{-iEt/\hbar}\ \Phi(\vec{r})$, such that the gravitational potential becomes time *independent*. $\Phi(\vec{r})$ obeys the equation

$$E\ \Phi(\vec{r}) \ = \ - \ \frac{\hbar^2}{2m}\nabla^2\ \Phi(\vec{r}) \ - \ \left(Gm^2 \int \frac{|\Phi(\vec{r}')|^2}{|\vec{r} - \vec{r}'|}\ d^3r' \right)\ \Phi(\vec{r}) \tag{5}$$

The authors [9], [10] found *numerical* spherically-symmetric solutions to eq-(5). Variational forms of the stationary Newton-Schroedinger equation to find a lower bound for the ground state energy have been studied by several authors, see references in [8], and compared to numerical values in the literature.

If one replaces a delta function point-mass source distribution $m\delta^3(\vec{r}) = m\delta(r)/4\pi r^2$ for a normalized Gaussian mass distribution of width σ

$$\rho(r) \ = \ \frac{m}{\pi^{3/2}}\ \frac{e^{-r^2/\sigma^2}}{\sigma^3} \ \Rightarrow \ m \ = \ \int_0^\infty \ \rho(r)\ 4\pi r^2\ dr \tag{6}$$

the solution to Poisson's equation

$$\nabla^2 V_G(r) \ = \ 4\pi\ G\ \rho(r) \tag{7}$$

is given in terms of the error function Erf(r) [2] as follows

$$V_G(r) \ = \ - \ \frac{Gm}{r}\ Erf(\frac{r}{\sigma}) \ = \ - \ \frac{Gm}{r}\ \frac{1}{\sqrt{\pi}} \int_{-r/\sigma}^{r/\sigma} \ e^{-t^2}\ dt \tag{8}$$

In the asymptotic regime $r \rightarrow \infty$, $Erf(\infty) \rightarrow 1$, the potential recovers the Newtonian form $-\frac{Gm^2}{r}$. Because the error function Erf(r) admits a series expansion around $r = 0$ as

$$Erf(r) \simeq \ \frac{2r}{\sqrt{\pi}} \ - \ \frac{r^3}{\sqrt{\pi}} \ + \ \frac{11r^5}{20\sqrt{\pi}} \ - \ \frac{241r^7}{840\sqrt{\pi}} \ + \ \ldots \tag{9}$$

the potential $V_G(r)$ is *no* longer *singular* at the origin $r = 0$, but it behaves as

$$V_G(r) \simeq \ - \ \frac{2Gm}{\sqrt{\pi}\ \sigma}\ [\ 1 \ - \ \frac{r^2}{2\sigma^2} \ + \ \frac{11r^4}{40\sigma^4} \ + \ \ldots\ ] \tag{10}$$

The normalized Gaussian wave function

$$\Phi(\vec{r}) \ \equiv \ \frac{1}{\pi^{3/4}}\ \frac{e^{-r^2/2\sigma^2}}{\sigma^{3/2}} \tag{11}$$

[1] This may be in conflict with Born's rule of interpreting $|\Psi(\vec{r}', t)|^2$ as the probability density of finding a particle at the *point* $\vec{r}$ if one has abandoned the notion of point-particles. At the moment we shall not be concerned with this.

[2] Which is also related to the incomplete gamma function $\gamma(\frac{1}{2}; r) = \sqrt{\pi}\ Erf(r)$

satisfies

$$\int_0^\infty \ |\Phi(\vec{r})|^2 \ 4\pi \ r^2 \ dr \ = \ 1 \tag{12}$$

Let us minimize the energy functional

$$E \ = \ -\frac{\hbar^2}{2m} \int_0^\infty \ (\Phi(\vec{r}) \ \nabla^2 \ \Phi(\vec{r})) \ 4\pi \ r^2 \ dr \ + \ \int_0^\infty \ m \ V_G[\Phi(\vec{r})] \ |\Phi(\vec{r})|^2 \ 4\pi \ r^2 \ dr \tag{13}$$

using this Gaussian as a trial function, and which generates the regular potential at the origin $V_G[\Phi(\vec{r})] \ = \ - \frac{Gm}{r} \ Erf(\frac{r}{\sigma})$ given by eq-(8) after solving Poisson's equation (7) . The integrals to be evaluated in eq-(13) are of the form

$$\int_0^\infty \ (x^4 - 3x^2) \ e^{-x^2} \ dx \ = \ -\frac{3}{8} \sqrt{\pi} \ Erf(x) \ - \ \frac{1}{4} \ e^{-x^2} \ x \ (2x^2 - 3) \tag{14}$$

$$\int_0^\infty \ x \ Erf(x) \ e^{-x^2} \ dx \ = \ \frac{1}{4} \ \left(\sqrt{2} \ Erf(\sqrt{2} \ x) \ - \ 2 \ e^{-x^2} \ Erf(x) \right) \tag{15}$$

After performing the definite integrals we find that

$$E(\sigma) \ = \ \frac{3\sqrt{\pi}}{4} \ \frac{\hbar^2}{m \ \sigma^2} \ - \ \sqrt{\frac{2}{\pi}} \ \frac{Gm^2}{\sigma} \tag{16}$$

Minimizing $dE(\sigma)/d\sigma = 0$ yields

$$\sigma_{min} \ = \ \frac{3\pi}{2\sqrt{2}} \ \frac{\hbar^2}{Gm^3} \tag{17}$$

and inserting this value of σ into $E(\sigma)$ gives

$$E_{min} \ = \ - \ \frac{2}{3\pi^{3/2}} \ \frac{G^2 m^5}{\hbar^2} \ = \ - \ 0.119 \ \frac{G^2 m^5}{\hbar^2} \tag{18}$$

which is a satisfactory value since it is *above* the *lower* energy bound

$$E_{bound} \ = \ - \ \frac{32}{9\pi^2} \ \frac{G^2 m^5}{\hbar^2} \ = \ - \ 0.360 \ \frac{G^2 m^5}{\hbar^2} \tag{19}$$

derived by [8], and it is also close to the value $-0.163 \ (G^2 m^5/\hbar^2)$ obtained numerically by [9], [10]. We can also verify that the value of σ_{min} given by eq-(17) is consistent with the virial theorem $|U| = 2T$, stating that the absolute value of the potential energy is twice the kinetic energy. It is also worth mentioning that a trial exponential $\psi(r) = (k^3/\pi)^{1/2} e^{-kr}$ leads to an upper energy bound for the ground state of [8]

$$E_0 \ \leq \ - \ \frac{75}{512} \ \frac{G^2 m^5}{\hbar^2} \ = \ - \ 0.146 \ \frac{G^2 m^5}{\hbar^2} \tag{20}$$

which is close to the value found numerically by [9], [10]. However the series expansion of e^{-kr} involves even and odd powers of r which is not the case of the numerical solutions provided by [9], [10]. The spherically-symmetric solutions of the stationary Schroedinger-Newton equations have been solved *numerically* by many authors [9], [10]. They have demonstrated numerically the existence of a discrete set of "bound-state" solutions which are everywhere finite and smooth (and which are associated with finite energy eigenvalues), but which separate ever decreasing intervals of partial solutions which diverge alternately to plus or minus infinity.

The value found for the ground state energy turned out to be

$$E_0 \;=\; -\;0.163\;\frac{G^2m^5}{\hbar^2}\;=\;-\;0.163\;(\frac{m}{M_{Planck}})^5\;M_{Planck}\;c^2 \tag{21}$$

these numerical values for the energy are very small for masses much smaller than the Planck mass.

The width (spread) σ of the Gaussian wave-function is measured in units of $(\hbar^2/Gm^3)$, which translated into Planckian units is $(L_P/L_S)^3(2L_P)$, where L_P is the Planck length $(2\hbar G/c^3)^{1/2}$, and $L_S = 2Gm/c^2$ is the Schwarzschild radius. It was argued by the authors [9] that for a nucleon mass m the value of σ is vast and is of order of 10^{24} cm. The corresponding time $\Delta t = \hbar/\Delta E$ for the largest (in magnitude) energy is of the order of 10^{53} seconds for the mass of the nucleon, and is of the order of 1 second for 10^{11} nucleon masses. This is perfectly satisfactory for the state vector reduction of [3], because it tells us that the state of a single nucleon will not self-reduce on a timescale of relevance to any actual particle, in agreement with observation. For large collections of particles, on the other hand, the reduction time can become important.

The coupled system of differential equations (1,2) can be recast after integrating twice, and using *dimensionless* variables S, V, as [9]

$$S(r)\;=\;S_o\;-\frac{1}{l^2}\;\int_0^r\;x\;(1-\frac{x}{r})\;S(x)\;V(x)\;dx \tag{22}$$

$$V(r)\;=\;V_o\;-\frac{1}{l^2}\;\int_0^r\;x\;(1-\frac{x}{r})\;S^2(x)\;dx \tag{23}$$

The value of V_o must be $V_o > 0$ to ensure convergence of $S(r)$ and $V(r)$ at $r \rightarrow \infty$. Scaling arguments allow one to choose $V_o = 1$. The numerical solutions to eqs-(22,23) found by [9], when $V_o = 1$, are

$$S(r)\;=\;S_o\;\left(1\;-\;\frac{1}{6}\;\frac{r^2}{l^2}\;+\;\frac{(S_o^2+1)}{120}\;\frac{r^4}{l^4}\;+\;\ldots\right) \tag{24}$$

$$V(r)\;=\;\left(1\;-\;\frac{S_o^2}{6}\;\frac{r^2}{l^2}\;+\;\frac{S_o^2}{60}\frac{r^4}{l^4}\;+\;\ldots\right) \tag{25}$$

The wavefunctions are $\Psi(r) = \frac{\alpha}{l^2}S(r)$, and where α, β are defined by

$$\alpha\;\equiv\;[\frac{\hbar^2}{8\pi Gm^3}]^{1/2},\quad \beta\;\equiv\;\frac{\hbar^2}{2m} \tag{26}$$

and l is a physical length parameter (the gravitational analog of the Bohr's radius) given by $l = \hbar^2/Gm^3$, such that $\beta/l^2 = \frac{1}{2}(G^2m^5/\hbar^2)$.

The energy eigenvalues are $E = U(r) + \frac{\beta}{l^2} V(r)$. Because $U(r=0)$ is of the form $\infty \times 0$ which is undefined, it is more convenient to evaluate the energy at $r = \infty$ where $U(\infty)$ vanishes, as follows [9], [10], [11]

$$E \ = \ U(r=\infty) \ + \ \frac{\beta}{l^2} V(r=\infty) \ = \ \frac{\beta}{l^2} \ \left(\ V_o \ - \frac{1}{l^2} \ \int_0^\infty \ x \ S^2(x) \ dx \ \right), \quad V_o = 1 \qquad (27)$$

The values of S_o are fine-tuned such that the ground state wave-function has *no* zeros, it is bounded at the origin and vanishes at infinity. By demanding that Ψ and U are finite and smooth everywhere, the authors showed that the n-th eigenfunction has n zeros and the wave-functions are normalizable. The corresponding energy eigenvalues are negative, converging monotonically to zero as n increases. Each bound-state is unstable in the sense that infinite precision is required in the initial value of $S(r=0) = S_o$ to ensure that the solutions do not diverge to infinity as r increases. For the ground state, the numerical value of S_o fell in the range given by $1.088 < S_o < 1.090$ and it led to the ground state energy $E_0 = -0.163 \ \frac{G^2 m^5}{\hbar^2}$. For further details of how to obtain other energy eigenvalues and eigenfunctions by choosing different values for S_o we refer to [10]. The study of the Newton-Schroedinger equations in D dimensions other than $D = 3$ can be found in [11].

A careful inspection of the work by [9], [10], [11] inspired us to find the remarkable numerical coincidence provided by the definite integral

$$- \ \int_0^\infty \ e^{-(y^2+y^4)} \ y^2 \ dy \ = \ - \ 0.160 \ \simeq \ -0.163 \ = \ \frac{1}{2} \ \left(\ 1 \ - \frac{1}{l^2} \ \int_0^\infty \ x \ S^2(x) \ dx \ \right) \qquad (28)$$

(we set $V_o = 1$). The normalization of the wavefunction $\Psi = \frac{\alpha}{l^2} S(r)$

$$\frac{\alpha^2}{l^4} \ \int_0^\infty \ 4\pi r^2 \ S^2 \ dr \ = \ 1 \qquad (29a)$$

allows to recast the right-hand side of (28) as

$$\frac{\alpha^2}{2l^4} \ \int_0^\infty \ 4\pi r^2 \ S^2 \ dr \ - \frac{1}{2l^2} \ \int_0^\infty \ x \ S^2(x) \ dx \qquad (29b)$$

after relabeling the variable r for x in the above expression, eq-(28) becomes

$$- \ 0.160 \ = \ - \ \int_0^\infty \ e^{-(y^2+y^4)} \ y^2 \ dy \ \simeq \ \frac{1}{2} \int_0^\infty \ \left(\ \frac{4\pi \ \alpha^2}{l} \ y \ - 1 \ \right) \ y \ S^2(y) \ dy \ = \ - \ 0.163 \qquad (30)$$

which allows to identify the variables $y \equiv \frac{x}{l} = \frac{r}{l}$, and basically equate the dimensionless integrals. However this does *not* mean that the *integrands* are the same. The integral in the right-hand side of eq-(30) is evaluated after inserting the value of $4\pi\alpha^2/l = \frac{1}{2}$. The expression for $S^2(r)$ corresponding to the ground state is derived from eq-(24) after inserting a value of $S_o \simeq 1.088$ (lying in the interval $[1.088, 1.090]$). The other "bound" states require *different* values of $S_{o,n}$ yielding different expressions for $S_n(r)$ ($S_n(r = 0) = S_{o,n}$), and which have n nodes (zeros).

Next we shall present the novel and most salient results of this section. To our knowledge these findings below are new. The following remarkable numerical coincidences for the values of the remaining (negative) energy eigenvalues $E_1, E_2, E_3, ...$ have also been found. To obtain the values for the first excited state we replace the integral in the left-hand side of (28) for :

$$n = 1 : \quad - \frac{1}{2^2} \int_0^\infty e^{-(y^2+y^4+y^6)} \ y^2 \ dy \ = \ - \ 0.0302 \ \simeq \ - 0.0308 \tag{31}$$

to get the second one

$$n = 2 : \quad - \frac{1}{3^2} \int_0^\infty e^{-(y^2+y^4+y^6+y^8)} \ y^2 \ dy \ = \ - \ 0.0120 \ \simeq \ - 0.0125 \tag{32}$$

the third one

$$n = 3 : \quad - \frac{1}{4^2} \int_0^\infty e^{-(y^2+y^4+y^6+y^8+y^{10})} \ y^2 \ dy \ = \ - \ 0.00641 \ \simeq \ - 0.00675 \tag{33}$$

the fourth one

$$n = 4 : \quad - \frac{1}{5^2} \int_0^\infty e^{-(y^2+y^4+y^6+y^8+y^{10}+y^{12})} \ y^2 \ dy \ = \ - \ 0.00399 \ \simeq \ - 0.00421 \tag{34}$$

and so forth ..... the pre-factors in front of the integrals for the n-th energy eigenvalue are $(n+1)^{-2}$, which are "reminiscent" of the Balmer series for the Hydrogen atom, and the arguments of the negative exponentials are $y^2 + y^4 + \cdots + y^{2(n+2)}$. The authors [9] plotted the ten energy eigenvalues as functions of n and found that the slope of $log(n)$ versus $log(E(n))$ was not -2 [9]. However, the slope is asymptotically close to -2 [12] which would be the exact value for the Hydrogen atom as previously note by Bernstein et al [10].

Using these expressions in eq-(28), and eqs-(31-34), we find that energy values are remarkably close to the numerical results obtained in [12], and given by the numbers in the right-hand side of eqs-(28, 31-34). We don't believe this is a numerical coincidence and could point to the fact that an integrability may underlie the solutions of the stationary spherically symmetric Newton-Schroedinger equations. It was emphasized by [9], [10], [11] that one would need to know the values of $S_n(r=0) = S_{o,n}$ with $infinite$ precision in order to ensure that $S_n(r)$ does not shoot off to $\pm\infty$ at a finite value of r. All the numerical results appearing in the right-hand side of eqs-(28, 31-34) are obtained by narrowing in the values of $S_{o,n}$ within certain domains. For this reason we find the analytical results in the left-hand side of eqs-(28,31-34) very appealing.

To conclude this section we shall recur to the *virial* theorem $< U > = -2 < T >$ $\Rightarrow \ E = < T + U > = - < T >$ to write the most general expression for the energy after integration by parts, in the form

$$E_n \ = \ - < T >_n \ = \frac{\hbar^2}{2m} \int_{-\infty}^\infty (\Psi_n(\vec{r}))^* \ \nabla^2 \ \Psi_n(\vec{r}) \ d^3r \ = \ - \ \frac{\hbar^2}{2m} \int_{-\infty}^\infty |(\nabla \Psi_n(\vec{r}))|^2 \ d^3r \ \simeq$$

$$- \frac{1}{(n+1)^2} \frac{G^2 m^5}{\hbar^2} \int_0^\infty e^{-(y^2+y^4+y^6+ \cdots + y^{2(n+2)})} y^2 \, dy \qquad (35)$$

the wave-functions must be normalized and vanish at $\pm\infty$. One must emphasize that one is comparing (roughly equating) the values of the integrals in (35) and *not* the integrands.

If one were to equate the *integrands* one would arrive at a *contradiction*. In the spherically symmetric case, for real-valued wave-functions, and after setting $y = r/l$, it gives

$$\frac{\hbar^2 \, l}{2m} \, (\partial_y \Psi_n(y))^2 \, 4\pi y^2 \simeq \frac{1}{(n+1)^2} \frac{G^2 m^5}{\hbar^2} e^{-(y^2+y^4+y^6+ \cdots + y^{2(n+2)})} y^2 \Rightarrow$$

$$(\partial_y \Psi_n(y))^2 \simeq \frac{1}{(n+1)^2} \frac{1}{2\pi l^3} e^{-(y^2+y^4+y^6+ \cdots + y^{2(n+2)})} \qquad (36)$$

Given Ψ_n an even, or an odd function, for $n =$ even, odd, respectively, the left hand side is always *even* due to the squaring of the derivatives, as it should, since the right hand side of (36) is an *even* function [3]. For this reason, if one is going to take the square root of eq-(36) we must choose $n =$ odd $\Rightarrow \partial\Psi =$ even. Hence

$$\partial_y \Psi_n(y) \simeq \frac{1}{(n+1)} \frac{1}{\sqrt{2\pi l^3}} e^{-\frac{1}{2}(y^2+y^4+y^6+ \cdots + y^{2(n+2)})}, \quad n = odd \qquad (37)$$

integrating (37) between 0 and y we arrive finally

$$\Psi_n(y) - \Psi_n(0) \simeq \frac{1}{(n+1)} \frac{1}{\sqrt{2\pi l^3}} \int_0^y e^{-\frac{1}{2}(y^2+y^4+y^6+ \cdots + y^{2(n+2)})} \, dy, \quad n = odd \qquad (38)$$

we still need to check that Ψ vanishes at $\pm\infty$, and include a normalization constant to enforce $\int |\Psi|^2 d^3r = 1$. When $n =$ odd, $\Psi(0) = 0$ because antisymmetric functions must *vanish* at the origin. Here is where the *contradiction* arises. The integral in the right-hand side is *not* zero when $y = \infty$. It is given by a *finite* number. This would force $\Psi_n(\infty) \neq 0$ since $\Psi_n(0) = 0$, for $n =$ odd. Since the wave-functions do not vanish at ∞ these solutions are unphysical. For this reason one *cannot* equate the *integrands* in eq-(35). It is only the integrals which can be equated.

To conclude this section, based on the numerical results by [9], [10], [11] we have arrived at the integrals

$$\int_0^y (\partial_y \Psi_n(y))^2 \, y^2 \, dy \simeq \frac{1}{(n+1)^2} \frac{1}{2\pi l^3} \int_0^y e^{-(y^2+y^4+y^6+ \cdots + y^{2(n+2)})} y^2 \, dy, \quad n = 0, 1, 2, \ldots \qquad (39)$$

which furnish *implicitly* the stationary spherically symmetric wave-function solutions to the Newton-Schroedinger equation in terms of integrals involving *generalized* Gaussians.

[3]This assumes that one can extend r to negative values $r < 0$, which is not unreasonable because $r = \pm\sqrt{x^2+y^2+z^2}$. Black-hole solutions can be extended to $r < 0$. The Schwarzschild metric solution is invariant under $r \to -r, m \to -m$

The energy eigenvalues are provided by the left hand side of eqs-(28,31-34). Eqs-(39) must be supplemented by the normalization condition of the wave-function. The time-dependent evolution of the Schroedinger-Newton equations has been studied by many authors, in particular [13], [14]. The experimental tests of the validity of the nonlinear Newton-Schroedinger equation pose a technologically formidable challenge due to the weakness of gravity and the difficulty of controlling quantum coherence [14]

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2 Infinite Derivative Gravity and The Modified Newton-Schroedinger Equation

Since Quantum Mechanics is notoriously non-local, it is not farfetched that a theory of Quantum Gravity may require to modify Einstein's theory of gravity (and other local theories of gravitation) to include a modified gravitational theory involving *infinite* derivatives, which by construction, is non-local. It turned out that the infinite derivative gravity (IDG) can resolve the problem of massive ghosts as well as it may avoid the singularity of the Newtonian potential at the origin, when one chooses the exponential of an entire function. This model is also named super-renormalizable quantum gravity [17]. However, one does not understand fully how this non-local gravity could provide a regular potential. It was argued that the cancellation of the singularity at the origin is an effect of an infinite amount of hidden ghost-like complex poles [16].

A regular potential at the origin of the form (8) has been studied by several authors [15] in connection to infinite derivative gravity [16] which is ghost-free and renormalizable when one choses the exponential of an entire function in the construction of the infinite-derivative gravitational (IDG) action S_{IDG} [16]. For this IDG case, the corresponding Newtonian potential generated from the delta function is non-singular at the origin. However, the authors [15] explicitly showed that the source generating this non-singular potential is given not by the delta-function due to the point-like source of mass, but by the Gaussian mass distribution. This explains clearly why the IDG with the exponential of an entire function yields the finite potential at the origin.

Because the Fourier transform of a Gaussian is a Gaussian, one can infer that the infinite-derivative (non-local) *modified* Poisson equation

$$(e^{-\frac{\sigma^2}{4}\nabla^2}\nabla^2)\ V \ = \ 4\pi\ Gm\ \delta^3(\vec{r}) \ = \ Gm\ \frac{\delta(r)}{r^2} \tag{40}$$

sourced by a point-mass at $r = 0$, leads to *identical* solutions to the potential as the ordinary Poisson equation sourced by a Gaussian mass distribution [15]. This can be shown by taking the $3D$ Fourier transform of eq-(40)

$$e^{\frac{\sigma^2}{4}\mathbf{k}^2}\ \mathbf{k}^2\ \tilde{V}(\mathbf{k}) = \ Gm \ \Rightarrow \ \tilde{V}(\mathbf{k}) \ = \ Gm\ \frac{e^{-\frac{\sigma^2}{4}\mathbf{k}^2}}{\mathbf{k}^2} \tag{41}$$

and then performing the inverse $3D$ Fourier transform it yields $V_G(r) = -\frac{Gm}{r}\ Erf(\frac{r}{\sigma})$, see [15] for further details.

One can *generalize* the Newton-Schroedinger coupled system of equations (1,2) by *replacing* the ordinary Poisson equation with the *modified* non-local Poisson equation associated with infinite-derivative gravity (IDG)

$$(e^{-\frac{\sigma^2}{4}\nabla^2}\ \nabla^2)\ \mathbf{V}_{IDG}(\vec{r}) \ = \ 4\pi\ G\rho \ = \ 4\pi\ Gm\ |\Phi(\vec{r})|^2 \tag{42}$$

In this case

$$\mathbf{V}_{IDG}(\vec{r}) \ \neq \ \ Gm\ \int\ \frac{|\Phi(\vec{r}')|^2}{|\vec{r}-\vec{r}'|}\ d^3r' \tag{43}$$

therefore to find (numerical) solutions of the highly nonlinear and nonlocal modified Newton-Schrodinger equation

$$E\ \Phi(\vec{r}) \ = \ -\frac{\hbar^2}{2m}\nabla^2\Phi(\vec{r}) \ + \ m\mathbf{V}_{IDG}[\Phi(\vec{r})]\ \Phi(\vec{r}) \tag{44}$$

becomes more problematic. However, solutions to the infinite-derivative modified Poisson equation are not that difficult to find. For example, in the case when $\rho(r)$ is given by a Gaussian profile, after performing the Fourier transform procedure it gives $V_{IDG} = -\frac{Gm^2}{r}\ Erf(\frac{r}{\sqrt{2}\sigma})$, which is almost identical to $-\frac{Gm^2}{r}\ Erf(\frac{r}{\sigma})$, the main difference being that $Erf(\frac{r}{\sigma}) \neq Erf(\frac{r}{\sqrt{2}\sigma})$.

Bohm's quantum potential $V_Q = (\nabla^2\sqrt{\rho}/\sqrt{\rho})$ has a *geometrical* derivation as the Weyl scalar curvature produced by an ensemble density of paths associated with one, and only one particle [18]. This geometrization process of quantum mechanics allowed to derive the Schroedinger, Klein-Gordon [18] and Dirac equations [19]. Most recently, a related geometrization of quantum mechanics was proposed [20] that describes the time evolution of particles as geodesic lines in a curved space, whose curvature is induced by the quantum potential. This formulation allows therefore the incorporation of all quantum effects into the geometry of space-time, as it is the case for gravitation in the general relativity. Based on these results we propose the following nonlinear quantum-like Bohm-Poisson equation

$$\nabla^2 V_Q \ = \ 4\pi G\rho \ \Rightarrow \ \nabla^2\ (\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}) \ = \ 4\pi\rho \tag{44}$$

such that one could replace the nonlinear Newton-Schroedinger equation for the above non-linear quantum-like Bohm-Poisson equation (44) where the fundamental quantity is *no* longer the wave-function Ψ (complex-valued in general) but the real-valued probability density $\rho = \Psi^*\Psi$.

It has been proposed by [21], [22] to give up the description of physical states in terms of ensembles of state vectors with various probabilities, relying instead solely on the density matrix as the description of reality. The time evolution of ρ is governed by the Lindblad equation. The authors [22] also investigated a number of unexplored features of quantum theory, including an interesting *geometrical* structure- which they called subsystem space- that they believed merits further study. We leave the geometry of quantum information theory, fractal spacetimes, Moyal-Fedosov deformation quantization (based on non-local star products), κ-deformed Poincare symmetry, curvature of momentum spaces, $\cdots$ for future investigations.

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