Laws of General Solutions of Partial Differential Equations

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Abstract

In this paper, four kinds of Z Transformations are used to get many laws of general solutions of mth-order linear and nonlinear partial differential equations with n variables. Some general solutions of first-order linear partial differential equations, which cannot be obtained by using the characteristic equation method, can be solved by the Z Transformations. By comparing, we find that the general solutions of some first-order partial differential equations got by the characteristic equation method are not complete.

keywords:
Z Transformations; banal PDEs; non-banal PDEs; general solutions; the characteristic equation method.

Introduction

In recent years, some numerical methods have been developed to solve linear partial differential equations (PDEs), such as Finite integration method [1, 2], Bernoulli matrix method [3], Chebyshev matrix method [4] and so on, the existence [5], uniqueness [6, 7] and stability [8] of their solutions are also the focus of research.

In [9], we used new analytical methods to preliminarily study some laws of general solutions of the mth-order linear PDEs with n variables. In this paper, we will use four kinds of Z Transforms to further study the mth-order linear and nonlinear PDEs with n variables.

1. New principles and methods

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In order to obtain the general solutions of PDEs in various orthogonal coordinate systems, we proposed the concepts and laws of the independent variable transformational equations (IVTEs) and the dependent variable transformational equations (DVTEs) in [9]. About the IVTEs, there is an important new theorem:

Theorem 1. *In the domain D, ($D \subset \mathbb{R}^n$), if* $G(y_1, y_2, \ldots y_n, u, u_{y_1}, u_{y_2} \ldots u_{y_n}, u_{y_1 y_2}, u_{y_1 y_3} \ldots) = 0$ *is an arbitrary independent variable transformational equation of a mth-order PDE* $F(x_1, x_2 \ldots x_n, u, u_{x_1}, u_{x_2} \ldots u_{x_n}, u_{x_1 x_2}, u_{x_1 x_3} \ldots) = 0$*, so*
1.If F=0 is a linear PDE, then G=0 is an mth-order linear PDE.
2.If F=0 is a nonlinear PDE, then G=0 is an mth-order nonlinear PDE.

Proof. Since

$$u_{x_i} = \sum_{p=1}^{n} u_{y_p} \frac{\partial y_p}{\partial x_i}, \tag{1}$$

$$u_{x_i x_j} = \sum_{p=1}^{n} u_{y_p} \frac{\partial^2 y_p}{\partial x_i \partial x_j} + \sum_{p=1}^{n}\sum_{q=1}^{n} u_{y_p y_q} \frac{\partial y_p}{\partial x_i}\frac{\partial y_q}{\partial x_j}, \tag{2}$$

$$\begin{aligned} u_{x_i x_j x_k} =& \sum_{p=1}^{n} u_{y_p} \frac{\partial^3 y_p}{\partial x_i \partial x_j \partial x_k} + \sum_{p=1}^{n}\sum_{q=1}^{n} u_{y_p y_q}\left(\frac{\partial^2 y_p}{\partial x_i \partial x_j}\frac{\partial y_q}{\partial x_k} + \frac{\partial^2 y_p}{\partial x_i \partial x_k}\frac{\partial y_q}{\partial x_j} + \frac{\partial y_p}{\partial x_i}\frac{\partial^2 y_q}{\partial x_j \partial x_k}\right) \\ &+ \sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{r=1}^{n} u_{y_p y_q y_r} \frac{\partial y_p}{\partial x_i}\frac{\partial y_q}{\partial x_j}\frac{\partial y_r}{\partial x_k}, \end{aligned} \tag{3}$$

$$\cdots$$

where $i, j, k, p, q, r \in \{1, 2, \cdots n\}$, namely $u_{x_i}, u_{x_i x_j}, u_{x_i x_j x_k} \cdots$ are linear relationship with $u_{y_p}, u_{y_p y_q}, u_{y_p y_q y_r} \cdots$, and the highest order of the partial derivatives on both sides of the equations are equal. Therefore each linear term in $F = 0$ is transformed into a new linear term; every non-linear term in $F = 0$ is transformed into a new nonlinear term; and the highest order of the partial derivative of the dependent variable of each term is constant. So the independent variable transformation not only does not change the linearity or non-linearity of the original PDEs, but also does not change their order. Then Theorem 1 is proved. □

For the DVTEs, the specific lth-order transformation $v = h(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1 x_2}, \cdots)$ may be linear or non-linear. For the linear transformation, each linear term of an mth-order linear PDE is transformed into a new linear term, so a $(l+m)$th-order linear PDE will be transformed. For an mth-order nonlinear PDE, each nonlinear term will transform a new nonlinear term, normally it will be transformed a $(l+m)$th-order nonlinear PDE.

For the nonlinear transformation, an mth-order linear PDE is usually transformed into a $(l+m)$th-order nonlinear PDE, an mth-order nonlinear PDE may be transformed into a $(l+m)$th-order nonlinear PDE or a $(l+m)$th-order linear PDE.

In $\mathbb{R}^n$ whose independent variables are $x_1, x_2, \cdots x_n$, if the solution of a PDE contains an arbitrary function, the number of independent variables of the arbitrary function is generally not equal to n, unless the essence of the PDE is $0 = 0$, such as

$$u_{x_i} - u_{x_i} = 0. \tag{4}$$

It is obvious that the general solution of Eq. (4) can be an arbitrary first differentiable function with n variables, but this PDE has no practical meaning. We call a PDE whose essence is $0 = 0$ a **banal PDE** , and a PDE whose nature is not $0 = 0$ a **non-banal PDE** .

For non-banal PDEs, we propose a conjecture:

Conjecture 1. *In $\mathbb{R}^n$,$(n \geq 2)$, if the solution of a non-banal PDE contains an arbitrary function, the number l of independent variables of the arbitrary function satisfies $1 \leq l \leq n-1$.*

In [10], we presented $Z_1 - Z_3$ transformations with the following contents.

Z_1 **Transformation.** *In the domain D, $(D \subset \mathbb{R}^n)$, any established mth-order PDE with n space variables $F(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1x_2}, \cdots) = 0$, set $y_i = y_i(x_1, \cdots x_n)$ and $u = f(y_1, \cdots y_l)$ are both undetermined mth-differentiable functions $(u, y_i \in C^m(D), 1 \leq i \leq l \leq n)$, $y_1, y_2 \cdots y_l$ are independent of each other, then substitute $u = f(y_1, \cdots y_l)$ and its partial derivatives into $F = 0$,*
1. In case of working out $y_i = y_i(x_1, \cdots x_n)$ and $f(y_1, \cdots y_l)$, then $u = f(y_1, \cdots y_l)$ is the solution of $F = 0$,
2. In case of dividing out $u = f(y_1, \cdots y_l)$ and its partial derivative, also working out $y_i = y_i(x_1, \cdots x_n)$, then $u = f(y_1, \cdots y_l)$ is the solution of $F = 0$, and f is an arbitrary mth-differentiable function,
3. In case of dividing out $u = f(y_1, \cdots y_l)$ and its partial derivative, also getting $k = 0$, but in fact $k \neq 0$, then $u = f(y_1, \cdots y_l)$ is not the solution of $F = 0$, and f is an arbitrary mth-differentiable function.

Z_2 **Transformation.** *In the domain D, $(D \subset \mathbb{R}^n)$, any established mth-order PDE with n space variables $F(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1x_2}, \cdots) = 0$, set $y_i = y_i(x_1, \cdots x_n)$ known and $u = f(y_1, \cdots y_l)$ undetermined $(u, y_i \in C^m(D), 1 \leq i \leq l \leq n)$, $y_1, y_2, \cdots y_l$ are independent of each other, then substitute $u = f(y_1, \cdots y_l)$ and its partial derivatives into $F = 0$*
1. In case of working out $f(y_1, \cdots y_l)$, then $u = f(y_1, \ldots y_l)$ is the solution of $F = 0$,
2. In case of dividing out $u = f(y_1, \ldots y_l)$ and its partial derivative, also getting $0 = 0$, then $u = f(y_1, \cdots y_l)$ is the solution of $F = 0$, and f is an arbitrary mth-differentiable function,
3. In case of dividing out $u = f(y_1, \cdots y_l)$ and its partial derivative, also getting $k = 0$, but in fact $k \neq 0$, then $u = f(y_1, \cdots y_l)$ is not the solution of $F = 0$, and f is an arbitrary mth-differentiable function.

Z_3 **Transformation.** *In the domain D, $(D \subset \mathbb{R}^n)$, any established mth-order PDE with n space variables $F(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1x_2}, \cdots) = 0$, setting $g(x_1, \cdots x_n), h(y_1, \cdots y_l)$ and $y_i = y_i(x_1, \cdots x_n)$ are all undetermined function, $y_1, y_2, \cdots y_l$ are independent of each other, $(g, h, y_i \in C^m(D), 1 \leq i \leq l \leq n)$, then substitute $u = gh(y_1, \cdots y_l)$ and its partial derivatives into $F = 0$,*
1. In case of working out h, g and y_i, then $u = gh(y_1, \cdots y_l)$ is the solution of $F = 0$,
2. In case of dividing out h and its partial derivative, also working out g and y_i, then $u = gh(y_1, \cdots y_l)$ is the solution of $F = 0$, and h is an arbitrary mth-differentiable function,
3. In case of getting $k = 0$, but in fact $k \neq 0$, then $u = gh(y_1, \cdots y_l)$ is not the solution of $F = 0$.

In order to obtain general solutions or exact solutions of some PDEs, we further propose Z_4 transformation.

Z_4 **Transformation.** *In the domain D, $(D \subset \mathbb{R}^n)$, any established mth-order PDE with n space variables $F(x_1, \cdots x_n, u, u_{x_1}, \cdots u_{x_n}, u_{x_1x_2}, \cdots) = 0$, set $y_i = y_i(x_1, \cdots x_k, u)$ and $f(y_1, y_2, \cdots y_k, x_{k+1}, x_{k+2}, \cdots x_n)$ are both undetermined mth-differentiable functions $(f, y_i \in C^m(D), 1 \leq i \leq k \leq n)$, $y_1, y_2, \cdots y_k$ are independent of each other, and set $f(y_1, y_2, \cdots y_k, x_{k+1}, x_{k+2}, \cdots x_n) = 0$, then substitute $u, u_{x_1}, \cdots u_{x_n}, u_{x_1x_2}, \cdots$ into $F = 0$*

1. In case of working out $y_i = y_i(x_1, \cdots x_k, u)$ *and* $f(y_1, y_2, \cdots y_k, x_{k+1}, x_{k+2}, \cdots x_n)$*, then* $f(y_1, y_2, \cdots y_k, x_{k+1}, x_{k+2}, \cdots x_n) = 0$ *is the solution of* $F = 0$*,*
2. In case of dividing out all the partial derivatives of $f(y_1, y_2, \cdots y_k, x_{k+1}, x_{k+2}, \cdots x_n)$*, also working out* $y_i = y_i(x_1, \cdots x_k, u)$*, then* $f(y_1, y_2, \cdots y_k, x_{k+1}, x_{k+2}, \cdots x_n) = 0$ *is the solution of* $F = 0$*, and* f *is an arbitrary* m*th-differentiable function,*
3. In case of dividing out all the partial derivatives of $f(y_1, y_2, \cdots y_k, x_{k+1}, x_{k+2}, \cdots x_n)$*, also getting* $k = 0$*, but in fact* $k \neq 0$*, then* $f(y_1, y_2, \cdots y_k, x_{k+1}, x_{k+2}, \cdots x_n) = 0$ *is not the solution of* $F = 0$*, and* f *is an arbitrary* m*th-differentiable function.*

In Z_4 Transformation, $y_i = y_i(x_1, \cdots x_k, u)$ and $f(y_1, \cdots y_k, x_{k+1}, \cdots x_n)$ are both undetermined, $y_i(x_1, \cdots x_k, u)$ may be an unknown function completely or has a determinate form with unknown constants, the solution of $f(y_1, \cdots y_k, x_{k+1}, \cdots x_n)$ may be an arbitrary or a certain mth-differentiable function, the solution of y_i and f may not be single, etc., which are determined by the PDE and the specific solving process.

According to $y_i = y_i(x_1, \cdots x_k, u)$ and $f(y_1, \cdots y_k, x_{k+1}, \cdots x_n) = 0$, we get

$$f_{x_i} = \sum_{j=1}^{l} f_{y_j} \left(y_{j\,x_i} + y_{j\,u} u_{x_i} \right) = 0,$$

$$\begin{aligned} f_{x_i x_k} =& \sum_{j=1}^{l} (f_{y_j} (y_{j\,x_i x_k} + y_{j\,x_i u} u_{x_k} + y_{j\,u} u_{x_i x_k} + y_{j\,uu} u_{x_i} u_{x_k} + y_{j\,u x_k} u_{x_i}) \\ &+ y_{j\,x_i} + y_{j\,u} u_{x_i}) \sum_{s=1}^{l} f_{y_j y_s} (y_{s\,x_k} + y_{s\,u} u_{x_k})) = 0 \end{aligned}$$

So

$$u_{x_i} = -\frac{\sum_{j=1}^{l} f_{y_j} y_{j\,x_i}}{\sum_{j=1}^{l} f_{y_j} y_{j\,u}}, \tag{5}$$

$$\begin{aligned} u_{x_i x_k} =& \frac{1}{\sum_{j=1}^{l} f_{y_j} y_{j\,u}} (-\sum_{j=1}^{l} f_{y_j} y_{j\,x_i x_k} + \frac{\sum_{j=1}^{l} f_{y_j} y_{j\,x_k}}{\sum_{j=1}^{l} f_{y_j} y_{j\,u}} \sum_{j=1}^{l} f_{y_j} y_{j\,x_i u} \\ &- \frac{\sum_{j=1}^{l} f_{y_j} y_{j\,x_i}}{\sum_{j=1}^{l} f_{y_j} y_{j\,u}} \frac{\sum_{j=1}^{l} f_{y_j} y_{j\,x_k}}{\sum_{j=1}^{l} f_{y_j} y_{j\,u}} \sum_{j=1}^{l} f_{y_j} y_{j\,uu} + \frac{\sum_{j=1}^{l} f_{y_j} y_{j\,x_i}}{\sum_{j=1}^{l} f_{y_j} y_{j\,u}} \sum_{j=1}^{l} f_{y_j} y_{j\,u x_k} \\ &- \sum_{j=1}^{l} ((y_{j\,x_i} - y_{j\,u} \frac{\sum_{j=1}^{l} f_{y_j} y_{j\,x_i}}{\sum_{j=1}^{l} f_{y_j} y_{j\,u}}) \sum_{s=1}^{l} f_{y_j y_s} y_{s\,x_k} \\ &- \frac{\sum_{j=1}^{l} f_{y_j} y_{j\,x_k}}{\sum_{j=1}^{l} f_{y_j} y_{j\,u}} (y_{j\,x_i} - y_{j\,u} \frac{\sum_{j=1}^{l} f_{y_j} y_{j\,x_i}}{\sum_{j=1}^{l} f_{y_j} y_{j\,u}}) \sum_{s=1}^{l} f_{y_j y_s} y_{s\,u})) \end{aligned} \tag{6}$$

$$\begin{aligned} u_{x_i x_k} =& \frac{1}{\sum_{j=1}^{l} f_{y_j} y_{j\,u}} (-\sum_{j=1}^{l} f_{y_j} y_{j\,x_i x_k} + 2\frac{\sum_{j=1}^{l} f_{y_j} y_{j\,x_k}}{\sum_{j=1}^{l} f_{y_j} y_{j\,u}} \sum_{j=1}^{l} f_{y_j} y_{j\,x_i u} - \left(\frac{\sum_{j=1}^{l} f_{y_j} y_{j\,x_i}}{\sum_{j=1}^{l} f_{y_j} y_{j\,u}} \right)^2 \sum_{j=1}^{l} f_{y_j} y_{j\,uu} \\ &- \sum_{j=1}^{l} ((y_{j\,x_i} - y_{j\,u} \frac{\sum_{j=1}^{l} f_{y_j} y_{j\,x_i}}{\sum_{j=1}^{l} f_{y_j} y_{j\,u}}) \sum_{s=1}^{l} f_{y_j y_s} y_{s\,x_i} \\ &- \frac{\sum_{j=1}^{l} f_{y_j} y_{j\,x_i}}{\sum_{j=1}^{l} f_{y_j} y_{j\,u}} (y_{j\,x_i} - y_{j\,u} \frac{\sum_{j=1}^{l} f_{y_j} y_{j\,x_i}}{\sum_{j=1}^{l} f_{y_j} y_{j\,u}}) \sum_{s=1}^{l} f_{y_j y_s} y_{s\,u})) \end{aligned} \tag{7}$$

By Eqs. (5-7), we can preliminarily judge that using Z_4 transformation to solve the first order PDEs may be convenient, and to solve the second order or greater than the second order PDEs may encounter difficulties.

2. General solutions' laws of partial differential equations with variable coefficients

2.1. General solutions' laws of linear PDEs with variable coefficients

In this section, if there is no special interpretation, the acquiescent independent variables of $\mathbb{R}^n$ are $x_1, x_2, \cdots x_n$. $a_i = a_i(x_1, \cdots x_k), a_{j_i} = a_{j_i}(x_1, \cdots x_k), a_{i_1 i_2 \cdots i_k} = a_{i_1 i_2 \cdots i_k}(x_1, \cdots x_k), b_i = b_i(x_1, \cdots x_k)$, and $b_{j_i} = b_{j_i}(x_1, \ldots x_k)$ are arbitrary known functions. $y_i = y_i(x_1, \ldots x_k), D_{x_i} \equiv \frac{\partial}{\partial x_i}$. c_i and c_{j_i} are arbitrary constants, f and f_i are arbitrary smooth functions.$(i, j = 1, 2, \cdots), (1 \leq k \leq n)$

Proposition 1. *In $\mathbb{R}^n$, if the exact solutions $u = y_i(x_1, \ldots x_k)$ of Eq. (8), which are independent of each other, are known, $(1 \leq i \leq l \leq k-1)$,*

$$a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_k u_{x_k} = 0, \tag{8}$$

then the general solution of Eq. (8) is

$$u = f(y_1, y_2, \ldots y_l, x_{k+1}, x_{k+2}, \ldots x_n). \tag{9}$$

Prove. By Z_1 Transformation, set $u(x_1, \ldots x_n) = f(y_1, y_2, \ldots y_l, x_{k+1}, x_{k+2}, \ldots x_n)$, then

$$\begin{aligned}
&a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_k u_{x_k} \\
&= a_1 \left(f_{y_1} y_{1x_1} + f_{y_2} y_{2x_1} + \ldots + f_{y_l} y_{lx_1}\right) + a_2 \left(f_{y_1} y_{1x_2} + f_{y_2} y_{2x_2} + \ldots + f_{y_l} y_{lx_2}\right) + \ldots \\
&+ a_k \left(f_{y_1} y_{1x_k} + f_{y_2} y_{2x_k} + \ldots + f_{y_l} y_{lx_k}\right) \\
&= f_{y_1} \left(a_1 y_{1x_1} + a_2 y_{1x_2} + \ldots + a_k y_{1x_k}\right) + f_{y_2} \left(a_1 y_{2x_1} + a_2 y_{2x_2} + \ldots + a_k y_{2x_k}\right) + \ldots \\
&+ f_{y_l} \left(a_1 y_{lx_1} + a_2 y_{lx_2} + \ldots + a_k y_{lx_k}\right) = 0.
\end{aligned}$$

Since $a_1 y_{ix_1} + a_2 y_{ix_2} + \ldots + a_k y_{ix_k} = 0, (1 \leq i \leq l \leq k-1)$, the above equation is correct eternally. So Proposition 1 is proved. According to the characteristic equation method and Conjecture 1 we can know: Usually $l = k-1$. □

According to Proposition 1, if the exact solutions $u = y_i(x_1, \cdots x_n)$ of $a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_n u_{x_n} = 0$, which are independent of each other, are known, $(1 \leq i \leq l \leq n-1)$, then its general solution is $u = f(y_1, y_2, \cdots y_l)$.

Proposition 2. *In $\mathbb{R}^n$, if the general solution $u = f(y_1, y_2, \ldots y_l, x_{k+1}, x_{k+2}, \ldots x_n)$ of $(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k})u = 0$ is known, then*

$$(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k})^2 u = 0 \tag{10}$$

The general solution of Eq. (10) is

$$u = f_1(y_1, \ldots y_l, x_{k+1}, \ldots x_n) + (c_1 x_1 + c_2 x_2 + \ldots + c_k x_k) f_2(y_1, \ldots y_l, x_{k+1}, \ldots x_n). \tag{11}$$

Prove. Because the general solution $u = f(y_1, y_2, \ldots y_l, x_{k+1}, x_{k+2}, \ldots x_n)$ of $(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k})u = 0$ is known, apparently $u = f(y_1, y_2, \ldots y_l, x_{k+1}, x_{k+2}, \ldots x_n)$ is also the solution of $(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_kD_{x_k})^2u=0$, setting $u=g(x_1,\cdots x_k)f=c_sx_sf$ is the solution of $(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_kD_{x_k})^2u=0$ too, that is $g(x_1,\cdots x_k)=c_sx_s$, and c_s are arbitrary constants, $(s=1,2,\cdots k)$, then

$$(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_kD_{x_k})^2u=\left(\sum_{i=1}^{k}b_i^2D_{x_i}^2+2\sum_{i<j}b_ib_jD_{x_i}D_{x_j}\right)(gf)$$

$$=\sum_{i=1}^{k}b_i^2\left(g_{x_ix_i}f+2g_{x_i}f_{x_i}+gf_{x_ix_i}\right)+2\sum_{i<j}b_ib_j\left(g_{x_ix_j}f+g_{x_i}f_{x_j}+g_{x_j}f_{x_i}+gf_{x_ix_j}\right)$$

$$=g\left(\sum_{i=1}^{k}b_i^2f_{x_ix_i}+2\sum_{i<j}b_ib_jf_{x_ix_j}\right)+f\left(\sum_{i=1}^{k}b_i^2g_{x_ix_i}+2\sum_{i<j}b_ib_jg_{x_ix_j}\right)+2\sum_{i=1}^{k}b_i^2g_{x_i}f_{x_i}$$

$$+2\sum_{i<j}b_ib_j\left(g_{x_i}f_{x_j}+g_{x_j}f_{x_i}\right)=2\sum_{i=1}^{k}b_i^2g_{x_i}f_{x_i}+2\sum_{i<j}b_ib_jg_{x_i}f_{x_j}+2\sum_{i<j}b_ib_jg_{x_j}f_{x_i}$$

$$=2b_s^2c_sf_{x_s}+2c_sb_s\sum_{s<j}b_jf_{x_j}+2c_sb_s\sum_{i<s}b_if_{x_i}=2c_sb_s\sum_{i=1}^{k}b_if_{x_i}=0.$$

That $u=c_sx_sf$ is the solution of $(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_kD_{x_k})^2u=0$ is proved. So its general solution is (11). □

According to Proposition 2, we present a conjecture.

Conjecture 2. *In $\mathbb{R}^n$, if the general solution $u=f(y_1,y_2,\cdots y_l,x_{k+1},x_{k+2},\cdots x_n)$ of $(b_1D_{x_1}+b_2D_{x_2}+\cdots+b_kD_{x_k})u=0$ is known, then the general solution of*

$$(b_1D_{x_1}+b_2D_{x_2}+\ldots+b_kD_{x_k})^mu=0 \tag{12}$$

is

$$u=\sum_{j=1}^{m}(c_{j_1}x_1+c_{j_2}x_2+\ldots+c_{j_k}x_k)^{j-1}f_j\left(y_1,\ldots y_l,x_{k+1},\ldots x_n\right). \tag{13}$$

Theoretically Conjecture 2 can be proved by mathematical induction, we shall not analyse it further.

In $\mathbb{R}^n$, for the mth-order linear PDE with variable coefficients

$$\sum_{i_1+i_2+\ldots+i_k=m}a_{i_1i_2\ldots i_k}u_{x_1x_2\ldots x_k}^{(i_1i_2\ldots i_k)}=0, \tag{14}$$

where i_j are natural number, $1\le j\le k\le n$, If Eq. (14) can be translated into

$$(b_{1_1}D_{x_1}+b_{1_2}D_{x_2}+\ldots+b_{1_k}D_{x_k})(b_{2_1}D_{x_1}+b_{2_2}D_{x_2}+\ldots+b_{2_k}D_{x_k})\ldots(b_{m_1}D_{x_1}+b_{m_2}D_{x_2}+\ldots+b_{m_k}D_{x_k})u=0. \tag{15}$$

For

$$(b_{j_1}D_{x_1}+b_{j_2}D_{x_2}+\ldots+b_{j_k}D_{x_k})\,u=b_{j_1}u_{x_1}+b_{j_2}u_{x_2}+\ldots+b_{j_k}u_{x_k}=0\ \ ,(j=1,2,\ldots m). \tag{16}$$

If the particular solutions $u = y_{j_s}(x_1, \cdots x_k)$ of Eq. (16), which are independent of each other, are all known, $(1 \le s \le l_j)$, by Proposition 1 the general solution of Eq. (14) is

$$u = \sum_{j=1}^{m} f_j \left(y_{j_1}, y_{j_2}, \ldots y_{j_{l_j}}, x_{k+1}, x_{k+2}, \ldots x_n \right). \tag{17}$$

If Eq. (14) can be translated into:

$$\prod_{j=1}^{q} \left(b_{j_1} D_{x_1} + b_{j_2} D_{x_2} + \ldots + b_{j_k} D_{x_k} \right)^{p_j} u = 0, \tag{18}$$

where $\sum\limits_{j=1}^{q} p_j = m$, its general solution of conjecture may be written by Conjecture 2.

Proposition 3. *In* $\mathbb{R}^n$*, if the particular solution* $g = g(x_1, \cdots x_k)$ *of* $a_1 g_{x_1} + a_2 g_{x_2} + \cdots + a_k g_{x_k} + a_{k+1} g = 0$ *and the exact solutions* $u = y_i(x_1, \cdots x_k)$ *of* $a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} = 0$*, which are independent of each other, are all known,* $(1 \le i \le l \le k-1)$*, then*

$$a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_k u_{x_k} + a_{k+1} u = 0, \tag{19}$$

then the general solution of Eq. (19) is

$$u = g\left(x_1, \ldots x_k\right) f\left(y_1, \ldots y_l, x_{k+1}, \ldots x_n\right), \tag{20}$$

Prove. By Z_3 Transformation, set $u = g\left(x_1, \ldots x_k\right) f\left(y_1, \ldots y_l, x_{k+1}, \ldots x_n\right)$, then

$$\begin{aligned} & a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_k u_{x_k} + a_{k+1} u \\ & = a_1 \left(f g_{x_1} + g \sum_{i=1}^{l} f_{y_i} y_{i x_1} \right) + a_2 \left(f g_{x_2} + g \sum_{i=1}^{l} f_{y_i} y_{i x_2} \right) + \ldots + a_k \left(f g_{x_k} + g \sum_{i=1}^{l} f_{y_i} y_{i x_k} \right) \\ & + a_{k+1} g f \\ & = f\left(a_1 g_{x_1} + a_2 g_{x_2} + \ldots + a_k g_{x_k} + a_{k+1} g\right) + g \sum_{i=1}^{l} f_{y_i} \left(a_1 y_{i x_1} + a_2 y_{i x_2} + \ldots + a_k y_{i x_k} \right) = 0 \end{aligned}$$

Since $a_1 g_{x_1} + a_2 g_{x_2} + \ldots + a_k g_{x_k} + a_{k+1} g = 0$ and $a_1 y_{i x_1} + a_2 y_{i x_2} + \ldots + a_k y_{i x_k} = 0$. The above equation is correct eternally, so Proposition 3 is proved. □

Proposition 4. *In* $\mathbb{R}^n$*, if the general solution* $u = g(x_1, \cdots x_k) f(y_1, \cdots y_l, x_{k+1}, \cdots x_n)$ *of* $\left(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k} + b_{k+1}\right) u = 0$ *is known, then*

$$\left(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k} + b_{k+1}\right)^2 u = 0 \tag{21}$$

The general solution of Eq. (21) is

$$u = g\left(x_1, \ldots x_k\right) \left(f_1\left(y_1, \ldots y_l, x_{k+1}, \ldots x_n\right) + \left(c_1 x_1 + c_2 x_2 + \ldots + c_k x_k\right) f_2\left(y_1, \ldots y_l, x_{k+1}, \ldots x_n\right) \right). \tag{22}$$

Prove. Because the general solution $u = g(x_1, \cdots x_k) f(y_1, \cdots y_l, x_{k+1}, \cdots x_n)$ of $(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1}) u = 0$ is known, apparently $u = g(x_1, \cdots x_k) f(y_1, \cdots y_l, x_{k+1}, \cdots x_n)$ is also the solution of $\left(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k} + b_{k+1}\right)^2 u = 0$, setting

$$u = ht = c_s x_s g\left(x_1, \ldots x_k\right) f\left(y_1, \ldots y_l, x_{k+1}, \ldots x_n\right), \quad (1 \le s \le k), \tag{23}$$

where $h = c_s x_s, t = g(x_1, \cdots x_k) f(y_1, \cdots y_l, x_{k+1}, \cdots x_n)$, and c_s are arbitrary constants. Assuming (23) is a solution of (21), then

$$\begin{aligned}
&(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k} + b_{k+1})^2 u \\
&= \left(\sum_{i=1}^{k} b_i^2 D_{x_i}^2 + b_{k+1}^2 + 2 \sum_{1 \le i < j \le k} b_i b_j D_{x_i} D_{x_j} + 2 b_{k+1} \sum_{i=1}^{k} b_i D_{x_i} \right) (ht) \\
&= \sum_{i=1}^{k} b_i^2 \left(h_{x_i x_i} t + 2 h_{x_i} t_{x_i} + h t_{x_i x_i} \right) + b_{k+1}^2 ht \\
&+ 2 \sum_{1 \le i < j \le k} b_i b_j \left(h_{x_i x_j} t + h_{x_i} t_{x_j} + h_{x_j} t_{x_i} + h t_{x_i x_j} \right) + 2 b_{k+1} \sum_{i=1}^{k} b_i \left(h_{x_i} t + h t_{x_i} \right) \\
&= h \left(\sum_{i=1}^{k} b_i^2 t_{x_i x_i} + b_{k+1}^2 t + 2 \sum_{1 \le i < j \le k} b_i b_j t_{x_i x_j} + 2 b_{k+1} \sum_{i=1}^{k} b_i t_{x_i} \right) + t \sum_{i=1}^{k} b_i^2 h_{x_i x_i} \\
&+ 2 \sum_{i=1}^{k} b_i^2 h_{x_i} t_{x_i} + 2t \sum_{1 \le i < j \le k} b_i b_j h_{x_i x_j} + 2 \sum_{1 \le i < j \le k} b_i b_j h_{x_i} t_{x_j} + 2 \sum_{1 \le i < j \le k} b_i b_j h_{x_j} t_{x_i} \\
&+ 2 b_{k+1} t \sum_{i=1}^{k} b_i h_{x_i} \\
&= 2 \sum_{i=1}^{k} b_i^2 h_{x_i} t_{x_i} + 2 \sum_{1 \le i < j \le k} b_i b_j h_{x_i} t_{x_j} + 2 \sum_{1 \le i < j \le k} b_i b_j h_{x_j} t_{x_i} + 2 b_{k+1} t \sum_{i=1}^{k} b_i h_{x_i} \\
&= 2 b_s^2 h_{x_s} t_{x_s} + 2 b_s h_{x_s} \sum_{1 \le s < j \le k} b_j t_{x_j} + 2 b_s h_{x_s} \sum_{1 \le i < s \le k} b_i t_{x_i} + 2 b_{k+1} b_s h_{x_s} t \\
&= 2 b_s h_{x_s} \sum_{i=1}^{k} b_i t_{x_i} + 2 b_{k+1} b_s h_{x_s} t = 2 b_s h_{x_s} \left(\sum_{i=1}^{k} b_i t_{x_i} + b_{k+1} t \right) = 0.
\end{aligned}$$

That (23) is a solution of (21) is proved. So its general solution is (22). □

According to Proposition 4, we present Conjecture 3.

Conjecture 3. *In $\mathbb{R}^n$, if the general solution $u = g(x_1, \cdots x_k) f(y_1, \cdots y_l, x_{k+1}, \cdots x_n)$ of $(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1}) u = 0$ is known, then the general solution of*

$$(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k} + b_{k+1})^m u = 0 \tag{24}$$

is

$$u = g\left(x_1, \ldots x_k\right) \sum_{j=1}^{m} \left(c_{j_1} x_1 + c_{j_2} x_2 + \ldots + c_{j_n} x_n\right)^{j-1} f_j\left(y_1, \ldots y_l, x_{k+1}, \ldots x_n\right), (m \ge 2). \tag{25}$$

In $\mathbb{R}^n$, for the mth-order linear PDE with variable coefficients

$$\sum_{0 \le i_1 + i_2 + \ldots + i_k \le m} a_{i_1 i_2 \ldots i_k} u_{x_1 x_2 \ldots x_k}^{(i_1 i_2 \ldots i_k)} = 0, \tag{26}$$

where i_j are natural number, $1 \le j \le k \le n$. If Eq. (26) can be translated into

$$\begin{aligned}
&\left(b_{1_1} D_{x_1} + b_{1_2} D_{x_2} + \ldots + b_{1_k} D_{x_k} + b_{1_{k+1}}\right) \left(b_{2_1} D_{x_1} + b_{2_2} D_{x_2} + \ldots + b_{2_k} D_{x_k} + b_{2_{k+1}}\right) \ldots \\
&\left(b_{m_1} D_{x_1} + b_{m_2} D_{x_2} + \ldots + b_{m_k} D_{x_k} + b_{m_{k+1}}\right) u = 0.
\end{aligned} \tag{27}$$

If the particular solution $u = g_j(x_1, \cdots x_k)$ of $(b_{j_1}D_{x_1} + b_{j_2}D_{x_2} + \cdots + b_{j_k}D_{x_k} + b_{j_{k+1}})u = 0$ and the exact solutions $u = y_{j_s}(x_1, \cdots x_k)$ of $(b_{j_1}D_{x_1} + b_{j_2}D_{x_2} + \cdots + b_{j_k}D_{x_k})u = 0$, which are independent of each other, are all known, $(1 \le j \le m, 1 \le s \le l_j)$, by Proposition 3 the general solution of Eq. (26) is

$$u\left(x_1, \ldots x_n\right) = \sum_{j=1}^{m} \left(g_j\left(x_1, \ldots x_k\right) f_j\left(y_{j_1}, y_{j_2}, \ldots y_{j_{l_j}}, x_{k+1}, x_{k+2}, \ldots x_n\right)\right). \tag{28}$$

If Eq. (26) can be translated into:

$$\prod_{j=1}^{q} \left(b_{j_1}D_{x_1} + b_{j_2}D_{x_2} + \ldots + b_{j_n}D_{x_n} + b_{j_{n+1}}\right)^{p_j} u = 0, \tag{29}$$

where $\sum\limits_{j=1}^{q} p_j = m$, its general solution of conjecture may be written by Conjecture 3.

In $\mathbb{R}^n$, for the mth-order linear PDE with variable coefficients

$$\sum_{0 \le i_1 + i_2 + \ldots + i_k \le m} a_{i_1 i_2 \ldots i_k} u_{x_1 x_2 \ldots x_k}^{(i_1 i_2 \ldots i_k)} = A\left(x_1, x_2, \ldots x_n\right), \tag{30}$$

where $A(x_1, x_2, \cdots x_n)$ is an arbitrary known function. In general, we need to solve the particular solution of (30) first, and then use the general solution of its homogeneous equation to obtain its general solution. For some PDEs, we can get the general solution by Z Transformation directly, such as

Proposition 5. *In $\mathbb{R}^n$, if the particular solutions $u = y_i(x_1, \cdots x_k)$ of $a_1u_{x_1} + a_2u_{x_2} + \cdots + a_ku_{x_k} = 0$ are all known, $(1 \le i \le k-1)$, then*

$$a_1u_{x_1} + a_2u_{x_2} + \ldots + a_ku_{x_k} + a_{k+1} = 0, \tag{31}$$

the general solution of Eq. (31) is

$$u = f\left(y_1, \ldots y_{k-1}, x_{k+1}, \ldots x_n\right) + \frac{\int a_{k+1}\left(y_1, \ldots y_k, x_{k+1}, \ldots x_n\right) dy_k}{a_1 y_{k_{x_1}} + a_2 y_{k_{x_2}} + \ldots + a_k y_{k_{x_k}}}, \tag{32}$$

where $y_1, y_2, \cdots y_k$ are independent of each other, and $x_j = x_j(y_1, y_2, \cdots y_k)$ can be solved, $(1 \le j \le k)$.

Prove. By Z_1 Transformation, set $u(x_1, x_2, \cdots x_n) = u(y_1, y_2, \cdots y_k, x_{k+1}, x_{k+2}, \cdots x_n)$ and $a_i(x_1, x_2, \cdots x_n) = a_i(y_1, y_2, \cdots y_k, x_{k+1}, x_{k+2}, \cdots x_n)$, where $y_t = y_t(x_1, x_2, \cdots x_k)$, $(t = 1, 2, \cdots k)$, and $y_1, y_2, \cdots y_k$ are independent of each other, then

$$\begin{aligned} & a_1u_{x_1} + a_2u_{x_2} + \ldots + a_ku_{x_k} + a_{k+1} = a_1\sum_{t=1}^{k} y_{t_{x_1}} u_{y_t} + a_2\sum_{t=1}^{k} y_{t_{x_2}} u_{y_t} + \ldots + a_k\sum_{t=1}^{k} y_{t_{x_k}} u_{y_t} + a_{k+1} \\ & = \left(a_1 y_{1_{x_1}} + a_2 y_{1_{x_2}} + \ldots + a_k y_{1_{x_k}}\right) u_{y_1} + \left(a_1 y_{2_{x_1}} + a_2 y_{2_{x_2}} + \ldots + a_k y_{2_{x_k}}\right) u_{y_2} + \ldots \\ & + \left(a_1 y_{k_{x_1}} + a_2 y_{k_{x_2}} + \ldots + a_k y_{k_{x_k}}\right) u_{y_k} + a_{k+1} = 0. \end{aligned} \tag{33}$$

If the particular solutions $u = y_i(x_1, \cdots x_k)$ of $a_1u_{x_1} + a_2u_{x_2} + \cdots + a_ku_{x_k} = 0$ are all known $(1 \le i \le k-1)$, then Eq. (33) can be translated into

$$\left(a_1 y_{k_{x_1}} + a_2 y_{k_{x_2}} + \ldots + a_k y_{k_{x_k}}\right) u_{y_k} + a_{k+1} = 0. \tag{34}$$

A particular solution of Eq. (34) is

$$u = \frac{\int a_{k+1}(y_1, \dots y_k, x_{k+1}, \dots x_n)\, dy_k}{a_1 y_{k_{x_1}} + a_2 y_{k_{x_2}} + \dots + a_k y_{k_{x_k}}}. \tag{35}$$

To compute (35) we need to find a first differentiable $y_k = y_k(x_1, x_2, \cdots x_k)$ and make $y_1, y_2, \cdots y_k$ independent of each other, and can solve $x_j = x_j(y_1, y_2, \cdots y_k)$, then the general solution of (31) is (32). So the proposition is proved. □

Proposition 6. *In $\mathbb{R}^n$, if the particular solutions $y_i = y_i(x_1, x_2, \cdots x_k, u)$ of $a_1 y_{x_1} + a_2 y_{x_2} + \cdots + a_k y_{x_k} - a_{k+1} y_u = 0$, which are independent of each other, are all known, $(1 \le i \le k \le n)$, then*

$$a_1 u_{x_1} + a_2 u_{x_2} + \dots + a_k u_{x_k} + a_{k+1} = 0, \tag{36}$$

the general solution of Eq. (36) is

$$f(y_1, y_2, \dots y_k, x_{k+1}, x_{k+2}, \dots x_n) = 0, \tag{37}$$

where $a_j = a_j(x_1, x_2, \cdots x_k, u)$ are arbitrary known functions $(1 \le j \le k+1)$.

Prove. By Z_4 Transformation, set $f(y_1, y_2, \cdots y_k, x_{k+1}, x_{k+2}, \cdots x_n) = 0$ and

$$y_i = y_i(x_1, x_2, \dots x_k, u), \quad (1 \le i \le k \le n) \tag{38}$$

$y_1, y_2, \cdots y_k$ are independent of each other. According to (5), we get

$$\begin{aligned}
& a_1 u_{x_1} + a_2 u_{x_2} + \dots + a_k u_{x_k} + a_{k+1} \\
& = -a_1 \frac{\sum_{i=1}^k f_{y_i} y_{i x_1}}{\sum_{i=1}^k f_{y_i} y_{i_u}} - a_2 \frac{\sum_{i=1}^k f_{y_i} y_{i x_2}}{\sum_{i=1}^k f_{y_i} y_{i_u}} - \dots - a_k \frac{\sum_{i=1}^k f_{y_i} y_{i x_k}}{\sum_{i=1}^k f_{y_i} y_{i_u}} + a_{k+1} = 0 \\
& \Longrightarrow a_1 \sum_{i=1}^k f_{y_i} y_{i x_1} + a_2 \sum_{i=1}^k f_{y_i} y_{i x_2} + \dots + a_k \sum_{i=1}^k f_{y_i} y_{i x_k} - a_{k+1} \sum_{i=1}^k f_{y_i} y_{i_u} \\
& = \left(a_1 y_{1_{x_1}} + a_2 y_{1_{x_2}} + \dots + a_k y_{1_{x_k}} - a_{k+1} y_{1_u}\right) f_{y_1} \\
& + \left(a_1 y_{2_{x_1}} + a_2 y_{2_{x_2}} + \dots + a_k y_{2_{x_k}} - a_{k+1} y_{2_u}\right) f_{y_2} + \dots \\
& + \left(a_1 y_{k_{x_1}} + a_2 y_{k_{x_2}} + \dots + a_k y_{k_{x_k}} - a_{k+1} y_{k_u}\right) f_{y_k} = 0.
\end{aligned}$$

Set

$$a_1 y_{i_{x_1}} + a_2 y_{i_{x_2}} + \dots + a_k y_{i_{x_k}} - a_{k+1} y_{i_u} = 0, \quad (i = 1, 2, \dots k). \tag{39}$$

For

$$a_1 y_{x_1} + a_2 y_{x_2} + \dots + a_k y_{x_k} - a_{k+1} y_u = 0. \tag{40}$$

If the particular solutions $y_i = y_i(x_1, x_2, \cdots x_k, u)$ of (40), which are independent of each other, are all known, then the general solution of (36) is (37). So the proposition is proved. □

2.2. General solutions' laws of some nonlinear partial differential equations with variable coefficients

Methods for solving exact solutions of nonlinear PDEs are complex and diverse, such as homogeneous balance method [11-13], multiple exp-function method [14,15], tanh-sech method [16-18] and so on. For certain seemingly complex nonlinear PDEs, we can obtain their general solution by using algebraic algorithm and some conclusions in the previous section. For example, according to Proposition 1 we can get Proposition 7 directly:

Proposition 7. *In $\mathbb{R}^n$, if the general solutions $u = f_i(y_{i_1}, y_{i_2}, \cdots y_{i_{l_i}}, x_{k+1}, x_{k+2}, \cdots x_n)$ of $a_{i_1}u_{x_1} + a_{i_2}u_{x_2} + \cdots + a_{i_k}u_{x_k} = 0$ are known, $(1 \le i \le m, 1 \le l_i \le k-1)$, then*

$$\begin{aligned}&(a_{1_1}u_{x_1} + a_{1_2}u_{x_2} + \ldots + a_{1_k}u_{x_k})^{c_1}(a_{2_1}u_{x_1} + a_{2_2}u_{x_2} + \ldots + a_{2_k}u_{x_k})^{c_2}\ldots\\ &(a_{m_1}u_{x_1} + a_{m_2}u_{x_2} + \ldots + a_{m_k}u_{x_k})^{c_m} = 0,\end{aligned} \tag{41}$$

the general solution of Eq. (41) is

$$\prod_{i=1}^{m}\left(u - f_i\left(y_{i_1}, y_{i_2}, \ldots y_{i_{l_i}}, x_{k+1}, x_{k+2}, \ldots x_n\right)\right) = 0. \tag{42}$$

According to Proposition 3 we can obtain directly Proposition 8:

Proposition 8. *In $\mathbb{R}^n$, if the general solutions $u = g_i(x_1, \ldots x_k)\, f_i\left(y_{i_1}, y_{i_2}, \ldots y_{i_{l_i}}, x_{k+1}, x_{k+2}, \ldots x_n\right)$ of $a_{i_1}u_{x_1} + a_{i_2}u_{x_2} + \cdots + a_{i_k}u_{x_k} + a_{i_{k+1}}u = 0$ are known, $(1 \le i \le m, 1 \le l_i \le k-1)$, then*

$$\begin{aligned}&\left(a_{1_1}u_{x_1} + a_{1_2}u_{x_2} + \ldots + a_{1_k}u_{x_k} + a_{i_{k+1}}u\right)^{c_1}\left(a_{2_1}u_{x_1} + a_{2_2}u_{x_2} + \ldots + a_{2_k}u_{x_k} + a_{2_{k+1}}u\right)^{c_2}\\ &\ldots\left(a_{m_1}u_{x_1} + a_{m_2}u_{x_2} + \ldots + a_{m_k}u_{x_k} + a_{m_{k+1}}u\right)^{c_m} = 0,\end{aligned} \tag{43}$$

the general solution of Eq. (43) is

$$\prod_{i=1}^{m}\left(u - g_i(x_1, \ldots x_k)\, f_i\left(y_{i_1}, y_{i_2}, \ldots y_{i_{l_i}}, x_{k+1}, x_{k+2}, \ldots x_n\right)\right) = 0. \tag{44}$$

According to Proposition 6 we can get directly Proposition 9:

Proposition 9. *In $\mathbb{R}^n$, if the general solutions $f_i(y_{i_1}, y_{i_2}, \cdots y_{i_k}, x_{k+1}, x_{k+2}, \cdots x_n) = 0$ of $a_{i_1}u_{x_1} + a_{i_2}u_{x_2} + \cdots + a_{i_k}u_{x_k} + a_{i_{k+1}} = 0$ are known, where $a_{i_j} = a_{i_j}(x_1, x_2, \cdots x_k, u)$ are arbitrary known functions $(1 \le i \le m, 1 \le j \le k+1)$, then*

$$\begin{aligned}&\left(a_{1_1}u_{x_1} + a_{1_2}u_{x_2} + \ldots + a_{1_k}u_{x_k} + a_{i_{k+1}}\right)^{c_1}\left(a_{2_1}u_{x_1} + a_{2_2}u_{x_2} + \ldots + a_{2_k}u_{x_k} + a_{2_{k+1}}\right)^{c_2}\\ &\ldots\left(a_{m_1}u_{x_1} + a_{m_2}u_{x_2} + \ldots + a_{m_k}u_{x_k} + a_{m_{k+1}}\right)^{c_m} = 0,\end{aligned} \tag{45}$$

the general solution of Eq. (45) is

$$\prod_{i=1}^{m} f_i\left(y_{i_1}, y_{i_2}, \ldots y_{i_k}, x_{k+1}, x_{k+2}, \ldots x_n\right) = 0. \tag{46}$$

3. General solutions laws of partial differential equations with constant coefficients

Here we will research the general solutions' laws of PDEs with constant coefficients, which are the special cases of PDEs with variable coefficients. In this section, if there is no special interpretation, the acquiescent independent variables of $\mathbb{R}^n$ are $x_1, x_2, \cdots x_n, (2 \le k \le n)$, $D_{x_i} \equiv \frac{\partial}{\partial x_i}$. a_i, b_i, b_{i_j} and $a_{i_1 i_2 \cdots i_n}$ are arbitrary known constants, $c_i, c_{ij}, c_{i_s j}, l_i$ and l_{j_i} are arbitrary constants, f and f_i are arbitrary smooth functions $(i, j, s = 1, 2, \cdots)$.

Proposition 10. *In $\mathbb{R}^n$,*

$$a_1u_{x_1} + a_2u_{x_2} + \ldots + a_ku_{x_k} = A(x_1, x_2, \ldots x_n), \quad (2 \le k \le n), \tag{47}$$

the general solution of Eq. (47) is

$$u = f\left(y_1, y_2, \ldots y_{k-1}, x_{k+1}, x_{k+2}, \ldots x_n\right) + \frac{\int A\left(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n\right) dy_k}{a_1 c_{k1} + a_2 c_{k2} + \ldots + a_k c_{kk}}, \quad (48)$$

where

$$y_i = c_{i1} x_1 + c_{i2} x_2 + \ldots + c_{ik} x_k, \quad (1 \le i \le k), \quad (49)$$

$$\frac{\partial\left(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n\right)}{\partial\left(x_1, x_2, \ldots x_n\right)} = \frac{\partial\left(y_1, y_2, \ldots y_k\right)}{\partial\left(x_1, x_2, \ldots x_k\right)} \neq 0, \quad (50)$$

$$c_{i1} = \frac{-a_2 c_{i2} - a_3 c_{i3} - \ldots - a_k c_{ik}}{a_1}, \quad (1 \le i \le k-1). \quad (51)$$

Prove. According to Z_1 Transformation, set $u\left(x_1, x_2, \ldots x_n\right) = u\left(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n\right)$, $A\left(x_1, x_2, \ldots x_n\right) = A\left(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n\right)$ and

$$\begin{cases} y_1 = c_{11} x_1 + c_{12} x_2 + \ldots + c_{1k} x_k \\ y_2 = c_{21} x_1 + c_{22} x_2 + \ldots + c_{2k} x_k, \\ \cdots \\ y_k = c_{k1} x_1 + c_{k2} x_2 + \ldots + c_{kk} x_k \end{cases} \quad (52)$$

and

$$\frac{\partial\left(y_1, y_2, \ldots y_k\right)}{\partial\left(x_1, x_2, \ldots x_k\right)} \neq 0, \quad (50)$$

where c_{ij} are undetermined constants. According to (50, 52), $x_i = x_i(y_1, y_2, \cdots y_k)$ always has a unique solution $(1 \le i, j \le k)$. So

$$\begin{aligned} a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_k u_{x_k} &= a_1 \sum_{i=1}^{k} c_{i1} u_{y_i} + a_2 \sum_{i=1}^{k} c_{i2} u_{y_i} + \ldots + a_k \sum_{i=1}^{k} c_{ik} u_{y_i} \\ &= \left(a_1 c_{11} + a_2 c_{12} + \ldots + a_k c_{1k}\right) u_{y_1} + \left(a_1 c_{21} + a_2 c_{22} + \ldots + a_k c_{2k}\right) u_{y_2} + \ldots \\ &+ \left(a_1 c_{k1} + a_2 c_{k2} + \ldots + a_k c_{kk}\right) u_{y_k}. \end{aligned}$$

Set

$$a_1 c_{11} + a_2 c_{12} + \ldots + a_k c_{1k} = a_1 c_{21} + a_2 c_{22} + \ldots + a_k c_{2k} = a_1 c_{(k-1)1} + a_2 c_{(k-1)2} + \ldots + a_k c_{(k-1)k} = 0.$$

We obtain

$$c_{i1} = \frac{-a_2 c_{i2} - a_3 c_{i3} - \ldots - a_k c_{ik}}{a_1}, \quad (1 \le i \le k-1). \quad (51)$$

Then

$$a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_k u_{x_k} = \left(a_1 c_{k1} + a_2 c_{k2} + \ldots + a_k c_{kk}\right) u_{y_k} = A\left(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n\right). \quad (52)$$

The particular solution of (52) is $u = \frac{\int A(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n) dy_k}{a_1 c_{k1} + a_2 c_{k2} + \ldots + a_k c_{kk}}$, so the general solution of Eq. (47) is (48). □

According to Proposition 10, in $\mathbb{R}^n$,

$$a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_n u_{x_n} = A\left(x_1, x_2, \ldots x_n\right), \quad (53)$$

the general solution of Eq. (53) is

$$u = f\left(y_1, y_2, \ldots y_{n-1}\right) + \frac{\int A\left(y_1, y_2, \ldots y_n\right) dy_n}{a_1 c_{n1} + a_2 c_{n2} + \ldots + a_n c_{nn}}, \tag{54}$$

where

$$y_i = c_{i1} x_1 + c_{i2} x_2 + \ldots + c_{in} x_n, \tag{55}$$

$$c_{i1} = \frac{-a_2 c_{i2} - a_3 c_{i3} - \ldots - a_k c_{ik}}{a_1}, \;\; (1 \leq i \leq n-1)\,. \tag{56}$$

According to Proposition 10 we can get Proposition 11:

Proposition 11. *In* $\mathbb{R}^n$*,*

$$a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_k u_{x_k} = 0, \tag{57}$$

the general solution of Eq. (57) is

$$u = f\left(y_1, y_2, \ldots y_{k-1}, x_{k+1}, x_{k+2}, \ldots x_n\right), \tag{58}$$

where y_i *satisfy (49-51).*

The proof of Proposition 11 is not complicated, the readers can try it.

According to Proposition 2 and 11, we can get directly Proposition 12:

Proposition 12. *In* $\mathbb{R}^n$*, if the general solution* $u = f\left(y_1, y_2, \ldots y_{k-1}, x_{k+1}, x_{k+2}, \ldots x_n\right)$ *of* $\left(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k}\right) u = 0$ *is known,* $y_i = y_i\left(x_1, \ldots x_k\right), (\; i = 1, 2, \ldots k-1)$*, then*

$$\left(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k}\right)^2 u = 0 \tag{59}$$

the general solution of Eq. (59) is

$$u = f_1\left(y_1, \ldots y_{k-1}, x_{k+1}, \ldots x_n\right) + \left(c_1 x_1 + c_2 x_2 + \ldots + c_k x_k\right) f_2\left(y_1, \ldots y_{k-1}, x_{k+1}, \ldots x_n\right). \tag{60}$$

According to Conjecture 2, we may present Conjecture 4:

Conjecture 4. *In* $\mathbb{R}^n$*, if the general solution* $u = f(y_1, \cdots y_{k-1}, x_{k+1}, \cdots x_n)$ *of* $(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})u = 0$ *is known, then the general solution of*

$$\left(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k}\right)^m u = 0 \tag{61}$$

is

$$u = \sum_{i=1}^{m} \left(c_{i1} x_1 + c_{i2} x_2 + \ldots + c_{ik} x_k\right)^{i-1} f_i\left(y_1, \ldots y_{k-1}, x_{k+1}, \ldots x_n\right). \tag{62}$$

In $\mathbb{R}^n$, for the mth-order linear PDE with constant coefficients

$$\sum_{i_1 + i_2 + \ldots + i_k = m} a_{i_1 i_2 \ldots i_k} u_{x_1 x_2 \ldots x_k}^{(i_1 i_2 \ldots i_k)} = 0, \tag{63}$$

where i_j are natural number, $1 \leq j \leq k \leq n$. If Eq. (63) can be translated into

$$\begin{aligned}&\left(b_{1_1} D_{x_1} + b_{1_2} D_{x_2} + \ldots + b_{1_k} D_{x_k}\right)\left(b_{2_1} D_{x_1} + b_{2_2} D_{x_2} + \ldots + b_{2_k} D_{x_k}\right)\\&\ldots\left(b_{m_1} D_{x_1} + b_{m_2} D_{x_2} + \ldots + b_{m_k} D_{x_k}\right) u = 0.\end{aligned} \tag{64}$$

According to Proposition 11, we can get the general solution of Eq. (63) is

$$u = \sum_{r=1}^{m} f_r\left(y_{r_1}, y_{r_2}, \ldots y_{r_{k-1}}, x_{k+1}, \ldots x_n\right), \quad (1 \leq r \leq m), \tag{65}$$

where

$$y_{r_s} = c_{r_{s1}} x_1 + c_{r_{s2}} x_2 + \ldots + c_{r_{sk}} x_k, \quad (1 \leq s \leq k-1), \tag{66}$$

$$c_{r_{s1}} = \frac{-b_{s2} c_{r_{s1}} - b_{s3} c_{r_{s2}} - \ldots - b_{sk} c_{r_{sk}}}{b_{s1}}. \tag{67}$$

If Eq. (63) can be translated into:

$$\prod_{j=1}^{q} \left(b_{j_1} D_{x_1} + b_{j_2} D_{x_2} + \ldots + b_{j_k} D_{x_k}\right)^{p_j} u = 0, \tag{68}$$

where $\sum_{j=1}^{q} p_j = m$, its general solution of conjecture may be written by Conjecture 3.

Proposition 13. *In* $\mathbb{R}^n$*,*

$$a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_k u_{x_k} + a_{k+1} u = 0 \tag{69}$$

the general solution of Eq. (69) is

$$u = f\left(y_1, y_2, \ldots y_{k-1}, x_{k+1}, x_{k+2}, \ldots x_n\right) \sum_{i=1}^{k} l_i e^{\frac{-a_{k+1} x_i}{a_i}}, \tag{70}$$

where l_i *are arbitrary constants, and* y_i *satisfy (49-51).*

Prove. According to Z_3 Transformation, set $u\left(x_1, \ldots x_n\right) = g\left(x_1, \ldots x_k\right) h\left(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n\right)$, $y_i = c_{i1} x_1 + c_{i2} x_2 + \ldots + c_{ik} x_k$ and

$$\frac{\partial\left(y_1, y_2, \ldots y_k\right)}{\partial\left(x_1, x_2, \ldots x_k\right)} \neq 0. \tag{50}$$

So

$$\begin{aligned}&a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_k u_{x_n} + a_{k+1} u\\&= a_1 g_{x_1} h + a_1 g \sum_{i=1}^{k} c_{i1} h_{y_i} + a_2 g_{x_2} h + a_2 g \sum_{i=1}^{k} c_{i2} h_{y_i} + \ldots + a_k g_{x_k} h + a_k g \sum_{i=1}^{k} c_{ik} h_{y_i} + a_{k+1} g h\\&= \left(a_1 c_{11} + a_2 c_{12} + \ldots + a_k c_{1k}\right) g h_{y_1} + \left(a_1 c_{21} + a_2 c_{22} + \ldots + a_k c_{2k}\right) g h_{y_2} + \ldots\\&+ \left(a_1 c_{k1} + a_2 c_{k2} + \ldots + a_k c_{kk}\right) g h_{y_k} + \left(a_1 g_{x_1} + a_2 g_{x_2} + \ldots + a_k g_{x_k} + a_{k+1} g\right) h = 0.\end{aligned}$$

Set

$$c_{i1} = \frac{-a_2 c_{i2} - a_3 c_{i3} - \ldots - a_k c_{ik}}{a_1}, \quad (1 \leq i \leq k-1). \tag{51}$$

And set

$$a_1 g_{x_1} + a_2 g_{x_2} + \ldots + a_k g_{x_k} + a_{k+1} g = 0, \tag{71}$$

And set $g(x_1, \cdots x_k) = g(x_i)$, $(i = 1, 2, \cdots k)$, Then

$$a_1 g_{x_1} + a_2 g_{x_2} + \ldots + a_k g_{x_k} + a_{k+1} g = a_i g_{x_i} + a_{k+1} g = 0 \Longrightarrow g(x_i) = l_i e^{\frac{-a_{k+1} x_i}{a_i}}, \tag{72}$$

where l_i are arbitrary constants. Namely $g(x_1, \ldots x_k) = \sum_{i=1}^{k} l_i e^{\frac{-a_{k+1} x_i}{a_i}}$ is a particular solution of $a_1 g_{x_1} + a_2 g_{x_2} + \ldots + a_k g_{x_k} + a_{k+1} g = 0$, Thus

$$a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_k u_{x_n} + a_{k+1} u = (a_1 c_{k1} + a_2 c_{k2} + \ldots + a_k c_{kk})\, g h_{y_k} = 0 \Longrightarrow h_{y_k} = 0.$$

Namely

$$h = f(y_1, y_2, \ldots y_{k-1}, x_{k+1}, x_{k+2}, \ldots x_n),$$

where f is an arbitrary first differentiable function, so the general solution of Eq. (69) is

$$u = gh = f(y_1, y_2, \ldots y_{k-1}, x_{k+1}, x_{k+2}, \ldots x_n) \sum_{i=1}^{k} l_i e^{\frac{-a_{k+1} x_i}{a_i}}.$$

□

According to Proposition 4 and 13, we can get Proposition 14 directly:

Proposition 14. *In* $\mathbb{R}^n$*,*

$$(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k} + b_{k+1})^2 u = 0, \tag{73}$$

the general solution of Eq. (73) is

$$u = \left(\sum_{j=1}^{2} (s_{j1} x_1 + s_{j2} x_2 + \ldots + s_{jk} x_k)^{j-1} f_j (y_1, \ldots y_{k-1}, x_{k+1}, \ldots x_n) \right) \sum_{i=1}^{k} l_i e^{\frac{-b_{k+1} x_i}{b_i}}, \tag{74}$$

where l_i, s_{jr} *are arbitrary constants* $(1 \le r, i \le k)$*, and* y_i *satisfy (49-51).*

According to Proposition 14, we may present Conjecture 5:

Conjecture 5. *In* $\mathbb{R}^n$*,*

$$(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k} + b_{k+1})^m u = 0 \tag{75}$$

the general solution of Eq. (75) is

$$u = \left(\sum_{j=1}^{m} (s_{j1} x_1 + s_{j2} x_2 + \ldots + s_{jk} x_k)^{j-1} f_j (y_1, \ldots y_{k-1}, x_{k+1}, \ldots x_n) \right) \sum_{i=1}^{k} l_i e^{\frac{-b_{k+1} x_i}{b_i}}, \tag{76}$$

where l_i, s_{jr} *are arbitrary constants* $(1 \le r, i \le k)$*, and* y_i *satisfy (49-51).*

In $\mathbb{R}^n$, for the mth-order linear PDE with constant coefficients

$$\sum_{0 \le i_1 + i_2 + \ldots + i_k \le m} a_{i_1 i_2 \ldots i_k} u^{(i_1 i_2 \ldots i_k)}_{x_1 x_2 \ldots x_k} = 0, \tag{77}$$

where i_j are natural number, $1 \leq j \leq k \leq n$. If Eq. (77) can be translated into

$$\begin{aligned}&\left(b_{1_1}D_{x_1}+b_{1_2}D_{x_2}+\ldots+b_{1_k}D_{x_k}+b_{1_{k+1}}\right)\left(b_{2_1}D_{x_1}+b_{2_2}D_{x_2}+\ldots+b_{2_k}D_{x_k}+b_{2_{k+1}}\right)\\&\ldots\left(b_{m_1}D_{x_1}+b_{m_2}D_{x_2}+\ldots+b_{m_k}D_{x_k}+b_{m_{k+1}}\right)u=0.\end{aligned} \tag{78}$$

By Proposition 13, the general solution of Eq. (77) is

$$u=\sum_{r=1}^{m}\left(f_r\left(y_{r_1},y_{r_2},\ldots y_{r_{k-1}},x_{k+1},\ldots x_n\right)\sum_{i=1}^{k}l_{j_i}e^{\frac{-b_{j_{k+1}}x_i}{b_{j_i}}}\right), \tag{79}$$

where y_{r_s} satisfies (66-67), $(1 \leq s \leq k-1)$.

If Eq. (77) can be translated into:

$$\prod_{j=1}^{q}\left(b_{j_1}D_{x_1}+b_{j_2}D_{x_2}+\ldots+b_{j_k}D_{x_k}+b_{j_{k+1}}\right)^{p_j}u=0, \tag{80}$$

where $\sum\limits_{j=1}^{q}p_j=m$, its general solution of conjecture may be written by Conjecture 5.

Proposition 15. *In* $\mathbb{R}^n$,

$$a_1u_{x_1}+a_2u_{x_2}+\ldots+a_ku_{x_k}=A\left(u\right) \tag{81}$$

the general solution of Eq. (81) is

$$f\left(y_1,y_2,\ldots y_k,x_{k+1},x_{k+2},\ldots x_n\right)=0, \tag{37}$$

where $A(u)$ *is an arbitrary known function, and*

$$y_i=c_{i1}x_1+c_{i2}x_2+\ldots+c_{ik}x_k-\int\frac{a_ic_{i1}+a_2c_{i2}+\ldots+a_kc_{ik}}{A\left(u\right)}du, \tag{82}$$

$$\frac{\partial\left(y_1,y_2,\ldots y_k\right)}{\partial\left(x_1,x_2,\ldots x_k\right)}\neq 0, \tag{50}$$

c_{ij} *which satisfy (50) are relatively arbitrary constants* $(i,j\in\{1,2,\cdots k\})$.

Prove. According to Z_4 Transformation, set $f(y_1,y_2,\cdots y_k,x_{k+1},x_{k+2},\cdots x_n)=0$ and

$$y_i=c_{i1}x_1+c_{i2}x_2+\ldots+c_{ik}x_k+B_i\left(u\right),\quad\left(i=1,2,\ldots k\right), \tag{83}$$

$y_1,y_2,\cdots y_k$ are independent of each other. According to (5), we get

$$\begin{aligned}&a_1u_{x_1}+a_2u_{x_2}+\ldots+a_ku_{x_k}=-a_1\frac{\sum_{i=1}^{k}c_{i1}f_{y_i}}{\sum_{i=1}^{k}B_{i_u}f_{y_i}}-a_2\frac{\sum_{i=1}^{k}c_{i2}f_{y_i}}{\sum_{i=1}^{k}B_{i_u}f_{y_i}}-\ldots-a_k\frac{\sum_{i=1}^{k}c_{ik}f_{y_i}}{\sum_{i=1}^{k}B_{i_u}f_{y_i}}=A\left(u\right)\\&\Longrightarrow a_1\sum_{i=1}^{k}c_{i1}f_{y_i}+a_2\sum_{i=1}^{k}c_{i2}f_{y_i}+\ldots+a_k\sum_{i=1}^{k}c_{ik}f_{y_i}+A\left(u\right)\sum_{i=1}^{k}B_{i_u}f_{y_i}\\&=\left(a_1c_{11}+a_2c_{12}+\ldots+a_kc_{1k}+B_{1_u}A\right)f_{y_1}+\left(a_1c_{21}+a_2c_{22}+\ldots+a_kc_{2k}+B_{2_u}A\right)f_{y_2}+\ldots\\&+\left(a_1c_{k1}+a_2c_{k2}+\ldots+a_kc_{kk}+B_{k_u}A\right)f_{y_k}=0,\end{aligned} \tag{84}$$

where $B_{i_u}\equiv\frac{dB_i}{du}$, set

$$a_ic_{i1}+a_2c_{i2}+\ldots+a_kc_{ik}+B_{i_u}A=0,\quad\left(i=1,2,\ldots k\right). \tag{85}$$

Then

$$B_i(u) = -\int \frac{a_i c_{i1} + a_2 c_{i2} + \ldots + a_k c_{ik}}{A(u)} du. \tag{86}$$

So the general solution of Eq. (81) is (37)

$$f(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n) = 0, \tag{37}$$

where y_i satisfy (50, 82). □

It is not difficult to verify that the incomplete general solution of (81) only be got by using the characteristic equation method.

For some special nonlinear PDEs with constant coefficients, their general solutions can be obtained by similar methods in 2.2 section, we will not specifically study here.

4. Conclusions

In this paper, we first prove a new theorem for the independent variable transformational equations, that is, the independent variable transformation not only does not change the linearity or non-linearity of the original PDEs, but also does not change their order.

We propose the concept of the banal PDE and the non-banal PDE, and then use the proposed four kinds of Z Transformations to obtain the plentiful laws of general solutions of the linear PDEs with variable coefficient and constant coefficients, and get some laws of general solutions of the nonlinear PDEs, and present five new guesses.

The characteristic equation method is a basic method to solve first order linear and quasilinear PDEs. By comparing with the Z Transformations, we can find that it has many limitations, such as using it cannot get the general solutions of the first order linear PDEs (31.47), cannot obtain the complete general solutions of (57,81) and so on.

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