# On Consistency in the Skyrme Topological Model

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**Abstract**

We point to a significant mismatch between the nature of the baryon number and of the electric charge of baryons in the Skyrme topological model. Requirement of consistency between these two then demands a significant improvement in how the electric charge is defined in this model. The Skyrme model thereafter has a consistent electric charge which has a unique colour dependence built into it. Its relationship with other theoretical model structures is also studied.

**Keywords:** Skyrme topological model, Wess-Zumino anomaly, QCD, colour, neutral pion decay, Standard Model, Quantized Charge Standard Model

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**1. Skyrme topological model**:

The Skyrme Lagrangian is [1,2]

$$L_S = \frac{{f_\pi}^2}{4} Tr(L_\mu L^\mu) + \frac{1}{32e^2} Tr[L_\mu, L_\nu]^2 \tag{1}$$

where $L_\mu = U^\dagger \partial_\mu U$ . The U field for the three flavour case is

$$U(x) = exp[\frac{i\lambda^a \phi^a(x)}{f_\pi}] \tag{2}$$

Here $\phi^a$ is the pseudoscalar octet of $\pi$, K and $\eta$ mesons. In the full topological Skyrme model this is supplemented with a Wess-Zumino effective action gievn as [1,3],

$$\Gamma_{WZ} = \frac{-i}{240\pi^2} \int_\Sigma d^5x \epsilon^{\mu\nu\alpha\beta\gamma} Tr[L_\mu L_\nu L_\alpha L_\beta L_\gamma] \tag{3}$$

Thus with this anomaly term the effective action is.

$$S_{eff} = \int d^4x \; Tr \; [L_\mu L^\mu] + n \; \Gamma_{WZ} + quartic \; term \tag{4}$$

where the winding numbber n is an integer $n \in Z$, the homotopy group of mapping being $\Pi_5(SU(3)) = Z$.

Under an electromagnetic gauge transformation, generated by the charge operator Q, the WZ term gives rise to an anomalous electromagnetic current. It turns out that this current is purely isoscalar and therefor it is proportional to the quark model baryon current. One finds the Noether current as [1,2]

$$J^\mu(x) = \frac{1}{48\pi^2} \epsilon^{\mu\nu\alpha\beta} Tr[Q(L_\alpha L_\beta L_\gamma - R_\alpha L_\beta R_\gamma)] \tag{5}$$

with $L_\mu = U \partial_\mu U^\dagger$

Demanding gauge invariance in the presence of the electromagnetic field, the effective action becomes,

$$\hat{S_{eff}} = \frac{{f_\pi}^2}{4} \int d^4x \; Tr \; [D_\mu U (D^\mu U)^\dagger] + n \; \hat{\Gamma_{WZ}} + quartic \; term \tag{6}$$

One of the terms in $n\hat{\Gamma_{WZ}}$ is applied [1,3] to calculate the anomalous decay rate of $\pi^0 \to \gamma\gamma$ with u- and d- quarks going around in the triangular loop as,

$$A = \frac{1}{8\pi^2} \frac{1}{f_\pi} [n(Q(u)^2 - Q(d)^2)] e^2 F_{\mu\nu} \bar{F}_{\mu\nu} \phi_{\pi^0} \tag{7}$$

**2. Static charges and the Standard Model**:

With static quark charges 2/3 and -1/3, in eqn. (7) the term $(Q(u)^2 - Q(d)^2) = \frac{1}{3}$. Then identifying the winding number with colour, we get $n = N_c = 3$. Note that the number of colours taken as 3 is what matches the experiment well. Thus again, note that it is only for three colours that the expermemntal data is matched.

Note that in eqn. (7) the square bracket has two terms - first the winding number "n" and the second one is the square charge differences. This charge part gives (as above) 1/3 for the static charges. By static we mean these are independent of the colour degree of freedom.

Wherefrom arises this unique static charge structure used in Witten's paper [1,3]? To get the static charges of quarks 2/3 and -1/3 one defines the electric charge operator in the standard way [3, 4 p. 309] as,

$$Q = T_3 + \frac{1}{6} \tag{8}$$

To get these static charges, the term $\frac{1}{6}$ is fixed, i.e. is independent of colour.

Actually these are the well know charges obtained in the Standard Model (SM). The electric charge that arise here are actually of the electro-weak group $SU(2)_W \otimes U(1)_W$, A commonly used definition of the electric charge in the SM is [2 p. 368,5 p. 346]:

$$Q = {T_3}^W + \frac{Y_W}{2} \tag{9}$$

The value of the weak-hypercharges is put in by hand to ensure that for example the electric charges of the u- and the d-quarks are:

$$Q(u) = \frac{2}{3} \quad , Q(d) = -\frac{1}{3} \tag{10}$$

Later the electro-weak group was extended to include the colour group to make the whole product group of the SM to be $SU(3)_c \otimes SU(2)_W \otimes U(1)_W$. This then was christened the Standard Model (SM). However the same static charges as above were taken over to hold for the bigger product group of the SM. Thus these static charges are also by definition assumed to hold for arbitrary number of colours in $SU(N)_c \otimes SU(2)_W \otimes U(1)_W$.

Note that the electric charge is pre-defined in the SM with respect to the Spontaneous Symmetry Breaking (SSB) with an Englert-Brout-Higgs (EBH) field. SSB then generates massses for the gauge fields and also for the matter particles. Posteriori these SM charges do satisfy the anomaly cancellations for each generation separately [2,5]. However these charges are arbitarily defined and as such also are not quantized - and this has been known to be a major weakness of the SM [2,5].

Next note that for say, with five colours, the association $n = N_c = 5$ will not match the experimental pion decay rate for fixed charges 2/3 and -1/3. But anyway Witten took eqn. (7) to hold good for

$$n = N_c \tag{11}$$

for arbitrary number of colours and "with three in the real world" [3].

Then it can be shown that from eqn. (8), the Noether current in eqn. (5) $J_\mu$, is an isoscalar. Thus Witten obtained the baryon current from eqn. (5)

by replacing Q by $\frac{1}{N_c}$, the baryon number carried by each quark in a baryon. Thus the composite baryon number is finite $B = \frac{1}{N_c} \times N_c$. Then he identified $nJ_\mu \to nJ_B^\mu$, as

$$\begin{aligned} nJ_B^\mu(x) &= \frac{1}{48\pi^2}\frac{n}{N_c}\epsilon^{\mu\nu\alpha\beta}Tr[(L_\alpha L_\beta L_\gamma - R_\alpha L_\beta R_\gamma)] \\ &= \frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}Tr(L_\alpha L_\beta L_\gamma) \end{aligned} \tag{12}$$

This is exactly the same as the topological current [1,2].

Now we have seen above that for arbitrary $N_c$, the composite baryon number is finite, i.e. 1. As well known in quark model, the electric charge depends upon baryon number as well. So as baryon number B=1 is finite, we would expect the electric charges of proton and neutron to be finite as well. In standard manner we take, $N_c = 2k + 1$ to ensure that the number of colours are odd. Then e.g. proton is made up of (k+1) number of u-quarks and k number of d-quarks [6,7,8]. Then for static individual quark charges 2/3 and -1/3, the composite proton and neutron charges are, ,

$$Q(p) = \frac{N_c + 1}{2}(\frac{2}{3}) + \frac{N_c - 1}{2}(-\frac{1}{3}) = \frac{N_c + 3}{6} \tag{13}$$

$$Q(n) = \frac{N_c - 1}{2}(\frac{2}{3}) + \frac{N_c + 1}{2}(-\frac{1}{3}) = \frac{N_c - 3}{6} \tag{14}$$

Surprisingly these charges of composite baryons are colour dependent. Only for $N_c = 3$ the correct baryon charges arise. In general these are blowing up with colour, and neutron does not even remain charge neutral.

This should be considered a major inconsistency in this analysis. So what is the source of the problem? Notice that from eqns. (8) (9), the non-$T_3$ part are static and colour independent. On the other hand in the baryon number above the hypercharge part was only contributing. We therefore surmise that the above inconsistency must be arisisng from the the static nature of the electric charges.

**3. Consistent electric charges of quarks**:

Let us propose a general definition of electric charge for quarks, taking cue from eqns. (8) and (9) as

$$Q(q) = T_3 + b \tag{15}$$

where b is an unknown. In keeping with the spirit of our discussion here, we fix it by making sure that the proton charge is 1 and neutron is charge neutral. So for proton,

$$Q(p) = \frac{N_c + 1}{2}(\frac{1}{2} + b) + \frac{N_c - 1}{2}(-\frac{1}{2} + b) = 1 \tag{16}$$

This gives $b = \frac{1}{2N_c}$. And thus, the proper definition of electric charge here is,

$$Q(q) = T_3 + \frac{1}{2N_c} \tag{17}$$

Now with the above colour dependent charges the the square bracket in eqn. (7) and with $n = N_c$ for arbitrary number of colours,

$$N_c(Q(u)^2 - Q(d)^2) = N_c\left[\left\{\frac{1}{2}\left(1+\frac{1}{N_c}\right)\right\}^2 - \left\{\frac{1}{2}\left(-1+\frac{1}{N_c}\right)\right\}^2\right] = 1 \tag{18}$$

And hence overall there is no $N_c-$ dependence left in the decay rate of $\pi^0 \to \gamma\gamma$ and the subsequent result matches the experiment well for any arbitrary colour. So when proper colour dependent electric charges of the quarks are taken, the decay rate is actually independent of the colour degrees of freedom. Thus this is a major success of the above colour dependent charges of the quarks for the Skyrme-Witten model.

Thus we have now with these colour dependent charges, complete consistency between baryon number and the electric charges, and which in additiuon also leads to consistency of the pion decay for arbitrary number of colours.

We would like to point out similar colour dependent charges had earlier been obtained by the author [8,9].

In line with the full group stucture for arbitary number of colours $SU(N)_c \otimes SU(2)_L \otimes U(1)_Y$ (also for $N_c = 3$ as well), and having the same generational structure as the above Standard Model (SM) and with the same Englert-Brout-Higgs mechanism of spontaneous symmetry breaking etc., but with the major difference that the anomaly cancellations play a more direct role in fixing the hypercharges, that one obtains a more general definition of the quark electric charges [8,9] as,

$$Q(u) = Q(c) = Q(t) = \frac{1}{2}(1 + \frac{1}{N_c}) \tag{19}$$

$$Q(d) = Q(s) = Q(b) = \frac{1}{2}(-1 + \frac{1}{N_c}) \tag{20}$$

Thus amazingly, in this model the electric charges of quarks intrinsically know of the colour degree of freedom. It was also shown convincingly [8] that the correct charge does not merely take the static values of 2/3 and -1/3 (i.e., independent of any colour), but can differ from these values for arbitrary number of colours as given above. Also note that the square factor as being equal to 1, and as given in eqn. [18] was also obtained in the Quantized Charge Standard Model (QCSM) [10] (also see below).

Now an important distinction may be emphasized. In the SM the charges are of the electro-weak sector only with no colour dependence. In addition they are put in by hand and are pre-defined with respect to the SSB. Thus, by its very definition, colour in electric charge is completely outside the purview of the SM.

In contrast in the new model above, if is fully quantized and has intrinsic colour dependence. It gets defined by the process of the SSB itself, with an

EBH field and where the anomalies play a basic role in fixing the unknown hypercharges [8,9]. Amazingly though the QCD is independent of QED, the electric charge however knows of the colour itself. Thus $SU(N)_c \otimes SU(2)_L \otimes U(1)_Y$ (also for $N_c = 3$) is actually already **an unified model**. So in this model not only is mass generated in the SSB, but the electric charge itself gets generated by the SSB. Thus this model is fundamentally different from the SM. Though it has the same family structure as the SM and uses similar EBH field structure for SSB, it goes beyond the limitations of the SM, and thus provides a more fundamental description of the physical reality. We should thus distinguish it from the SM and hence we call it as the Quantized Charge Standard Model (QCSM).

**4. Conclusions**:

Note that in this paper we obtain a satisfying consistency of two apparently distinct theoretical structures - the Skyrme topological model and the Quantized Charge Standard Model. What seems to connect the two are the anomalies. These anomalies manifest themselves quite differently in these models, but intrinsically these provide consistent baryon numbers and electric charges of the matter fields. Our work here prompts for further studies on the relationship of these two theoretical structures.

In summary, here we have shown that the static charges 2/3 and -1/3 of the Standard Model appear to be inconsistent when used in the Skyrme topological model. Consistency here, within the complete structure of the Skyrme model, demands that the electric charges of quarks have colour dependence in them, This colour dependence is exactly the same as that provided by the Quantized Charge Standard Model as well. Thus these new consistency structure effects in the Skyrme model provide support to the structures arising in the Quantized Charge Standard Model (QCSM).

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