The mathematical machinery of logical independence underlying quantum randomness and indeterminacy

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Abstract I Follow up on the 2008 experiments of Tomasz Paterek et al, which link quantum randomness with logical independence. Analysis reveals, that the Paterek formalism (unwittingly) relaxes a *Quantum Postulate*. That relaxation denies the axiomatic imposition of unitary, Hermitian and Hilbert space mathematics, while allowing these to arise freely, as logically independent structures. Surprisingly, the Paterek formalism demands a *non-unitary* environment — where unitary structures may freely switch on or off. The unitary environment is necessary in the formation of superposition states, but not eigenstates. This unitary condition is sustained by self-referential logical circularity around cyclic sequences of transformations. Amongst all possible self-referential systems, these generate stable, persistent structures we recognise as quantum mechanical vectors and operators. Circularity explains indeterminacy's non-causedness. Non-definiteness, stems from *geometric ambiguity* — typically, left|right handedness in the Bloch sphere. Collapse is *caused* when the unitary symmetry is deformed by some agency, such as a magnetic field or polariser.

Keywords foundations of quantum theory, axiomatised quantum theory, quantum mechanics, quantum randomness, quantum indeterminacy, quantum information, linear algebra, elementary algebra, imaginary unit, prepared state, measured state, eigenstate, superposition state, Hilbert space, unitary, redundant unitarity, orthogonal, scalar product, inner product, mathematical logic, logical independence, self-reference, logical circularity, mathematical undecidability.

1 Introduction

In *classical physics*, experiments of chance, such as coin-tossing and dice-throwing, are *deterministic,* in the sense that, perfect knowledge of the initial conditions would render outcomes perfectly predictable. 'Classical randomness' stems from ignorance of *physical information* in the detail of the initial toss or throw.

In diametrical contrast, in the case of *quantum physics*, the theorems of Kocken and Specker [18], the inequalities of John Bell [4], and experimental evidence of Alain Aspect [1,2], all indicate that *quantum randomness* does not stem from any such *physical information.*

As response, in 2008, Tomasz Paterek et al published experiments, proving that the origin of quantum randomness lies in *mathematical information* [19,20,21,22]. In experiments measuring photon polarisation, Paterek et al demonstrate statistics correlating *predictable* outcomes with *logical dependence,* between certain Boolean propositions; and *random* outcomes with proposition's *logical independen*ce. Briefly stated, logical independence refers to the null logical connectivity that exists between mathematical formulae, that neither prove nor disprove one another. It is commonplace throughout mathematics.

In 1944, this same problem had been confronted by Hans Reichenbach [24], whose ideas were later supported by Hilary Putnam [23]. Reichenbach constructed a purely theoretical logic, which now can be seen to agree with the mathematics discovered by Tomasz Paterek et al.

Logical Independence in Physics — Information flow & self-reference in Elementary Algebra.

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Reichenbach's book details a '3-valued logic' comprising values: true, false and *indeterminate;* possessing the feature: 'true' is not the same as 'not false'. He showed that this non-classical logic resolves 'causal anomalies' of quantum theory, including *complementarity,* and the *action at a distance* paradox, highlighted by Einstein, Podolsky & Rosen [7,17].

Reichenbach is not in opposition to the 'mainstream' quantum logics, based on Postulates of Hilbert space theory, such as Birkhoff and von Neumann [5]. The approach of Reichenbach was to design a logic, isomorphic to the epistemology for *prepared* and *measured* states – typically the question of what we may know about the state of a photon immediately before measurement. As argued by Hardegree, Reichenbach's logic is framework for an alternative formulation of quantum theory [15]. Reichenbach was predicting, or at least expecting, something of the nature of the Paterek findings.

This present paper is part of an ongoing project investigating logical independence in relation to Foundations of Physics. In an earlier article [10] I cover logical Independence of the imaginary unit, within Elementary Algebra, and especially how the Soundness and Completeness Theorems from Model Theory, a branch of Mathematical Logic, demonstrate this logical independence.

In other articles, I demonstrate redundancy of the *Unitary Postulate* in quantum mathematics of the free particle quantum system [13,9]. In another I demonstrate self-referential logical circularity in the free particle system [12]. And in yet another I discuss ambiguity of perfect symmetries [8].

The implication of the Paterek findings is that quantum randomness results from logical independence. But because this independence is seen evident in a *Boolean* system, the insight they offer for the Foundations of Physics, is made obscure. To understand the workings of quantum randomness, the same logical artifact must be expressed explicitly in the language and formalism of *Matrix Mechanics* — and in this case, in terms of the Pauli algebra $\mathsf{su}(2)$.

Close examination reveals that the success of the Paterek formalism (unwittingly) relies on the relaxation of a *Quantum Postulate.* That relaxation freely allows the Postulate's mathematical content, but contradicts its axiomatic imposition. Section 10 of this paper demonstrates that. Taking the view that the Paterek research is sound, if Matrix Mechanics is to faithfully represent quantum randomness it must be made consistent with Paterek by relaxing that Postulate.

Working from the Paterek formalism as starting point, this paper shows that quantum indeterminacy is due to inaccessible history through loss of information, during the measurement process – as opposed to a limitation on information held in a cubit. This is demonstrated in Section 8. It reveals a matrix mechanical environment which is non-unitary, that supports eigenstates — on which rests a logically independent, unitary, Hermitian, Hilbert space environment, which is invoked by the existence of superposition states. The switch between these two environments is facilitated by freely occurring self-referential logical circularity which is the source origin of the logical independence. This self-referential mechanism is demonstrated in Section 11.

In this article, all usage of the term orthogonal refers to orthogonality in the sense of orthogonal members of some vector space, not in the sense of an orthogonal group. All usage of the term unitary refers to members of some unitary group.

At the heart of the Paterek formalism is the use of a *density matrix.* Critically, unlike probability density, the density matrix conveys information about the full set of all complimentary states. But much more importantly, it is the following radical approach which reveals the Paterek logical independence.

The unique form of the Paterek density operator is crucially ingenious. It behaves as a memory-bit, registering orthogonality within the experiment, conveyed by photons. I call this register *orthogonality index.* This is a Boolean register holding information generated by commutators in algebraic processes, within the density matrix. Density operators whose index becomes *set* have ambiguous history because orthogonality index has no memory of how orthogonality was generated — whether for instance, it originated through the sequence $\sigma_{\mathsf{x}}\sigma_{\mathsf{z}}$, or the sequence $\sigma_{\mathsf{y}}\sigma_{\mathsf{z}}$. This means that in experiments involving polarisers aligned (Bloch) orthogonally, measurements cannot access that historical information. On the other hand, density operators whose orthogonality index is *not set* suffer no ambiguous history.

The above briefly describes processes from the viewpoint of a Boolean 'memory-bit' formalism. The detail is covered in the text. But there is better insight to gain by viewing the density matrix from the standpoint of algebras and symmetry.

An assumption made by the Paterek team (unwittingly) relaxes one of the *Quantum Postulates* — and therefore contradicts it. They assume that isomorphism exists between the Boolean system and the Pauli algebra. In fact the Boolean system asserts non-unitary[1], involutory[2] operators which may be freely restricted to the Pauli algebra. That restriction requires extra, new, logically independent information. The freedom passes into the density matrix. The non-unitary, involutory operators are easily capable of representing experiments whose polarisers are all aligned parallel, but not otherwise.

Involutory matrices:

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}^2 = \mathbb{1}_2 \qquad \text{for } \ a^2 + bc = 1$$

Cases of interest are:

$$\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}^2 = \mathbb{1}_2 \qquad \text{for} \qquad a^2 - b^2 = 1$$

$$\begin{pmatrix} 0 & b^{-1} \\ b & 0 \end{pmatrix}^2 = \mathbb{1}_2 \qquad \text{for all } \ b$$

If a matrix mechanics is to be written that is consistent with Paterek's formalism — which does represent randomness and predictability — then that new formalism must accept that extra information is needed in representing experiments aligned orthogonal, over information needed in representing experiments aligned parallel. This means that that new formalism must contradict the assumption that 'measurement on eigenstates', and 'measurement on superposition states, can <u>*both*</u> be represented <u>*isomorphically* and *faithfully*</u> by the same matrix operator.

Taking the view that the Paterek density matrix correctly portrays experiments, the *Quantum Postulate* requiring every quantum system be Unitary, Hermitian, and represented by Hilbert space, must be rethought. The constraint it imposes must be relaxed. That relaxation does not deny, but allows the mathematics stated in the *Postulate*; but it does deny its *axiomatic imposition.* Superpositions require the usual textbook unitary mathematics— but eigenstates are free of this unitary constraint.

Even from an intuitive viewpoint, eigenstate mathematics does not demand an orthogonal or unitary environment.

Freedom from imposed unitarity permits *'unitary on-off switching',* in transitions between eigenstates and superposition states. This switching is affected through a mechanism of self-referential cyclic and anti-cyclic transformation sequences. This logically dynamic system maintains isomorphic representation across all states and is also faithful to the logic of experiments.

Enhanced understanding is to be gained when all the above is seen from the context of *Elementary Algebra.* This is the algebra learnt at school. It is the algebra of rational, real and complex numbers. More formally, it's the algebra of the infinite fields of scalars. It is the algebra we take for granted throughout Applied Mathematics and upon which quantum mathematics rests.

In Mathematical Logic, logical independence of the imaginary unit is well-known and well-understood [26]. When Elementary Algebra is treated as a *formal axiomatised system*, this algebra's relationship, connecting axioms with the imaginary unit is a logically independent one [11]. This is in contrast to all *rationals*, which are *logically dependent.* This independence and dependence furnishes a logic, comprising values: *provable*, *negatable* and *'neither provable nor negatable'.* This is logic comparable to Reichenbach's.

As opposed to the standpoint of Linear Algebra, where orthogonality is a matter of definition, from the standpoint of Elementary Algebra, orthogonality (of function spaces) emerges, not by way of definition, but through logically circular self-reference. Such self-referential orthogonality is documented by Elemér E Rosinger and Gusti van Zyl [25]. The imaginary unit then exists unavoidably, as a demand of the consequent unitarity.

The reasons why self-reference has any significance here is it explains the origins of logical independence in quantum mathematics, and also, it persistently sustains the *stable existence* of structures, we recognise as, quantum mechanical vectors and operators. Axioms of Elementary Algebra assert existence of algebraic structures and objects, agreeing to specific relations. These axioms are constantly in force, acting on the whole system. This allows mathematical machines to develop which happen to circulate their output back as input. Amongst the plethora of circulating information which diverges and dissipates, certain machines result in persistent stability.

[1] Need note explaining meaning of unitary. IE in the sense of the Pauli operators.
[2] An *involutory* operator is one whose square is the identity operator. e.g. $\mathsf{a}^2 = \mathbb{1}$.

Interpretationally, unitarity deriving from these self-referential origins successfully explains features of indeterminacy:

▷ *Non-definiteness of indeterminacy*
Superpositions necessarily occupy a unitary environment. This unitarity is a *perfect symmetry* which therefore presents a *perfect geometrical ambiguity* in left|right handedness, in the Bloch sphere. So long as the symmetry remains perfect, the ambiguity persists.

▷ *Non-causedness of indeterminacy*
Logically independent unitarity materialises, not through *cause,* but through *non-prevention* of the self-referential circularity, permitted by not contradicting any information in the environment. So long as no information contradicts its denial, the condition can persist.

▷ *Caused Collapse*
Measurement experiments *cause* the unitary symmetry to break, by denying orthogonality, by distorting it – by way of a magnetic field or polariser, maybe. When the unitary symmetry is broken, superpositions are destroyed, but unitarity is not needed for eigenstates and they are left in tact. This *unitary switch-off* forces the denial and collapse of the non-definite ambiguity, and denial and collapse of all superpositions, allowing only eigenstates.

2 The Imaginary Unit

There are two 'routes' by which a logically independent imaginary unit enters Elementary Algebra.

In ***Route One,*** certain eigenvalue equations are to blame, those of certain *orthogonal rotations.* Somewhat paradoxically these seek to map vectors *parallel* to themselves ! For example:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} \mathsf{x} \\ \mathsf{y} \end{bmatrix} \mapsto \lambda \begin{bmatrix} \mathsf{x} \\ \mathsf{y} \end{bmatrix} \tag{1}$$

Written in the language of Elementary Algebra, this becomes the simultaneous pair of linear equations

$$\begin{aligned} \mathsf{y} &= -\lambda \mathsf{x} \\ \mathsf{y} &= \lambda^{-1} \mathsf{x} \end{aligned} \tag{2}$$

This equivalence requires:

$$\exists \lambda \left(\lambda^2 = -1 \right) \tag{3}$$

Axioms of Elementary Algebra

	ADDITIVE GROUP	
A0	$\forall\beta\forall\gamma\exists\alpha \mid \alpha = \beta + \gamma$	CLOSURE
A1	$\exists 0 \forall\alpha \mid \alpha + 0 = \alpha$	IDENTITY 0
A2	$\forall\alpha\exists\beta \mid \alpha + \beta = 0$	INVERSE
A3	$\forall\alpha\forall\beta\forall\gamma \mid (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$	ASSOCIATIVITY
A4	$\forall\alpha\forall\beta \mid \alpha + \beta = \beta + \alpha$	COMMUTATIVITY
	MULTIPLICATIVE GROUP	
M0	$\forall\beta\forall\gamma\exists\alpha \mid \alpha = \beta \times \gamma$	CLOSURE
M1	$\exists 1 \forall\alpha \mid \alpha \times 1 = \alpha$	IDENTITY 1
M2	$\forall\beta\exists\alpha \mid \alpha \times \beta = 1 \wedge \beta \neq 0$	INVERSE
M3	$\forall\alpha\forall\beta\forall\gamma \mid (\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$	ASSOCIATIVITY
M4	$\forall\alpha\forall\beta \mid \alpha \times \beta = \beta \times \alpha$	COMMUTATIVITY
AM	$\forall\alpha\forall\beta\forall\gamma \mid \alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma)$	DISTRIBUTIVITY
C0	$0 \neq 1; \quad 0 \neq p, \quad p = \text{any prime}$	CHARACTERISTIC 0

Table 1 Axioms of Elementary Algebra. These are written as sentences in *first-order logic.* They comprise the standard *Field Axioms* with an added axiom that excludes modulo arithmetic. Collectively, Axioms assert a definite set of information, deriving a definite set of theorems. Any proposition (in the language) is either a theorem or is otherwise logically independent. Theorems include *implications* and *negations.* All other statements in the language are logically independent.

But, in Elementary Algebra, no such λ exists as a consequence of Axioms. That is to say, the square root of minus one does not follow as a consequence of Axioms of Elementary Algebra. There is no theorem, deriving from Axioms, which asserts existence of this number. But equally, no theorem of these Axioms contradicts (3), and so it is not denied either [11]. The same applies to all irrational numbers. This 'neither implied nor denied' condition is in contrast to all rationals, whose existences all follow in consequence of Axioms. Another way of looking at this is: (3) is consistent with Axioms even though Axioms do not prove it.

This means that (3), and therefore (1), asserts logically independent information with respect to the Axioms of Elementary Algebra, and that that logical independence stems from the *assumed simultaneity* in (2) — which was asserted by (1). By the way, no scalar factor can be extracted from the matrix in (1), which can avoid the result (3).

For any system, logically independent information enters always by way of some assumption, extra to Axioms.

It remains for me to explain why the orthogonal rotation $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ should arise at all. The reason is: if some condition should demand '3-way orthogonality' where three 2×2 matrices should be all mutually orthogonal[3], then that matrix is unavoidable, and shall be one of the three (up to a scaling).

In ***Route Two,*** as well as imaginary eigenvalues being a consequence, the '3-way orthogonal' system is unavoidably unitary. So to demand a condition of '3-way orthogonality' is to demand a *group* in which the imaginary unit is an unavoidable, undeniable scalar, necessarily being a factor somewhere.

3 What it means for Pauli

The job of this paper is to show what the Paterek Boolean information means for the system of Pauli operators. The interesting surprise revealed, is that although every measurement of polarisation is representable by the Pauli algebra $\mathsf{su}(2)$, only the measurement of superposed states *requires* this algebra. Measurement of pure eigenstates does not. For eigenstates, the unitary component of the Pauli algebra is not involved.

In predictable experiments, where measurement is on eigenstates, unitarity is shown to be 'redundant' — *possible* but *not necessary.* And in experiments whose outcomes are random, where measurement is on superposed states, unitarity is shown unavoidably *necessary.* My deduction is that there is a *unitary switch-on* in passing from eigenstates to superpositions and a *unitary switch-off* in passing from superpositions to pure. I show that this switch-on entails unopposed, freely-occurring logical circularity, in transformation information. This circularity is uncaused, but unprevented. By that, I mean, no information already present in the system implies or denies it. The switch-on process is *symmetry forming*[4], but does not involve energy or any other conservation rule. The newly formed symmetry results in new ambiguity, between left|right handedness, in a space where before, for the eigenstate, handedness had been definite. That ambiguously handed space is the Bloch sphere. It is worth noting that the creation of any perfect symmetry always introduces geometrical ambiguity of some sort [8].

Briefly said: when a photon is prepared in a definite pure state, when subsequently transformed (prepared) to a complementary state, it must convey the ambiguous information, inherent in the newly created unitary symmetry.

The logical regime for this unitary switching can be viewed in two ways. It can be viewed as a system that is always unitary, but where unitarity switches between possible and necessary: such a *possible/necessary* system constitutes a *modal logic.* Or otherwise, it can be seen as a complete switch between different symmetries, where unitarity is new, *logically independent*, extra information required for the transition. To adequately describe the transition between pure and superposed states, either modal logic is needed, or logical independence. The classical logic of *true* and *false* is not an option.

In measurement experiments made on superposed states, whose outcomes are random, in the usual well-known way, the system symmetry is isomorphically and

[3] Matrix operators a and b are orthogonal when they satisfy the condition $\mathsf{ab} + \mathsf{ba} = \mathbb{0}$.

[4] *Spontaneous symmetry breaking* involves energy.

faithfully represented by the (unitary) Pauli matrices:

$$\sigma_{\mathsf{x}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_{\mathsf{y}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_{\mathsf{z}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4}$$

But, for measurements made on eigenstates, the Paterek experiments prove they are faithfully represented by this set of non-unitary, matrices:

$$\mathsf{s}_{\mathsf{x}}(\eta) = \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} \qquad \mathsf{s}_{\mathsf{y}}(\eta) = \begin{pmatrix} 1 & 0 \\ 0 & \eta^2 \end{pmatrix} \qquad \mathsf{s}_{\mathsf{z}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

where η is scalar of any value. Note that $\{\sigma_{\mathsf{x}}, \sigma_{\mathsf{y}}, \sigma_{\mathsf{z}}\}$ is a particular instance of $\{\mathsf{s}_{\mathsf{x}}(\eta), \mathsf{s}_{\mathsf{y}}(\eta), \mathsf{s}_{\mathsf{z}}\}$.

The three matrices (5) are tailored by the author of this present paper to exactly match orthogonality and involutory information taken directly from Boolean information differentiating between *random* and *non-random* experiments of the Paterek paper. Whereas, in the three Pauli matrices (4) there is *3-way orthogonality*, that is, each is orthogonal with both the others, in the non-unitary matrices (5), there is altogether zero orthogonality, except in the accidental coincidence of $\eta = \pm i$. Also, each Pauli matrix (4) is *involutory*, but in (5) only $\mathsf{s}_{\mathsf{x}}(\eta)$ and s_{z} are involutory.

This unconventional ordering of Pauli matrices is chosen to agree with representation chosen by Paterek. It stems from his choice of $\{\sigma_{\mathsf{x}}, \sigma_{\mathsf{z}}\}$ information, used to represent each and every polarisation alignment, as opposed to $\{\sigma_{\mathsf{y}}, \sigma_{\mathsf{x}}\}$ say. Though this introduces asymmetry into the mathematical viewpoint, it simply reflects the choice of label selected for the axis along the beam direction, in experiments.

$$\begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} \text{ has eigenvectors } \begin{bmatrix} 1 \\ \eta \end{bmatrix} \ \& \ \begin{bmatrix} \eta^{-1} \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \eta^2 \end{pmatrix} \text{ has eigenvectors } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ \& \ \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with eigenvalues 1 & η^2

4 Information and logic

In Mathematical Logic, a *formal system* is a system of mathematical formulae, treated as propositions, where focus in on *provability* and *non-provability*.

A formal system comprises: a precise language, rules for writing formulae, and further rules of deduction. Within such a formal system, any two propositions are **either** *logically dependent* — in which case, one proves, or disproves the other — **or otherwise** they are *logically independent*, in which case, neither proves, nor disproves the other.

A helpful perspective on this is the viewpoint of Gregory Chaitin's information-theoretic formulation [6]. In that, logical independence is seen in terms of information content. If a proposition contains information, not contained in some given set of axioms[5], then those axioms can neither prove nor disprove the proposition.

Edward Russell Stabler explains logical independence in the following terms. A formal system is a postulate-theorem structure; the term postulate being synonymous with axiom. In this structure, there is discrimination, separating *assumed* from *provable* statements. Any statement labelled as a postulate which is capable of being proved from other postulates should be relabelled as a theorem. And if retained as a postulate, it is logically superfluous and redundant [26]. If incapable of being proved or disproved from other postulates, it is logically independent.

Central to the formal system used in the Paterek et al research are these Boolean functions of a binary argument:

$$x \in \{0,1\} \mapsto f(x) \in \{0,1\}$$

Typical propositions, stemming from those functions, are these:

$$\begin{array}{lll} f(0) = 0 & f(1) = 0 & f(0) = f(1) \\ f(0) = 1 & f(1) = 1 & f(0) \neq f(1) \end{array} \tag{6}$$

Such propositions are items of information, taken as being openly true or openly false. Our interest lies, not so much, in their truth or falsity, but in, which statements prove which, which disprove which, and which do neither. In other words, which are logically dependent and which are logically independent.

As illustration, if $f(0) = 0$ were considered to be true, the statement $f(0) = 1$ would be proved false. More simply, we could say: $f(0) = 0$ disproves $f(0) = 1$, and accordingly, $f(0) = 1$ is *logically dependent* on $f(0) = 0$.

On the other hand, again, if $f(0) = 0$ were considered to be true, that would not prove, or disprove $f(1) = 0$. We could say: $f(0) = 0$ neither proves, nor disproves $f(1) = 0$, and accordingly, $f(0) = 0$ and $f(1) = 0$ are *logically independent*.

[5] *Axioms* are propositions presupposed to be 'true' and adopted *a priori*.

Notation: The functions in (6) are shown as Paterek wrote them. In sections that follow, I replace that notation, by writing $\overset{\square}{f}(0)$ & $\overset{\square}{f}(1)$ to denote definite information determined and written by the Blackbox, and $\overset{\diamond}{f}(0)$ & $\overset{\diamond}{f}(1)$ to denote information, as read by Measurement. The $\square$ and $\diamond$ notation is borrowed from Modal Logic, respectively meaning *necessary* and *possible*.

5 The Paterek et al experiments

The Paterek et al research concerns polarised photons as information carriers through measurement experiments. The experiment hardware comprises a sequence of three segments, which in accordance with Paterek, I denote: Preparation, Blackbox and Measurement. These *prepare*, then *transform*, then *measure* polarisation states. Informationally, the experiment apparatus can be thought of as hardware being fed with hard input data, in the form of the hardware configuration; and expressing output data, in the form of measurement outcome. The hardware configuration is the experiment's orientational alignment of interchangeable hardware filters, read from an X–Y–Z reference system fixed to the hardware. The Y axis is aligned along the direction of photon propagation. Measured states of polarisation are the experiment's output data. Experiments were performed very many times and statistics accumulated. Finally, correlations are found evident, relating configuration input with experiments' output, being either random or predictable. Details of the experiments' setups are taken from Johannes Kofler's Dissertation [19].

The Johannnes Kofler article [19] best explains the experimental setup.

1. Preparation
   Photons prepared, either as $|z+\rangle$, $|x+\rangle$ or $|y+\rangle$ eigenstates, by filtering, directly after one of these Pauli transformations:
   (a) $|z+\rangle$, Linear polariser aligned at 0° against Z axis.
   (b) $|x+\rangle$, Linear polariser aligned at 45° against Z axis.
   (c) $|y+\rangle$, Linear polariser aligned at 0° against Z axis — plus Quarter wave plate.

2. Blackbox
   The prepared eigenstates are altered through one of these Pauli transformations:
   (a) $\mathbb{1}$, no waveplate
   (b) σ_z, Half wave plate aligned at 45° against Z axis.,
   (c) σ_x Half wave plate aligned at 0° against Z axis.,
   (d) $\sigma_x\sigma_z$, Half wave plate aligned at 45°+Half wave plate aligned at 0° against Z axis.

3. Measurement
   Measurement is performed, by detecting photon capture, directly after one of these Pauli transformations:
   (a) σ_z, no waveplate
   (b) σ_x, Half wave plate aligned at 22.5° against Z axis.
   (c) σ_y, Quarter wave plate aligned at 45° against Z axis.

6 The Boolean representation of experiments

Paterek et al represent their experiment configurations, using *Boolean pairs* $(0, 1)$, $(1, 0)$, $(1, 1)$. Information held in these pairs is taken directly from the indices, in the product $\sigma_x^i\sigma_z^j$, where i and j are interpreted as integers, modulo 2. Thus:

$$\sigma_z = \sigma_x^0\sigma_z^1 \mapsto (0,1) \qquad \sigma_x = \sigma_x^1\sigma_z^0 \mapsto (1,0) \qquad -i\sigma_y = \sigma_x^1\sigma_z^1 \mapsto (1,1) \tag{7}$$

By way of these three mappings, Boolean pairs $(0, 1)$, $(1, 0)$, $(1, 1)$ are *linked* to the operators: σ_z, σ_x, σ_y, respectively. The action (configuration) of each individual segment: Preparation, Blackbox and Measurement, is represented by its own Boolean pair. Action of the Preparation is written thus:

$$\sigma_x^m\sigma_z^n \mapsto (m,n)$$

Action of the Blackbox is written thus:

$$\sigma_x^{\overset{\square}{f}(0)}\sigma_z^{\overset{\square}{f}(1)} \mapsto \left(\overset{\square}{f}(0)\,,\overset{\square}{f}(1)\right) \tag{8}$$

where $\overset{\square}{f}(0)$ and $\overset{\square}{f}(1)$ are the Boolean functions relating to propositions written in (6). And action of the Measurement is written thus:

$$\sigma_{\mathsf{x}}^{p}\sigma_{\mathsf{z}}^{q} \mapsto (p, q)$$

Variables p and q are not used by Paterek. I introduce them to keep reasoning surrounding Measurement and Preparation, clearly distinct.

By comparing the three mappings in (7) against functions in (8) we get three propositions, each Pauli operator uniquely specific to one:

$$\boxed{\sigma_{\mathsf{z}} \Rightarrow \overset{\square}{f}(0) = 0 \qquad \sigma_{\mathsf{x}} \Rightarrow \overset{\square}{f}(1) = 0 \qquad \sigma_{\mathsf{y}} \Rightarrow \overset{\square}{f}(0) + \overset{\square}{f}(1) = 0} \tag{9}$$

Remark The converse of implications in (9) would not be valid, because any of these three propositions could imply $\sigma_{\mathsf{x}}^{0}\sigma_{\mathsf{z}}^{0} = \mathbb{1}_2$ the unit operator.

Critically, depending on its Pauli configuration, the Blackbox sets precisely one of these formulae as an axiom. During the run of an experiment, the Blackbox writes its Boolean information, along with that axiom, onto the photon's density matrix. The density matrix is perfectly capable of holding all that information, *complete*. Subsequently, Measurement attempts to read that information, and depending on its own configuration, Measurement's reading will either agree or disagree with the Blackbox axiom — OR do neither.

7 Logical independence from the viewpoint of Boolean propositions

The Paterek paper is concerned with the *fact* of logical independence, and not the question of its origins. In this paper the direction is different; here, focus is on *tracing lines of dependency and implication,* flowing through experiments — with the aim of revealing the point where events depart from dependency and logical independence enters. That is of interest because, whatever 'anomaly' occurs at that specific point will shed light on the workings and machinery of indeterminacy.

This section charts the progress of logical dependence through the experiment hardware, in order to reveal the origin and generation of logical independence, wheresoever it may arise.

The flow of dependency is considered in two stages. ***Stage 1*** considers the ingress and egress of information passing through the Blackbox; this is the account from the Blackbox, viewpoint. ***Stage 2*** deals with the reading of that information, by the Measurement hardware; this is the account from the Measurement viewpoint. ***Stage 1 + Stage 2*** are shown schematically in Figure 1 and written out fully in Section 8.

Stage 1 Density matrix determined by Preparation and Blackbox.

From the perspective of the Blackbox, polarisation states from Preparation are seen either as superpositions, or as eigenstates, depending on the relative polarisation alignments of the Blackbox and Preparation. That relative alignment feeds into the density matrix as Boolean information, taken from the Preparation and Blackbox hardware configurations. The density matrix conveys that alignment information, and thus propagates, whether the state is a superposition, or an eigenstate.

On entry into the Blackbox, the *input* density matrix (from Preparation) is:

$$\rho_{\mathsf{P}} = \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} i^{mn} \sigma_{\mathsf{x}}^{m} \sigma_{\mathsf{z}}^{n}\right]$$

with $\lambda = \pm 1$. The suffix P stands for 'after Preparation'. Under the action of the Blackbox the density matrix evolves to:

$$U_{\mathsf{B}}\rho_{\mathsf{P}}\, U_{\mathsf{B}}^{\dagger} = \frac{1}{2}\left[\mathbb{1} + \lambda_{mn}\,(-1)^{n\overset{\square}{f}(0)+m\overset{\square}{f}(1)}\, i^{mn}\sigma_{\mathsf{x}}^{m}\sigma_{\mathsf{z}}^{n}\right] \tag{10}$$

The suffix B stands for 'after the Blackbox'. The index, on the factor $(-1)^{n\overset{\square}{f}(0)+m\overset{\square}{f}(1)}$, I call *orthogonality index* and give it the label $\mathscr{N}_{\mathsf{B}}$, thus:

$$\mathscr{N}_{\mathsf{B}} = n\overset{\square}{f}(0) + m\overset{\square}{f}(1)$$

When alignment is parallel, $\mathscr{N}_{\mathsf{B}} = 0$ and consequently $\rho = U\rho U^{\dagger}$ there is no evolved change in ρ.

Determined by the relative alignments of the Blackbox and Preparation, the value of $\mathscr{N}_{\mathsf{B}}$ shall be either 0 or 1. All sums are taken modulo 2. When

$$\mathscr{N}_{\mathsf{B}} = n\overset{\square}{f}(0) + m\overset{\square}{f}(1) = 0$$

zero orthogonality was imparted by the relative alignments, and downstream of the Blackbox, the density matrix will convey *eigenstate* information. When

$$\mathscr{N}_{\mathsf{B}} = n\overset{\square}{f}(0) + m\overset{\square}{f}(1) = 1$$

unit orthogonality was imparted by the relative alignments, and downstream of the Blackbox, the density matrix will convey *superposition* information.

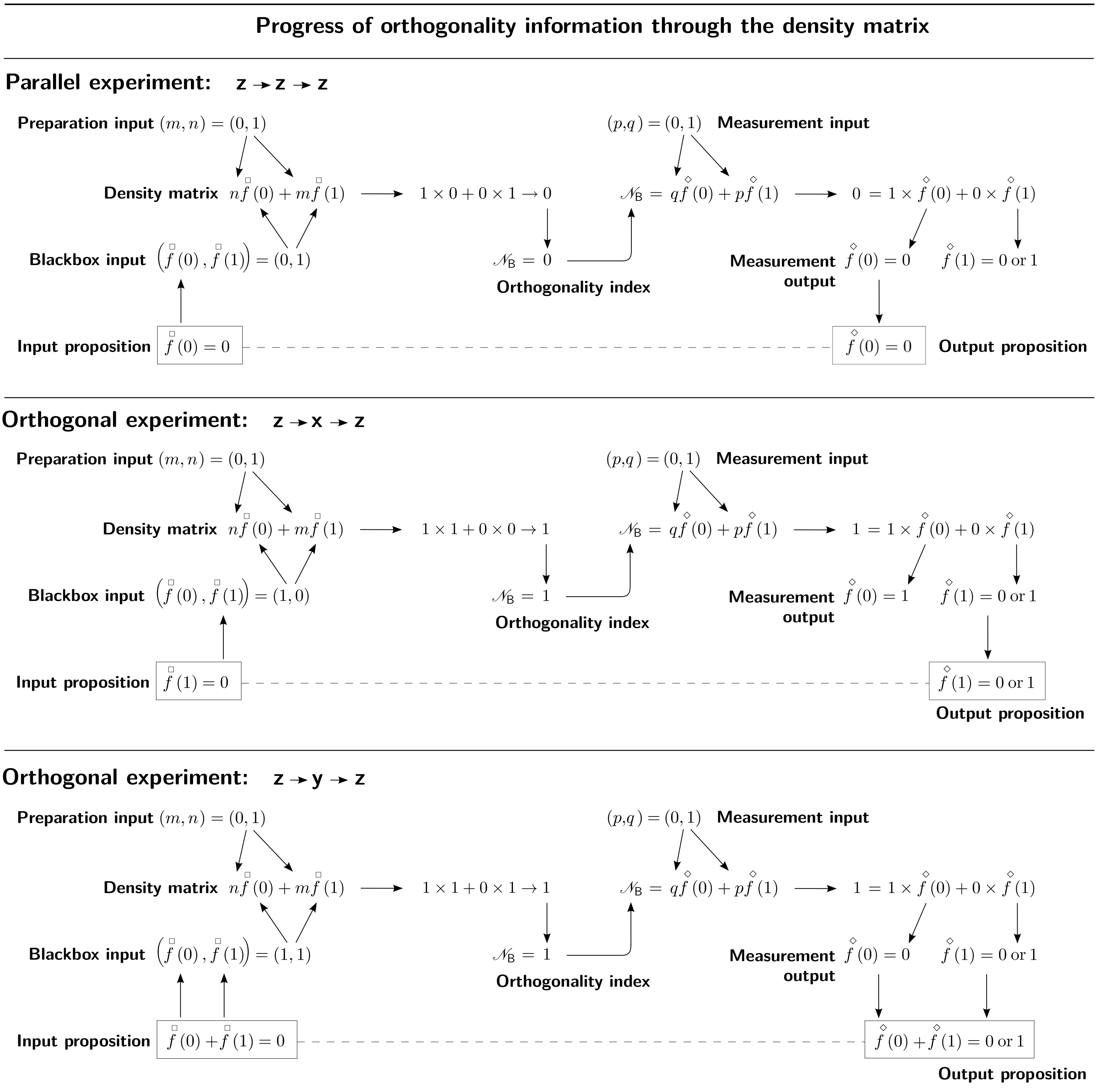

Figure 1 The Paterek research involves polarised photons as information carriers. The figure shows lines of dependency for three experiments. Orthogonality index $\mathscr{N}_{\mathsf{B}} = nf(0) + mf(1)$ is a Boolean quantity, that registers historical orthogonality as the density matrix evolves. The overall dependency or independency is seen by comparing the boxed propositions. The full computation is laid out in Section 8.

Leaving the Blackbox, $\mathscr{N}_\mathsf{B}$ has a definite, *deterministic* value, logically dependent on, and computed from (m,n) and $\left(\overset{\square}{f}(0), \overset{\square}{f}(1)\right)$. That determination might be thought of as a computation process where (m,n) and $\left(\overset{\square}{f}(0), \overset{\square}{f}(1)\right)$ are copied from the Preparation and Blackbox, then given as numerical input to $n\overset{\square}{f}(0)+m\overset{\square}{f}(1)$, from which $\mathscr{N}_\mathsf{B}$ is computed, as numerical output.

$$n\overset{\square}{f}(0) + m\overset{\square}{f}(1) \rightarrow \mathscr{N}_\mathsf{B} \tag{11}$$

Remark It is worth noting that orthogonality registered in the orthogonality index $\mathscr{N}_\mathsf{B}$ has no memory of how it was generated — for instance, whether orthogonality originated through the sequence $\sigma_\mathsf{x}\sigma_\mathsf{z}$, or the sequence $\sigma_\mathsf{y}\sigma_\mathsf{z}$.

Stage 2 Measurement attempts to read the Blackbox configuration.

Leaving the Blackbox, the definite, deterministic quantity $\mathscr{N}_\mathsf{B}$, continues its propagation through the experiment, to be read as input, into the Measurement hardware. Once the Measurement hardware knows the value $\mathscr{N}_\mathsf{B}$, given the Measurement alignment, set by:

$$\sigma_\mathsf{x}^p\sigma_\mathsf{z}^q \rightarrow (p,q)$$

the Measurement hardware attempts to compute $\overset{\diamond}{f}(0)$ and $\overset{\diamond}{f}(1)$. Thus:

$\left(\overset{\diamond}{f}(0), \overset{\diamond}{f}(1)\right) = (0,0)$ can never result from $\mathscr{N}_\mathsf{B} = 1$.

$$q\overset{\diamond}{f}(0) + p\overset{\diamond}{f}(1) \leftarrow \mathscr{N}_\mathsf{B} \tag{12}$$

However, $\overset{\diamond}{f}(0)$ and $\overset{\diamond}{f}(1)$ are not *both* determinable from $\mathscr{N}_\mathsf{B}$ and (p,q). In order to determine one, the other must be known. The upshot is that processes performed in (11) — (12), $\overset{\diamond}{f}(0)$ and $\overset{\diamond}{f}(1)$ are not *both* computable nor determinable from $\overset{\square}{f}(0)$ and $\overset{\square}{f}(1)$.

8 Algebraic processes in the density matrix

This section sets out, in detail, algebraic processes which go on from the point where the Blackbox writes the $\overset{\square}{f}(0)$ and $\overset{\square}{f}(1)$ information, upto the point where Measurement reads $\overset{\diamond}{f}(0)$ and $\overset{\diamond}{f}(1)$. Manipulations involve the exchange of operators to produce commutators. Section 8.1 shows the Blackbox processes (13)→(15) where the commutators accumulate in the orthogonality index $\mathscr{N}_\mathsf{B}$. Section 8.2 shows the Measurement processes (16)→(18) where 'reverse' algebraic processes 'un-exchange' the operators, so as to redistribute the commutator information held in the orthogonality index.

The overall process (13)→(18) is the conversion of (orthogonal) geometric information into scalar information, then the reversal of that. This reversal results in ambiguity because the geometry–scalar relationship is not one–one.

8.1 *The* Blackbox *determines the density matrix* ρ_{B} *(Stage 1)*

The Blackbox, takes $\overset{\square}{f}(0)$ and $\overset{\square}{f}(1)$ as input, then algebraic processes determine a definite value for the orthogonality index $\mathscr{N}_{\mathsf{B}}$, and hence, a definite density matrix ρ_{B}, at (15).

The underlining is intended to point out where the action is taking place.

$$\begin{aligned}
\rho_{\mathsf{B}} = U_{\mathsf{B}}\rho_{\mathsf{P}}\, U_{\mathsf{B}}^{\dagger} &= \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} i^{mn}\left[\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\sigma_{\mathsf{z}}^{\overset{\square}{f}(1)}\right]\left[\sigma_{\mathsf{x}}^{m}\sigma_{\mathsf{z}}^{n}\right]\left[\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\sigma_{\mathsf{z}}^{\overset{\square}{f}(1)}\right]^{\dagger}\right] && (13)\\
&= \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} i^{mn}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\sigma_{\mathsf{z}}^{\overset{\square}{f}(1)}\sigma_{\mathsf{x}}^{m}\underline{\sigma_{\mathsf{z}}^{n}\sigma_{\mathsf{z}}^{\overset{\square}{f}(1)}}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\right]\\
&= \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} i^{mn}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\sigma_{\mathsf{z}}^{\overset{\square}{f}(1)}\sigma_{\mathsf{x}}^{m}\underline{\sigma_{\mathsf{z}}^{\overset{\square}{f}(1)}\sigma_{\mathsf{z}}^{n}}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\right]\\
&= \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} i^{mn}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\sigma_{\mathsf{z}}^{\overset{\square}{f}(1)}\underline{\sigma_{\mathsf{x}}^{m}\sigma_{\mathsf{z}}^{\overset{\square}{f}(1)}}\sigma_{\mathsf{z}}^{n}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\right]\\
&= \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} \underline{(-1)^{m\overset{\square}{f}(1)}}\, i^{mn}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\sigma_{\mathsf{z}}^{\overset{\square}{f}(1)}\underline{\sigma_{\mathsf{z}}^{\overset{\square}{f}(1)}\sigma_{\mathsf{x}}^{m}}\sigma_{\mathsf{z}}^{n}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\right]\\
&= \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} (-1)^{m\overset{\square}{f}(1)}\, i^{mn}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\underline{\sigma_{\mathsf{z}}^{\overset{\square}{f}(1)}\sigma_{\mathsf{z}}^{\overset{\square}{f}(1)}}\sigma_{\mathsf{x}}^{m}\sigma_{\mathsf{z}}^{n}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\right]\\
&= \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} (-1)^{m\overset{\square}{f}(1)}\, i^{mn}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\,\underline{\mathbb{1}}\,\sigma_{\mathsf{x}}^{m}\sigma_{\mathsf{z}}^{n}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\right]\\
&= \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} (-1)^{m\overset{\square}{f}(1)}\, i^{mn}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\sigma_{\mathsf{x}}^{m}\underline{\sigma_{\mathsf{z}}^{n}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}}\right]\\
&= \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} (-1)^{m\overset{\square}{f}(1)}\, \underline{(-1)^{n\overset{\square}{f}(0)}}\, i^{mn}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\sigma_{\mathsf{x}}^{m}\underline{\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\sigma_{\mathsf{z}}^{n}}\right]\\
&= \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} (-1)^{n\overset{\square}{f}(0)+m\overset{\square}{f}(1)}\, i^{mn}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\underline{\sigma_{\mathsf{x}}^{m}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}}\sigma_{\mathsf{z}}^{n}\right]\\
&= \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} (-1)^{n\overset{\square}{f}(0)+m\overset{\square}{f}(1)}\, i^{mn}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\underline{\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\sigma_{\mathsf{x}}^{m}}\sigma_{\mathsf{z}}^{n}\right]\\
&= \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} (-1)^{n\overset{\square}{f}(0)+m\overset{\square}{f}(1)}\, i^{mn}\underline{\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}\sigma_{\mathsf{x}}^{\overset{\square}{f}(0)}}\sigma_{\mathsf{x}}^{m}\sigma_{\mathsf{z}}^{n}\right]\\
&= \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} (-1)^{n\overset{\square}{f}(0)+m\overset{\square}{f}(1)}\, i^{mn}\,\underline{\mathbb{1}}\,\sigma_{\mathsf{x}}^{m}\sigma_{\mathsf{z}}^{n}\right]\\
&= \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} (-1)^{n\overset{\square}{f}(0)+m\overset{\square}{f}(1)}\, i^{mn}\sigma_{\mathsf{x}}^{m}\sigma_{\mathsf{z}}^{n}\right] && (14)\\
\rho_{\mathsf{B}} &= \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} (-1)^{\mathscr{N}_{\mathsf{B}}}\, i^{mn}\sigma_{\mathsf{x}}^{m}\sigma_{\mathsf{z}}^{n}\right] && (15)
\end{aligned}$$

At this point there is no memory of $\overset{\square}{f}(0)$ & $\overset{\square}{f}(1)$ and they are lost.

8.2 Measurement *receives and processes the density matrix* ρ_{B} *(Stage 2)*

Measurement now takes the definite density matrix ρ_{B} from (15), as input, along with its orthogonality index $\mathscr{N}_{\mathsf{B}}$. From that, Measurement attempts to compute polarisation information from the density matrix's history. That means working backward through the reverse of the Blackbox. processes (13)→(15).

Upto this point, everything has been deterministic. But the first step into the reverse process is not. In the step (16)→(17), definite values for *both* $\overset{\square}{f}(0)$ and $\overset{\square}{f}(1)$

cannot be recovered because the step (14)→(15) is not reversible. It is interesting that, because of this 'impasse', Measurement processes do not stop here. The reason they do not is that, *existence* of variables is demanded by the logic — according to:

$$\forall \mathcal{N}_{\mathsf{B}} \forall p \forall q \exists \overset{\diamond}{f}(0)\, \exists \overset{\diamond}{f}(1) \quad | \quad q\overset{\diamond}{f}(0) + p\overset{\diamond}{f}(1) = \mathcal{N}_{\mathsf{B}}$$

and the ambiguous versions $\overset{\diamond}{f}(0)$ and $\overset{\diamond}{f}(1)$ are caused to enter in (17). Throughout this whole process, that is the extent of the anomaly, but its effect trickles through the remaining processes (17)→(18).

$$\rho_{\mathsf{B}} = \frac{1}{2}\left[\mathbb{1} + \lambda_{pq}\,(-1)^{\mathcal{N}_{\mathsf{B}}}\, i^{pq}\sigma_{\mathsf{x}}^{p}\sigma_{\mathsf{z}}^{q}\right] \tag{16}$$

$$= \frac{1}{2}\left[\mathbb{1} + \lambda_{pq}\,(-1)^{q\overset{\diamond}{f}(0)+p\overset{\diamond}{f}(1)}\, i^{pq}\sigma_{\mathsf{x}}^{p}\sigma_{\mathsf{z}}^{q}\right] \tag{17}$$

$$= \frac{1}{2}\left[\mathbb{1} + \lambda_{pq}\,(-1)^{q\overset{\diamond}{f}(0)+p\overset{\diamond}{f}(1)}\, i^{pq}\underbrace{\mathbb{1}}\,\sigma_{\mathsf{x}}^{p}\sigma_{\mathsf{z}}^{q}\right]$$

$$= \frac{1}{2}\left[\mathbb{1} + \lambda_{pq}\,(-1)^{q\overset{\diamond}{f}(0)+p\overset{\diamond}{f}(1)}\, i^{pq}\underbrace{\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}}\sigma_{\mathsf{x}}^{p}\sigma_{\mathsf{z}}^{q}\right]$$

$$= \frac{1}{2}\left[\mathbb{1} + \lambda_{pq}\,(-1)^{q\overset{\diamond}{f}(0)+p\overset{\diamond}{f}(1)}\, i^{pq}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\underbrace{\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\sigma_{\mathsf{x}}^{p}}\sigma_{\mathsf{z}}^{q}\right]$$

$$= \frac{1}{2}\left[\mathbb{1} + \lambda_{pq}\,(-1)^{q\overset{\diamond}{f}(0)+p\overset{\diamond}{f}(1)}\, i^{pq}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\underbrace{\sigma_{\mathsf{x}}^{p}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}}\sigma_{\mathsf{z}}^{q}\right]$$

$$= \frac{1}{2}\left[\mathbb{1} + \lambda_{pq}\,\underbrace{(-1)^{q\overset{\diamond}{f}(0)}}\,(-1)^{p\overset{\diamond}{f}(1)}\, i^{pq}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\sigma_{\mathsf{x}}^{p}\underbrace{\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\sigma_{\mathsf{z}}^{q}}\right]$$

$$= \frac{1}{2}\left[\mathbb{1} + \lambda_{pq}\,(-1)^{p\overset{\diamond}{f}(1)}\, i^{pq}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\sigma_{\mathsf{x}}^{p}\underbrace{\sigma_{\mathsf{z}}^{q}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}}\right]$$

$$= \frac{1}{2}\left[\mathbb{1} + \lambda_{pq}\,(-1)^{p\overset{\diamond}{f}(1)}\, i^{pq}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\underbrace{\mathbb{1}}\,\sigma_{\mathsf{x}}^{p}\sigma_{\mathsf{z}}^{q}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\right]$$

$$= \frac{1}{2}\left[\mathbb{1} + \lambda_{pq}\,(-1)^{p\overset{\diamond}{f}(1)}\, i^{pq}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\underbrace{\sigma_{\mathsf{z}}^{\overset{\diamond}{f}(1)}\sigma_{\mathsf{z}}^{\overset{\diamond}{f}(1)}}\sigma_{\mathsf{x}}^{p}\sigma_{\mathsf{z}}^{q}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\right]$$

$$= \frac{1}{2}\left[\mathbb{1} + \lambda_{pq}\,\underbrace{(-1)^{p\overset{\diamond}{f}(1)}}\, i^{pq}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\sigma_{\mathsf{z}}^{\overset{\diamond}{f}(1)}\underbrace{\sigma_{\mathsf{z}}^{\overset{\diamond}{f}(1)}\sigma_{\mathsf{x}}^{p}}\sigma_{\mathsf{z}}^{q}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\right]$$

$$= \frac{1}{2}\left[\mathbb{1} + \lambda_{pq}i^{pq}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\sigma_{\mathsf{z}}^{\overset{\diamond}{f}(1)}\underbrace{\sigma_{\mathsf{x}}^{p}\sigma_{\mathsf{z}}^{\overset{\diamond}{f}(1)}}\sigma_{\mathsf{z}}^{q}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\right]$$

$$= \frac{1}{2}\left[\mathbb{1} + \lambda_{pq}i^{pq}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\sigma_{\mathsf{z}}^{\overset{\diamond}{f}(1)}\sigma_{\mathsf{x}}^{p}\underbrace{\sigma_{\mathsf{z}}^{\overset{\diamond}{f}(1)}\sigma_{\mathsf{z}}^{q}}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\right]$$

$$= \frac{1}{2}\left[\mathbb{1} + \lambda_{pq}i^{pq}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\sigma_{\mathsf{z}}^{\overset{\diamond}{f}(1)}\sigma_{\mathsf{x}}^{p}\underbrace{\sigma_{\mathsf{z}}^{q}\sigma_{\mathsf{z}}^{\overset{\diamond}{f}(1)}}\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\right]$$

$$= \frac{1}{2}\left[\mathbb{1} + \lambda_{pq}i^{pq}\left[\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\sigma_{\mathsf{z}}^{\overset{\diamond}{f}(1)}\right]\left[\sigma_{\mathsf{x}}^{p}\sigma_{\mathsf{z}}^{q}\right]\left[\sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\sigma_{\mathsf{z}}^{\overset{\diamond}{f}(1)}\right]^{\dagger}\right]$$

$$= \overset{\diamond}{U}_{\mathsf{B}}\rho_{\mathsf{p}}\overset{\diamond}{U}_{\mathsf{B}}^{\dagger} \tag{18}$$

That is to say, from Measurement's viewpoint, evolution of the density matrix, due to the Blackbox is thus:

$$\rho_{\mathsf{p}} \longrightarrow \rho_{\mathsf{B}} = \overset{\diamond}{U}_{\mathsf{B}}\rho_{\mathsf{p}}\overset{\diamond}{U}_{\mathsf{B}}^{\dagger} \qquad \text{with} \qquad \overset{\diamond}{U}_{\mathsf{B}} = \sigma_{\mathsf{x}}^{\overset{\diamond}{f}(0)}\sigma_{\mathsf{z}}^{\overset{\diamond}{f}(1)}$$

In cases when $\mathcal{N}_{\mathsf{B}} = 1$, $\overset{\diamond}{U}_{\mathsf{B}}$ is indefinite. This means that Measurement is unable 'know' the alignments of polarisers upstream in the experiment. As illustration, referring to experiments depicted in Figure 1, a Blackbox alignment of σ_{x} (or σ_{y}) would be understood ambiguously by Measurement, either as σ_{x}, or as σ_{y}.

This $\sigma_{\mathsf{x}}|\,\sigma_{\mathsf{y}}$ ambiguity is isomorphic to $+|-$ ambiguity due to left|right handedness.

However, in cases when $\mathcal{N}_{\mathsf{B}} = 0$, in none of the Blackbox and Measurement algebraic processes, does any operator exchanging occur, and no commutators are produced

, because involutory cancellations occur instead. Referring to experiments depicted in Figure 1, a Blackbox alignment of σ_{z} would be understood by Measurement as σ_{z}, or that no polariser was in place, in the Blackbox, at all.

9 Aside: Information content of the Pauli algebra

For the sake of comparison with the above, it is instructive to consider a list of six statements which can be regarded as elements of information within the Pauli algebraic statement:

$$-i\mathsf{b} = \mathsf{ac} \tag{19}$$

The list consists of the six statements (25) — (30). It is capable of deriving (19) but is by no means unique. As a matter of interest, these six statements fail to be logically independent of one another, and so contain some degree of redundancy. Indeed, from the list, derivation of (43) above uses only (25), (27) and (30), and additionally, *closure* for products ac within the algebra.

The procedure of derivation following the list is an adaption of a proof given by W E Baylis, J Huschilt and Jiansu Wei [3], however, may originate with David Hestenes [16].

The Pauli algebra is a Lie algebra; and hence, is a linear vector space. Therefore, I begin with information inherited from the vector space axioms, and then add other information peculiar to the Pauli Lie algebra, $\mathsf{su}(2)$.

Closure: For any two vectors u and v, there exists a vector w such that

$$\mathsf{w} = \mathsf{u} + \mathsf{v}$$

Identities: There exist additive and multiplicative identities, $\mathbb{0}$ and $\mathbb{1}$. For any arbitrary vector v:

$$\mathsf{v}\mathbb{1} = \mathbb{1}\mathsf{v} = \mathsf{v} \tag{20}$$

$$\mathsf{v} + \mathbb{0} = \mathbb{0} + \mathsf{v} = \mathsf{v} \tag{21}$$

$$\mathsf{v}\mathbb{0} = \mathbb{0}\mathsf{v} = \mathbb{0} \tag{22}$$

Additive inverse: For any arbitrary vector v, there exists an additive inverse $-\mathsf{v}$ such that

$$(-\mathsf{v}) + \mathsf{v} = \mathbb{0} \tag{23}$$

Scaling: For any arbitrary vector v, and any scalar a, there exists a vector u such that

$$\mathsf{u} = a\mathsf{v} \tag{24}$$

Products: A feature of Lie algebras is that, between any two arbitrary vectors, u and v, there exist products uv and vu. Commutators of these products (Lie brackets) are members of the vector space.

Dimension: Assume a 3 dimensional vector space, with independent basis a, b, c.

The list of six statements

Involutory information: Assume all three basis vectors are involutory. Thus:

$$\mathsf{aa} = \mathbb{1} \qquad \mathsf{a} \text{ involutory} \tag{25}$$

$$\mathsf{bb} = \mathbb{1} \qquad \mathsf{b} \text{ involutory} \tag{26}$$

$$\mathsf{cc} = \mathbb{1} \qquad \mathsf{c} \text{ involutory} \tag{27}$$

Orthogonal information: Assume products between basis vectors are orthogonal. Thus:

$$\mathsf{ab} + \mathsf{ba} = \mathbb{0} \qquad \mathsf{ab} \text{ orthogonal} \tag{28}$$

$$\mathsf{bc} + \mathsf{cb} = \mathbb{0} \qquad \mathsf{bc} \text{ orthogonal} \tag{29}$$

$$\mathsf{ca} + \mathsf{ac} = \mathbb{0} \qquad \mathsf{ca} \text{ orthogonal} \tag{30}$$

Bringing items of information together, the Pauli algebra is constructed thus:

$$\begin{aligned} \mathsf{bc} + \mathsf{cb} &= \mathbb{0} && \text{by (29)} \quad , \quad \mathsf{bc} \text{ orthogonal} \\ \mathsf{b} + \mathsf{cbc} &= \mathbb{0} && \text{by (27)} \quad , \quad \mathsf{c} \text{ involutory} \\ \mathsf{ba} + \mathsf{cbca} &= \mathbb{0} && \text{by (22)} \end{aligned} \tag{31}$$

And similarly:

$$\begin{aligned} \mathsf{ca} + \mathsf{ac} &= \mathbb{0} && \text{by (30)} \quad , \quad \mathsf{ca} \text{ orthogonal} \\ \mathsf{cac} + \mathsf{a} &= \mathbb{0} && \text{by (27)} \quad , \quad \mathsf{c} \text{ involutory} \\ \mathsf{cacb} + \mathsf{ab} &= \mathbb{0} && \text{by (22)} \end{aligned} \tag{32}$$

Adding (32) and (31) gives:

$$\begin{aligned} \mathsf{cacb} + \mathsf{ab} + \mathsf{ba} + \mathsf{cbca} &= \mathbb{0} \\ \mathsf{cacb} + \mathsf{cbca} &= \mathbb{0} && \text{by (28)} \quad , \quad \mathsf{ab} \text{ orthogonal} \\ \mathsf{acb} + \mathsf{bca} &= \mathbb{0} && \text{by (27)} \quad , \quad \mathsf{c} \text{ involutory} \\ \mathsf{acba} + \mathsf{bc} &= \mathbb{0} && \text{by (25)} \quad , \quad \mathsf{a} \text{ involutory} \\ \mathsf{acbac} + \mathsf{b} &= \mathbb{0} && \text{by (27)} \quad , \quad \mathsf{c} \text{ involutory} \\ \mathsf{acbacb} + \mathbb{1} &= \mathbb{0} && \text{by (26)} \quad , \quad \mathsf{b} \text{ involutory} \\ (\mathsf{acb})^2 &= -\mathbb{1} && \text{by (23)} \\ (\mathsf{acb})^2 &= (-1)\,\mathbb{1} \\ \mathsf{acb} &= \pm i\mathbb{1} \\ \mathsf{ac} &= \pm i\mathsf{b} && \text{by (26)} \quad , \quad \mathsf{b} \text{ involutory} \end{aligned} \tag{33}$$

And a couple of extra steps gives the Pauli algebra:

$$\mathsf{ca} = \mp i\mathsf{b} \qquad \text{by (33)} \quad , \quad \mathsf{a}, \mathsf{b}, \mathsf{c} \text{ involutory} \tag{34}$$

$$\mathsf{ac} - \mathsf{ca} = \pm 2i\mathsf{b} \qquad \text{by (33) \& (34)} \tag{35}$$

10 Logical independence from the viewpoint of symmetry

Readers of the Paterek paper might infer that (7) suggests there is a one – one correspondence linking the Pauli products with Boolean pairs. The actual picture is one – way. Implication is only directed from the Pauli products, to the Boolean pairs, in the sense of the arrows shown here:

$$\sigma_{\mathsf{z}} = \sigma_{\mathsf{x}}^0 \sigma_{\mathsf{z}}^1 \longrightarrow (0,1) \qquad \sigma_{\mathsf{x}} = \sigma_{\mathsf{x}}^1 \sigma_{\mathsf{z}}^0 \longrightarrow (1,0) \qquad -i\sigma_{\mathsf{y}} = \sigma_{\mathsf{x}}^1 \sigma_{\mathsf{z}}^1 \longrightarrow (1,1) \tag{36}$$

If the Pauli system were to connect logically, one – one, with the Boolean system, we would witness a backwards implication, also, in the sense of these reverse arrows:

$$\sigma_{\mathsf{z}} = \sigma_{\mathsf{x}}^0 \sigma_{\mathsf{z}}^1 \longleftarrow (0,1) \qquad \sigma_{\mathsf{x}} = \sigma_{\mathsf{x}}^1 \sigma_{\mathsf{z}}^0 \longleftarrow (1,0) \qquad -i\sigma_{\mathsf{y}} = \sigma_{\mathsf{x}}^1 \sigma_{\mathsf{z}}^1 \longleftarrow (1,1) \tag{37}$$

But, as they stand, the formulae in (37) are invalid. Generally, the Boolean pairs do not imply the Pauli operators. They invoke operators that are not necessarily Paulian; they invoke operators belonging to some wider system. And they do not form a Lie algebra. The Pauli operators are merely the special case that happens to be unitary.

If the backwards implication is to be insisted upon we must accept certain freedoms for the operators showing in (37), that *would* maintain backwards validity.

The situation is made clearer when all Pauli notation is dropped and replaced by abstract symbols a, c, b. Formulae can then be seen for the information they *assert*, rather than content we *presume,* which stems from meaning we place on the symbols they contain.

Stating what (37) actually asserts:

$$\left[\forall \mathsf{c} \,|\, \mathsf{c} = \mathsf{a}^0\mathsf{c}^1\right] \leftarrow (0,1) \quad \left[\forall \mathsf{a} \,|\, \mathsf{a} = \mathsf{a}^1\mathsf{c}^0\right] \leftarrow (1,0) \quad \left[\forall \eta \exists \mathsf{b} \,|\, \eta^{-1}\mathsf{b} = \mathsf{a}^1\mathsf{c}^1\right] \leftarrow (1,1) \tag{38}$$

where a, c, b are linear operators and η, a scalar. Together these formulae assert $\mathsf{aa} = \mathsf{cc} = \mathbb{1}$, and that there is *algebraic closure* for the product ac; that is, ac is an element in the same algebra. The scalar η arises because there is no assurance that any particular value scalar is asserted here, let alone the value i; we can always extract an arbitrary factor from a linear operator b, without loss of validity. The inverse η^{-1} is chosen, purely for convenience, later in the text.

Written less formally, formulae asserted by the Boolean pairs are these:

$$\mathbb{1} = \mathsf{a}^0\mathsf{c}^0 \qquad \mathsf{c} = \mathsf{a}^0\mathsf{c}^1 \qquad \mathsf{a} = \mathsf{a}^1\mathsf{c}^0 \qquad \eta^{-1}\mathsf{b} = \mathsf{a}^1\mathsf{c}^1 \tag{39}$$

The first of these corresponds to the Boolean pair (0,0) which is never invoked in the forward sense of (36), but in the backward sense, I prefer not to ignore.

I now shall calculate what further information is needed if the Boolean pairs are to invoke the Pauli operators σ_x, σ_y, σ_z, rather than operators a, b, c.

Now, because inverses of a and c are guaranteed, I can use the standard result:

$$(\mathsf{ac})^{-1} \equiv \mathsf{c}^{-1}\mathsf{a}^{-1} \tag{40}$$

Taking $\eta^{-1}\mathsf{b} = \mathsf{a}^1\mathsf{c}^1$ from (38) and applying (40) gives:

$$\eta\mathsf{b}^{-1} = \mathsf{ca} \tag{41}$$

Now taking $\eta^{-1}\mathsf{b} = \mathsf{a}^1\mathsf{c}^1$ again and adding to (41), we get:

$$\eta^{-1}\mathsf{b} + \eta\mathsf{b}^{-1} = \mathsf{ac} + \mathsf{ca}$$

Now imposing the further new constraint:

$$\boxed{\mathsf{ac} + \mathsf{ca} = \mathbb{0}} \tag{42}$$

whose inclusion then implies Pauli:

$$\begin{aligned} \eta\mathsf{b} &= -\eta^{-1}\mathsf{b}^{-1} \\ \eta^2\mathsf{b}^2 &= (-1)\,\mathbb{1} \end{aligned} \tag{43}$$

with the result:

$$\eta = \pm i \qquad \mathsf{b}^2 = \mathbb{1}$$

11 Logical independence from the viewpoint of self-reference

An orthogonal vector space can be thought of as a composite of information — consisting of — information comprising a general vector space, plus information that renders that space orthogonal. More formally we might think of axioms imposing rules for a general vector space with additional axioms imposing orthogonality. That information of orthogonality need not originate in axioms or definitions; it can derive from *self-reference* or *logical circularity* [25].

The same should apply to orthogonal tensors.

Self-reference, as generator for orthogonality, has profound implications for the logical standing of Hilbert spaces: in particular – the logical standing of eigenstates in relation to superposition states. This self-reference takes place at the transition interface between these, and is the source origin logical independence in quantum systems.

Within Elementary Algebra, self-referential statements can express Linear Algebraic information, which would normally be asserted as axioms belonging to Linear Algebra. Thus, this self-reference moves Linear Algebra into the arena of Elementary Algebra – meaning that – Linear Algebra for quantum theory, is expressible as a single logio–algebraic system, rather than a composite amalgamation of Elementary Algebra plus Linear Algebra. And so, instead of information, normally expressed as definitions from Linear Algebra, equivalent information is expressed as self-reference in Elementary Algebra. And, instead of the usual demarcation of definitions and axioms, that separates the two algebras, there is now *logic* that interfaces them: wholly within Elementary Algebra. Thus, the whole information of the Hilbert space is expressed as a single integrated algebraic system — with logical structure *within*, that replaces definitions that were from *without*.

Matrices acting on vectors are notation for sets of simultaneous equations, within Elementary Algebra.

In the case of Pauli systems, before the self-reference may proceed, a triplet of non-orthogonal vector spaces, with no inner products (Banach spaces), forms up into a closed system. The self-reference consists of the passing of information, from

In momentum-position wave mechanics, a dual-pair of spaces forms into a closed system. The reason this is *dual* rather than a *triplet* is that the system algebra:

$$[\mathsf{p}, \mathsf{x}] = -i\mathbb{1}$$

each matrix's vector space to the next, in complete cycles, both cyclically and anti-cyclically. This restricts the system to unitary, Hermitian operators and Hilbert space vectors, because no other systems survive as stable.[14].

The self-reference is possible because all its component statements are *logically independent* of axioms; so no information in the system opposes them. Inherent in the self-reference is the implicit necessity for the imaginary unit. This acts as a marker, confirming logical independence [14].

In the derivations that follow, I begin with the 3 formulae, implied by the Paterek formalism, copied from (39), then adopt a faithful matrix representation of these, and then perform the self-reference. I start with the 3 formulae,

$$\mathsf{a}^2 = \mathbb{1} \qquad \mathsf{c}^2 = \mathbb{1} \qquad \eta^{-1}\mathsf{b} = \mathsf{ac} \tag{44}$$

Note that these correspond to (25) and (27) from the **list of six statements**, plus the statement of closure: $\eta^{-1}\mathsf{b} = \mathsf{ac}$. I emphasise: (44) includes no information corresponding to the statements (26), (28), (29), (30).

Now write down matrices that faithfully represent the algebraic system (44):

$$\mathsf{a}(\eta) = \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} \qquad \mathsf{b}(\eta) = \begin{pmatrix} 1 & 0 \\ 0 & \eta^2 \end{pmatrix} \qquad \mathsf{c} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{45}$$

My reason for choosing $\begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix}$ in preference to $\begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix}$, is to maintain η as a bounded variable. Whereas the former matrix is universally involutory under the quantifier $\forall\eta$, in the case of the latter, an involutory condition imposes the value $\eta^2 = -1$, so also imposes unitarity. Crucially, the former permits involutarity without also being unitary.

In the proof (39) − (43), at the end of Section 10, I showed that $\mathsf{ac} + \mathsf{ca} = \mathbb{0}$ is sufficient to restrict (39) [and (44)] to the Pauli operators. In the self-referential processes below, comparable information restricts the matrices (45), to the Pauli matrices — upto left|right ambiguity:

$$\mathsf{a} = \pm\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{b} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathsf{c} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{46}$$

11.1 *The self-referential processes*

Starting with the three matrices of (45), I begin by writing the most general arbitrary transformation of which each of these matrices is capable.

The self-reference is effectively the same as a statement of ac+ca=0, but without the logic.

$$\forall\eta\,\forall\alpha_1\forall\alpha_2\exists\psi_1\exists\psi_2 \;\Big|\; \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{47}$$

$$\forall\eta\,\forall\beta_1\forall\beta_2\exists\phi_1\exists\phi_2 \;\Big|\; \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \eta^2 \end{pmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \tag{48}$$

$$\forall\gamma_1\forall\gamma_2\exists\chi_1\exists\chi_2 \;\Big|\; \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \tag{49}$$

Note that these formulae do not assert equality, they assert existence. I now explore the possibility of (47), (48) and (49) accepting information, circularly, from one another, through a 'forward' cyclic mechanism where:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \text{feeds off} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \qquad \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \text{feeds off} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \qquad \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \text{feeds off} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{50}$$

and a 'backward' mechanism where:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \text{feeds off} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \qquad \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \text{feeds off} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \qquad \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \text{feeds off} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \tag{51}$$

These form closed, self-referential flows of information. There is no *cause* implying this self-reference; the idea is that no information in the system environment prevents it or contradicts it.

To proceed with the derivation, the strategy followed will be to make a formal assumption, by positing the hypothesis that such self-reference does occur; then investigate for conditionality implied. To properly document this assumption, the hypothesis is formally declared, thus:

Part One
Hypothesised forward coincidence:

$$\forall\phi_1\forall\phi_2\exists\alpha_1\exists\alpha_2 \;\Big|\; \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = (+1)\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \tag{52}$$

$$\forall\chi_1\forall\chi_2\exists\beta_1\exists\beta_2 \;\Big|\; \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} = (+1)\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} \tag{53}$$

$$\forall\psi_1\forall\psi_2\exists\gamma_1\exists\gamma c \;\Big|\; \begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix} = (+1)\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \tag{54}$$

Substitution involving quantifiers

$$\begin{aligned} \forall\beta\forall\underline{\gamma}\exists\alpha \;&|\; \alpha = \beta + \gamma \\ \forall\lambda\underline{\exists\gamma} \;&|\; \gamma = 2\lambda \\ \Rightarrow\; \forall\lambda\forall\beta\exists\alpha \;&|\; \alpha = \beta + 2\lambda \end{aligned}$$

An *existential* quantifier of one proposition is matched with a *universal* quantifier of the other. Those matched are underlined.

Note: there is no guarantee that any such coincidence should exist. We proceed to investigate.. In this block of manipulations, I begin with the transformation (47), then repeatedly make substitutions, cyclically.

$$\forall\eta\forall\beta_1\forall\beta_2\exists\phi_1\exists\phi_2 \;\Big|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \qquad \text{by (48)}$$

$$\forall\eta\forall\chi_1\forall\chi_2\exists\phi_1\exists\phi_2 \;\Big|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}(+1)\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} \qquad \text{by (53)}$$

$$\forall\eta\forall\gamma_1\forall\gamma_2\exists\phi_1\exists\phi_2 \;\Big|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}(+1)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix} \qquad \text{by (49)}$$

$$\forall\eta\forall\psi_1\forall\psi_2\exists\phi_1\exists\phi_2 \;\Big|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}(+1)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(+1)\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \qquad \text{by (54)}$$

$$\forall\eta\forall\alpha_1\forall\alpha_2\exists\phi_1\exists\phi_2 \;\Big|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}(+1)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(+1)\begin{pmatrix}0 & \eta^{-1}\\ \eta & 0\end{pmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} \qquad \text{by (47)}$$

$$\forall\eta\forall\phi_1\forall\phi_2\exists\phi_1\exists\phi_2 \;\Big|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}(+1)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(+1)\begin{pmatrix}0 & \eta^{-1}\\ \eta & 0\end{pmatrix}(+1)\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \qquad \text{by (52)}$$

In summary, assuming the **Hypothesised forward coincidence**, the overall result is the assertion:

$$\forall\eta\forall\phi_1\forall\phi_2\exists\phi_1\exists\phi_2 \;\Big|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \eta^{-1}\\ \eta & 0\end{pmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \tag{55}$$

The seemingly ambiguous quantification $\forall\phi_1\forall\phi_2\exists\phi_1\exists\phi_2$ indicates a *possible* $\forall\phi_1\forall\phi_2$, but *guaranteed* $\exists\phi_1\exists\phi_2$. Either way, (55) implies the following:

$$\begin{aligned} &\forall\eta \;\;|\;\; \begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \eta^{-1}\\ \eta & 0\end{pmatrix} = \mathbb{1}_2 \\ \Longrightarrow\;\; &\forall\eta \;\;|\;\; \begin{pmatrix}\eta & 0\\ 0 & \eta\end{pmatrix} = \mathbb{1}_2 \end{aligned} \tag{56}$$

The assertion (56) is self-contradictory, because the matrix cannot equal the identity for all values of η. This confirms there is something invalid about the **Hypothesised forward coincidence.** Nevertheless, it is important to retain the full information of (56) if valid conditionality is to be later revealed.

Part Two
Hypothesised backward coincidence:

$$\forall\chi_1\forall\chi_2\exists\alpha_1\exists\alpha_2 \;\Big|\; \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = (-1)\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} \tag{57}$$

$$\forall\psi_1\forall\psi_2\exists\beta_1\exists\beta_2 \;\Big|\; \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} = (-1)\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \tag{58}$$

$$\forall\phi_1\forall\phi_2\exists\gamma_1\exists\gamma c \;\Big|\; \begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix} = (-1)\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \tag{59}$$

Note: there is no guarantee that any such coincidence should exist. We proceed to investigate.. In this block of manipulations, I begin with the transformation (47), then repeatedly make substitutions, cyclically.

$$\forall\eta\,\forall\beta_1\forall\beta_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \qquad \text{by (48)}$$

$$\forall\eta\,\forall\psi_1\forall\psi_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}(-1)\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \qquad \text{by (58)}$$

$$\forall\eta\,\forall\alpha_1\forall\alpha_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}(-1)\begin{pmatrix}0 & \eta^{-1}\\ \eta & 0\end{pmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} \qquad \text{by (47)}$$

$$\forall\eta\,\forall\chi_1\forall\chi_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}(-1)\begin{pmatrix}0 & \eta^{-1}\\ \eta & 0\end{pmatrix}(-1)\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} \qquad \text{by (57)}$$

$$\forall\eta\,\forall\gamma_1\forall\gamma_2\beta_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}(-1)\begin{pmatrix}0 & \eta^{-1}\\ \eta & 0\end{pmatrix}(-1)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix} \qquad \text{by (54)}$$

$$\forall\eta\,\forall\phi_1\forall\phi_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}(-1)\begin{pmatrix}0 & \eta^{-1}\\ \eta & 0\end{pmatrix}(-1)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(-1)\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \qquad \text{by (59)}$$

In summary, assuming the **Hypothesised backward coincidence**, the overall result is the assertion:

$$\forall\eta\,\forall\phi_1\forall\phi_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = (-1)\begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}\begin{pmatrix}0 & \eta^{-1}\\ \eta & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \tag{60}$$

The seemingly ambiguous quantification $\forall\phi_1\forall\phi_2\exists\phi_1\exists\phi_2$ indicates a *possible* $\forall\phi_1\forall\phi_2$, but *guaranteed* $\exists\phi_1\exists\phi_2$. Either way, (60) implies the following:

$$\begin{aligned} &\forall\eta \;\mid\; (-1)\begin{pmatrix}1 & 0\\ 0 & \eta^2\end{pmatrix}\begin{pmatrix}0 & \eta^{-1}\\ \eta & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \mathbb{1}_2 \\ \Longrightarrow\; &\forall\eta \;\mid\; (-1)\begin{pmatrix}\eta^{-1} & 0\\ 0 & \eta^3\end{pmatrix} = \mathbb{1}_2 \end{aligned} \tag{61}$$

The assertion (61) is self-contradictory, because the matrix cannot equal the identity for all values of η. This confirms there is something invalid about the **Hypothesised backward coincidence.** Nevertheless, it is important to retain the full information of (61) if valid conditionality is to be later revealed.

Part Three

Noting the forward and backward self-references (56) and (61), both result in the identity, so they can be equated:

$$\forall\eta \;\mid\; \begin{pmatrix}\eta & 0\\ 0 & \eta\end{pmatrix} = (-1)\begin{pmatrix}\eta^{-1} & 0\\ 0 & \eta^3\end{pmatrix} \tag{62}$$

But (62) can be proved invalid because it is not true for arbitrary η. Replacement of the universal quantifiers $\forall\eta$ by existential quantifiers $\exists\eta$ removes the contradiction, thus:

$$\exists\eta \;\mid\; \begin{pmatrix}\eta^{-1} & 0\\ 0 & \eta^3\end{pmatrix} = (-1)\begin{pmatrix}\eta & 0\\ 0 & \eta\end{pmatrix} \tag{63}$$

Hence, overall joint conditionality on both the assumed **Hypothesised forward coincidence** and **Hypothesised backward coincidence** is:

$$\exists\eta \;\mid\; \eta^{\pm 2} = -1 \tag{64}$$

Whereas the transformations (47), (48), (49) are derivable from Axioms in Figure 1, and contain no new information, (64) is not derivable from Axioms but is, logically independent information of Axioms, indicating new logically independent information originates in the self-reference.

12 Conclusions

Quantum indeterminacy is not irreducible, but has deeper explanation. It resides in inaccessible history — of photon polarity alignments — left behind as the density matrix evolves. Definite history cannot be determined because it is geometrically ambiguous, owing to perfect symmetry. Superposition states require this symmetry for their existence; and only they suffer this historical ambiguity. The symmetry is 'fixed' by logically independent, self-referential circularity, permitting transition of eigenstates into superpositions. The symmetry in question is the *unitary* symmetry.

Experiments of Tomasz Paterek et al reveal that quantum randomness is manifest exclusively in measurement outcomes deriving from experiments involving logically independent information. Analysis of the mathematics used reveals that the Paterek method (unwittingly) ignores and relaxes a *Quantum Postulate*. That relaxation releases mathematical machinery which generates the same mathematical content of the *Postulate*, but, entering the system as logically independent information, rather than axiomatic. The successfulness of their mathematics is demonstration that standard theory does not fully and faithfully represent measurement experiments and that any complete quantum theory that does, must necessarily be made consistent with this new mathematics.

The said machinery is self-referential circularity, going on cyclically through sets of transformations. Viewing these from the perspective of Elementary Algebra tells us how we should amend our beliefs concerning *Causation.* In addition to *'Cause'* Fundamental Physics cannot avoid *'Non-prevention'* as logical reason for certain physical effects.

12.1 *Quantum Postulate*

Mathematics inherent in the Paterek method contradicts the *Quantum Postulate* imposing unitary, Hermitian, Hilbert space mathematics, by (unwittingly) relaxing these structures for eigenstates. This is possible because Banach space, having no inner product, is adequate for eigenstates. By contrast, existence of superposition states necessitates these structures. Clearing away this Postulate makes visible a logical step, missing from standard theory, that makes these two types of state logically distinct.

12.2 *Density matrix history & ambiguity of perfect symmetry*

In experiments, density matrix evolution is definite and deterministic. Measurement is the attempt to extract information that determined, that evolved density matrix, from upstream in its history. However, identical density matrix values can evolve from pairs of operators which are the reflections of one another. So density matrix does not have one-one dependency with its history. History is perfectly ambiguous. At this bifurcation, history becomes logically independent of its evolved density matrix.

For experiments concerning only parallel polarisation alignments, history is not ambiguous, but is bijective, and there is logical dependence, in both directions, between density matrix and operators. The reason for non-ambiguity, in the parallel case, is that there is no demand by states, for the perfect symmetry, and no pairs of reflected operators.

It is my recommendation that accessibility of density matrix history be studied in relation to the EPR problem.

Every perfect symmetry presents geometrical or quantitative ambiguity, by way of some degree of freedom or other — typically, the left|right handedness of perfect 3-space, or perfectly ambiguous position along an infinitely homogeneous line. Following the reasoning I have given, I predict that logical independencies shall be discovered, stemming from other perfect symmetries, which are the sources of indeterminacy in other quantum systems.

12.3 *Non-Prevention & Cause*

The classical philosophy that underlies Physical Theory is *cause* and *effect.* The connection between these is a mathematical matter. Theoreticians propose causes, in terms of *Principles* and *Postulates*, then Mathematical Physics determines, by proof, whatever effects they imply. A Physicist's viewpoint would be that the cause *determines* the effect; a Mathematical Logician would regard the effect as being *logically dependent* on the cause.

While *implication* conveys dependent consequences, so too does *negation* or *denial.* In classical theory, everything *implied* coincides with everything *not denied.* A planet's motion is determined by 'Laws of Motion', plus, some 'state of motion' at some time instant. But equally, these same Laws deny and prevent every other alternative motion. The singular motion which is *caused* is strictly identical to the singular motion which is *not prevented.*

In a physical system where there is *logical independence*, the caused effects and the non-prevented effects are not identical sets, because logically independent effects comprise an excluded middle that is not prevented, and not caused; and effects can materialise by way of not being denied, even if not implied. Quantum randomness is an effect which arises through being *not prevented* — not through being *caused.*

There is the question of why a polarisation measurement entails logical independence, whereas the motion of a planet does not. For the planet, the motion is reversible and determinable from any point in its history or future. Its history is determinable from its present. But history is not determinable for the density matrix. Ultimately, the reason goes back to whether the planet needs a unitary symmetry to represent its motion.

Conceptualisation of *Non-prevetion* is key. In absence of the unitary regime, imposed by Postulate, the unitary symmetry arises as result of the non-prevented, self-referential machine, along with its perfect geometrical ambiguity. Whereas indeterminacy arises through *non-prevention* of perfect symmetry, collapse is *caused* when perfect symmetry is broken by some distorting influence such as a magnetic field, or an optical polariser.

12.4 *The Self-referential machine*

The self-reference is the circulation of information around a system which happens to sustain itself – because it is stable. As well as stability, it has logical consequences because it is not caused, yet it does impose implied conditions. Circularity is allowed to initiate and progress because no condition in the environment prevents it. The circularity is within sets of involutory transformations, acting on the arbitrary general vector, which happens to impose a unitary condition overall.

With no Unitary Postulate in place, if this self-reference were not possible, no Hermitian operators or Hilbert space would be available. What the self-reference possesses which the Postulate does not is logical independence.

The self-referential mechanism is symmetry creating – the converse of 'symmetry breaking' – as in the Higgs mechanism. But unlike Higgs, which involves energy, self-reference is perfectly free and not subject to any conservation law. There is no resistance to its onset. That said, the similarity should be investigated further.

Concisely said, algebra of this paper is a 'flexible $\mathsf{su}\,(2)$'. This is interpetable as 3-space curved metrics which are instantaneously unitary. The extension of these ideas to spacetime may prove useful for quantisation of Gravity Theory.

12.5 *The Field Axioms: Elementary Algebra*

All these ideas best make sense when viewed in the context of Elementary Algebra. This is the algebra of rational, real and complex scalars – the infinite scalar fields. For one thing, it puts logical meaning onto the imaginary unit's presence in quantum theory. This algebra is subject to Soundness and Completeness theorems of Model Theory – a branch of Mathematical Logic, and these confirm logical independence where it exists.

Elementary Algebra is already there, within the mathematics of quantum theory. Indeed it is the foundation upon which quantum mathematics rests. Matrix transformations and their vectors are structures which can be expressed in this algebra.

I am not thinking here of unit vectors **i**, **j**, **k**, but this applies to them, even.

Some require simultaneity and some, circularity. The proposal here is that quantum mathematics should be seen, not as an extension of Elementary Algebra, but closed within it, and viewed from the logical perspective of a *formal axiomatised system.* Note that Paterek treated his Boolean formulae that way.

By realising that Linear Algebra of quantum theory, can be written as formulae in Elementary Algebra — but exhibiting logic — the logic of unitary and Hermitian operators, and Hilbert space superpositions comes to light. Instead of writing definitions for Linear Algebra on top of Elementary Algebra, logically independent machinery and structure take their place.

By doing so, the mathematics of quantum theory is written as a system of formulae, all under the *Field Axioms.*

Problem: This leave the problem of the *Field Axioms,* as *a priori* fundamental, and the question of where in Nature, do they come from?

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