Schrödinger's equation as an energy diffusion equation

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Abstract :

While the real and imaginary part of the quantum-mechanical wavefunction are, obviously, not to be looked as field vectors, the similarity between the geometry of the quantum-mechanical wavefunction and that of a linearly polarized electromagnetic wave remains intriguing: from a mathematical point of view, only the relative phase differs. Also, if the physical dimension of the electromagnetic field is expressed in newton per coulomb (force per unit charge), then one might explore the implications of associating the components of the wavefunction with a similar physical dimension: force per unit mass (newton per kg). This leads to a remarkably elegant interpretation of the physical significance of the wavefunction and the wave equation:

1. The calculated energy densities are proportional to the square of the absolute value of the wavefunction and, hence, to the probabilities.
2. Schrödinger's wave equation itself may then, effectively, be interpreted as a diffusion equation for energy itself.

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Introduction

The similarity between the geometry of the quantum-mechanical wavefunction and that of a linearly polarized electromagnetic wave is obvious and intriguing : the components of both waves are orthogonal to the direction of propagation and to each other. Only the relative phase differs : the electric and magnetic field vectors (**E** and **B**) have the same phase. In contrast, the phase of the real and imaginary part of the wavefunction ($a\cdot\cos\theta$ and $-a\cdot\sin\theta$) differ by 90 degrees ($\pi/2$).[1] Pursuing the analogy, this article explores the following question: if the oscillating electric and magnetic field vectors of an electromagnetic wave carry the energy that one associates with the wave, can we analyze the real and imaginary part of the wavefunction in a similar way?

This article shows the answer is positive. In fact, the analysis is remarkably straightforward. If the physical dimension of the electromagnetic field is expressed in newton per coulomb (force per unit charge), then the physical dimension of the components of the wavefunction may be associated with force per unit mass (newton per kg).[2] Of course, force over some distance is energy. The question then becomes: what is the energy concept here? Kinetic? Potential? Both?

The similarity between the energy of a (one-dimensional) linear oscillator ($E = m\cdot a^2\cdot\omega_0^2/2$) and Einstein's relativistic energy equation $E = mc^2$ inspires the author to interpret the energy as a *two*-dimensional oscillation of mass. To assist the reader, the author constructs a two-piston engine metaphor.[3] The formula for the electromagnetic energy density formula can then be used to calculate the energy densities for the wave function.

The result is elegant and intuitive: the energy densities are proportional to the square of the absolute value of the wavefunction and, hence, to the probabilities. Schrödinger's wave equation may then, effectively, be interpreted as a diffusion equation for energy itself.

Of course, such interpretation is also an interpretation of the wavefunction itself, and the immediate reaction of the reader is predictable: the electric and magnetic field vectors are real vectors. In contrast, the real and imaginary components of the wavefunction are not. However, this objection needs to be phrased more carefully. First, it may be noted that, in a classical analysis, the magnetic force is a pseudovector itself.[4] Second, a suitable choice of coordinates may make quantum-mechanical rotation matrices irrelevant.[5]

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[1] Of course, an *actual* particle is localized in space and can, therefore, *not* be represented by the elementary wavefunction $\psi = a\cdot e^{-i\cdot\theta} = a\cdot e^{-i[E\cdot t - \mathbf{p}\cdot\mathbf{x}]/\hbar} = a\cdot(\cos\theta - i\cdot a\cdot\sin\theta)$. –We must build a wave *packet* for that: a sum of wavefunctions, each with its own amplitude a_k and its own argument $\theta_k = (E_k\cdot t - \mathbf{p}_k\cdot\mathbf{x})/\hbar$. This is dealt with in this paper as part of the discussion on the mathematical and physical interpretation of the normalization condition.

[2] The N/kg dimension immediately, and naturally, reduces to the dimension of acceleration (m/s^2), thereby facilitating a direct interpretation in terms of Newton's force law.

[3] In physics, a two-*spring* metaphor is more common. Hence, the pistons in the author's *perpetuum mobile* may be replaced by springs.

[4] The magnetic force can be analyzed as a relativistic effect (see Feynman II-13-6). The dichotomy between the electric force as a polar vector and the magnetic force as an axial vector disappears in the relativistic four-vector representation of electromagnetic.

[5] For example, when using Schrödinger's equation in a central field (think of the electron around a proton), the use of polar coordinates is recommended, as it ensures the symmetry of the Hamiltonian under all rotations (see Feynman III-19-3)

Therefore, this article may provide some fresh perspective on the question, thereby further exploring Einstein's basic sentiment in regard to quantum mechanics, which may be summarized as follows: there must be some *physical* explanation for the calculated probabilities.[6]

Let us, therefore, start with Einstein's relativistic energy equation ($E = mc^2$) and wonder what it could possibly tell us.

I. Energy as a two-dimensional oscillation of mass

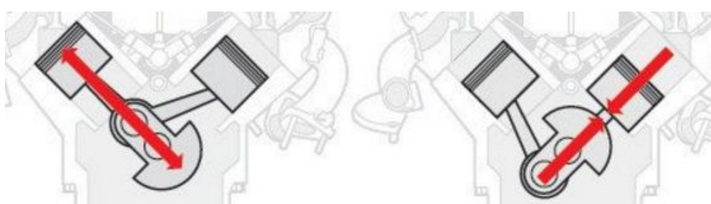
The structural similarity between the relativistic energy formula, the formula for the *total* energy of an oscillator, and the *kinetic* energy of a moving body, is striking:

1. $E = mc^2$
2. $E = m\omega_0^2/2$
3. $E = mv^2/2$

In these formulas, ω_0, v and c all describe some velocity.[7] Of course, there is the 1/2 factor in the $E = m\omega_0^2/2$ formula[8], but that's exactly the point we are going to explore here: can we think of an oscillation in *two* dimensions, so it stores an amount of energy that is equal to $E = 2 \cdot m \cdot \omega_0^2/2 = m \cdot \omega_0^2$?

That is easy enough. Think, for example, of a V-2 engine with the pistons at a 90-degree angle, as illustrated below. The 90° angle makes it possible to perfectly balance the counterweight and the pistons, thereby ensuring smooth travel at all times. With permanently closed valves, the air inside the cylinder compresses and decompresses as the pistons move up and down and provides, therefore, a restoring force. As such, it will store potential energy, just like a spring, and the motion of the pistons will also reflect that of a mass on a spring. Hence, we can describe it by a sinusoidal function, with the zero point at the center of each cylinder. We can, therefore, think of the moving pistons as harmonic oscillators, just like mechanical springs.

Figure 1: Oscillations in two dimensions



[6] This sentiment is usually summed up in the apocryphal quote: "God does not play dice."The actual quote comes out of one of Einstein's private letters to Cornelius Lanczos, another scientist who had also emigrated to the US. The full quote is as follows: "You are the only person I know who has the same attitude towards physics as I have: belief in the comprehension of reality through something basically simple and unified... It seems hard to sneak a look at God's cards. But that He plays dice and uses 'telepathic' methods... is something that I cannot believe for a single moment." (Helen Dukas and Banesh Hoffman, Albert Einstein, the Human Side: New Glimpses from His Archives, 1979)

[7] Of course, both are different velocities: ω_0 is an *angular* velocity, while v is a *linear* velocity: ω_0 is measured in *radians* per second, while v is measured in meter per second. However, the definition of a radian implies radians are measured in distance units. Hence, the physical dimensions are, effectively, the same. As for the formula for the total energy of an oscillator, we should actually write: $E = m \cdot a^2 \cdot \omega_0^2/2$. The additional factor (a) is the (maximum) amplitude of the oscillator.

[8] We also have a 1/2 factor in the $E = mv^2/2$ formula. It may be noted this is non-relativistic. Using the Lorentz factor (γ), we can write the relativistically correct formula for the kinetic energy as $K.E. = E - E_0 = m_v c^2 - m_0 c^2 = m_0 \gamma c^2 - m_0 c^2 = m_0 c^2(\gamma - 1)$. We may also note that the energy concept that is used in the context of the Principle of Least Action is equal to $E = mv^2$. The appendix provides some notes on that.

If we assume there is no friction, we have a *perpetuum mobile* here. The compressed air and the rotating counterweight (which, combined with the crankshaft, acts as a flywheel[9]) store the potential energy. The moving masses of the pistons store the kinetic energy of the system.[10]

At this point, it is probably good to quickly review the relevant math. If the magnitude of the oscillation is equal to a, then the motion of the piston (or the mass on a spring) will be described by $x = a\cdot\cos(\omega_0\cdot t + \Delta)$.[11] Needless to say, Δ is just a phase factor which defines our $t = 0$ point, and ω_0 is the *natural* angular frequency of our oscillator. Because of the 90° angle between the two cylinders, Δ would be 0 for one oscillator, and –π/2 for the other. Hence, the motion of one piston is given by $x = a\cdot\cos(\omega_0\cdot t)$, while the motion of the other is given by $x = a\cdot\cos(\omega_0\cdot t - \pi/2) = a\cdot\sin(\omega_0\cdot t)$.

The kinetic and potential energy of *one* oscillator (think of one piston or one spring only) can then be calculated as:

1. $\text{K.E.} = T = m\cdot v^2/2 = (1/2)\cdot m\cdot\omega_0^2\cdot a^2\cdot\sin^2(\omega_0\cdot t + \Delta)$
2. $\text{P.E.} = U = k\cdot x^2/2 = (1/2)\cdot k\cdot a^2\cdot\cos^2(\omega_0\cdot t + \Delta)$

The coefficient k in the potential energy formula characterizes the restoring force: $F = -k\cdot x$. From the dynamics involved, it is obvious that k must be equal to $m\cdot\omega_0^2$. Hence, the total energy is equal to:

$$E = T + U = (1/2)\cdot m\cdot\omega_0^2\cdot a^2\cdot[\sin^2(\omega_0\cdot t + \Delta) + \cos^2(\omega_0\cdot t + \Delta)] = m\cdot a^2\cdot\omega_0^2/2$$

To facilitate the calculations, we will briefly assume $k = m\cdot\omega_0^2$ and a are equal to 1. The motion of our first oscillator is given by the $\cos(\omega_0\cdot t) = \cos\theta$ function ($\theta = \omega_0\cdot t$), and its kinetic energy will be equal to $\sin^2\theta$. Hence, the (instantaneous) *change* in kinetic energy at any point in time will be equal to:

$$d(\sin^2\theta)/d\theta = 2\cdot\sin\theta\cdot d(\sin\theta)/d\theta = 2\cdot\sin\theta\cdot\cos\theta$$

Let us look at the second oscillator now. Just think of the second piston going up and down in the V-2 engine. Its motion is given by the sinθ function, which is equal to $\cos(\theta-\pi/2)$. Hence, its kinetic energy is equal to $\sin^2(\theta-\pi/2)$, and how it *changes* – as a function of θ – will be equal to:

$$2\cdot\sin(\theta-\pi/2)\cdot\cos(\theta-\pi/2) = = -2\cdot\cos\theta\cdot\sin\theta = -2\cdot\sin\theta\cdot\cos\theta$$

We have our *perpetuum mobile*! While transferring kinetic energy from one piston to the other, the crankshaft will rotate with a constant angular velocity: linear motion becomes circular motion, and vice versa, and the total energy that is stored in the system is $T + U = ma^2\omega_0^2$.

We have a great *metaphor* here. Somehow, in this beautiful interplay between linear and circular motion, energy is borrowed from one place and then returns to the other, cycle after cycle. We know the wavefunction consist of a sine and a cosine: the cosine is the real component, and the sine is the imaginary component. Could they be equally real? Could each represent *half* of the total energy of our particle? Should we think of the c in our $E = mc^2$ formula as an *angular* velocity?

These are sensible questions. Let us explore them.

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[9] Instead of two cylinders with pistons, one may also think of connecting two springs with a crankshaft.

[10] It is interesting to note that we may look at the energy in the rotating flywheel as *potential* energy because it is energy that is associated with motion, albeit *circular* motion. In physics, one may associate a rotating object with kinetic energy using the rotational equivalent of mass and linear velocity, i.e. rotational inertia (I) and angular velocity ω. The kinetic energy of a rotating object is then given by $\text{K.E.} = (1/2)\cdot I\cdot\omega^2$.

[11] Because of the sideways motion of the connecting rods, the sinusoidal function will describe the linear motion only *approximately,* but you can easily imagine the idealized limit situation.

II. The wavefunction as a two-dimensional oscillation

The elementary wavefunction is written as:

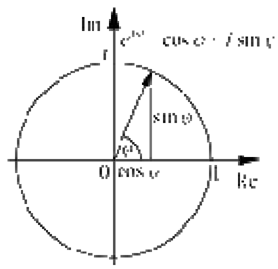
$$\psi = a\cdot e^{-i[\mathrm{E}\cdot\mathrm{t}\,-\,\mathbf{p}\cdot\mathbf{x}]/\hbar} = a\cdot e^{-i[\mathrm{E}\cdot\mathrm{t}\,-\,\mathbf{p}\cdot\mathbf{x}]/\hbar} = a\cdot\cos(\mathbf{p}\cdot\mathbf{x}/\hbar - \mathrm{E}\cdot t/\hbar) + i\cdot a\cdot\sin(\mathbf{p}\cdot\mathbf{x}/\hbar - \mathrm{E}\cdot t/\hbar)$$

When considering what happens at $\mathbf{x} = \mathbf{0}$, or considering a particle at rest ($\mathbf{p} = \mathbf{0}$) this reduces to:

$$\psi = a\cdot e^{-i\cdot\mathrm{E}\cdot\mathrm{t}/\hbar} = a\cdot\cos(-\mathrm{E}\cdot\mathrm{t}/\hbar) + i\cdot a\cdot\sin(-\mathrm{E}\cdot\mathrm{t}/\hbar) = a\cdot\cos(\mathrm{E}\cdot\mathrm{t}/\hbar) - i\cdot a\cdot\sin(\mathrm{E}\cdot\mathrm{t}/\hbar)$$

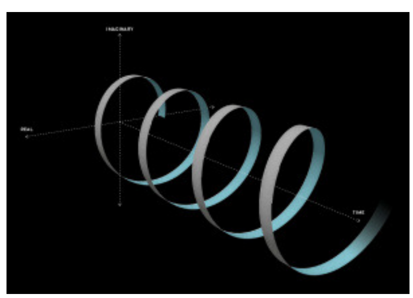
Let us remind ourselves of the geometry involved, which is illustrated below. Note that the argument of the wavefunction rotates *clockwise* with time, while the mathematical convention for measuring the phase angle (φ) is *counter*-clockwise.

Figure 2: Euler's formula



If we assume the momentum **p** is all in the *x*-direction, then the **p** and **x** vectors will have the same direction, and **p**·**x**/ħ reduces to p·x/ħ. Most illustrations – such as the one below – will either freeze **x** or, else, *t*. Alternatively, one can *google* web animations varying both. The point is: we do have a two-dimensional oscillation here. These two dimensions are perpendicular to the direction of propagation of the wavefunction. For example, if the wavefunction propagates in the *x*-direction, then the oscillations are along the *y*- and *z*-axis, which we may refer to as the real and imaginary axis. Note how the phase difference between the cosine and the sine – the real and imaginary part of our wavefunction – appear to give some spin to the whole. I will come back to this.

Figure 3: Geometric representation of the wavefunction



Hence, *if* we would say these oscillations carry half of the total energy of the particle, then we may refer to the real and imaginary energy of the particle respectively, and the interplay between the real and the imaginary part of the wavefunction may then describe how energy propagates over time through space.

Let us consider, once again, a particle at rest. Hence, p = 0 and the (elementary) wavefunction reduces to $\psi = a\cdot e^{-i\cdot\mathrm{E}\cdot\mathrm{t}/\hbar}$. Hence, the angular velocity of both oscillations, at some point **x**, is given by ω = –E/ħ. Now, the energy of our particle includes all of the energy – kinetic, potential and rest energy – and is, therefore, equal to $\mathrm{E} = \mathrm{m}c^2$.

Can we, somehow, relate this to the $m \cdot a^2 \cdot \omega_0^2$ energy formula for V-2 *perpetuum mobile*? Our wavefunction has an amplitude too. Now, if the oscillations of the real and imaginary wavefunction store the energy of our particle, then their amplitude will surely matter. In fact, the energy of an oscillation is, in general, proportional to the *square* of the amplitude: $E \propto a^2$. We may, therefore, think that the a^2 factor in the $E = m \cdot a^2 \cdot \omega_0^2$ energy will surely be relevant as well.

However, here is a complication: an *actual* particle is localized in space and can, therefore, *not* be represented by the elementary wavefunction. We must build a wave *packet* for that: a sum of wavefunctions, each with their own amplitude a_k, and their own $\omega_k = -E_k/\hbar$. Each of these wavefunctions will *contribute* some energy to the total energy of the wave packet. To calculate the contribution of each wave to the total, both a_k as well as E_k will matter.

What is E_k? E_k varies around some average E: the Uncertainty Principle kicks in here. The analysis becomes more complicated, but a formula such as the one below might make sense:

$$\mathrm{E} = \sum \mathrm{m} \cdot a_k^2 \cdot \omega_k^2 = \mathrm{m} \cdot \sum a_k^2 \cdot \omega_k^2$$

Now, because $E = mc^2$, this implies the following:

$$c^2 = \sum a_k^2 \cdot \omega_k^2$$

What is the meaning of this equation? We may look at it as a *physical* normalization condition. It says all of the $m \cdot a_k^2 \cdot \omega_k^2$ contributions have to add up to the total energy, as evidenced by the following alternative formulation of the identity:

$$1 = \frac{\mathrm{m} \cdot \sum a_k^2 \cdot \omega_k^2}{\mathrm{E}} = \frac{\mathrm{m}c^2}{\mathrm{E}} = 1$$

Of course, we should relate this to the *mathematical* normalization condition: the probabilities must be related to the energy *densities*.

III. What is mass?

We now have a meaningful interpretation for energy: it is a two-dimensional oscillation of mass. But what is mass? A new *aether* theory is, of course, not an option, but then what is it that is oscillating?

To understand the physics behind equations, it is always good to do an analysis of the physical dimensions in the equation. Let us start with Einstein's energy equation once again. If we want to look at mass, we should re-write it as $m = E/c^2$:

$$[m] = [E/c^2] = J/(m/s)^2 = N \cdot m \cdot s^2/m^2 = N \cdot s^2/m = kg$$

This is not very helpful. It only reminds us of Newton's definition of a mass: mass is that what gets accelerated by a force. At this point, we may want to think of the physical significance of the *absolute* nature of the speed of light. Einstein's $E = mc^2$ equation implies we can write the ratio between the energy and the mass of *any* particle is always the same, so we can write, for example:

$$\frac{E_{electron}}{m_{electron}} = \frac{E_{proton}}{m_{proton}} = \frac{E_{photon}}{m_{photon}} = \frac{E_{any\ particle}}{m_{any\ particle}} = c^2$$

This reminds us of the $\omega_0^2 = C^{-1}/L$ or $\omega_0^2 = k/m$ of harmonic oscillators once again.[12] The key difference is that the $\omega_0^2 = C^{-1}/L$ and $\omega_0^2 = k/m$ formulas introduce *two* or more degrees of freedom.[13] In contrast, $c^2=$

[12] The $\omega_0^2 = 1/LC$ formula gives us the natural or resonant frequency for a electric circuit consisting of a resistor (R),

E/m for *any* particle, *always*. However, that is exactly the point: we can modulate the resistance, inductance and capacitance of electric circuits, and the stiffness of springs and the masses we put on them, but we live in *one* physical space only: *our* spacetime. Hence, the speed of light *c* emerges here as *the* defining property of spacetime – the resonant frequency, so to speak. We have no further degrees of freedom here.

In this regard, we should highlight another interesting implication of *de Broglie*'s $E = h \cdot f = \omega \cdot \hbar$ equation for the matter wave, which boldly generalizes the Planck-Einstein relation for a photon $E = h \cdot \nu = \omega \cdot \hbar$ to encompass matter waves too: it gives us the *frequency* of the matter wave. Now, *f* is the number of oscillations per second, which we may write as $f = n/\text{s}$. Hence, we can write:

$$E/n = (E/f)\cdot(1\text{ s}) = h\cdot(1\text{ s}) = (6.626070040(81)\times10^{-34}\text{ J}\cdot\text{s})\cdot(1\text{ s}) = 6.626070040(81)\times10^{-34}\text{ J}$$

This is an amazing result: our particle – be it a photon or a matter-particle – will *always* pack $6.626070040(81)\times10^{-34}$ *joule* in *one* oscillation. Of course, the obvious question is: what *is* one oscillation? The matter wave comes as a wave packet and, therefore, the oscillations will not have the same amplitude. In fact, the Uncertainty Principle tells us we will not be able to define an exact period. Nevertheless, the result stands.[14]

The Planck-Einstein relation (for photons) and the *de Broglie* equation (for matter-particles) have another interesting feature: both imply that the *energy* of the oscillation is proportional to the frequency, with Planck's constant as the constant of proportionality. Now, for *one-dimensional* oscillations – think of a guitar string, for example – we know the energy will be proportional to the *square* of the frequency.[15] It is a remarkable observation: the two-dimensional matter-wave, or the electromagnetic wave, gives us *two* waves for the price of one, so to speak, each carrying *half* of the *total* energy of the oscillation but, as a result, we get an $E \propto f$ instead of an $E \propto f^2$ proportionality.

However, such reflections do not answer the fundamental question we started out with: what *is* mass? At this point, it is hard to go beyond the circular definition that is implied by Einstein's formula: energy is a two-dimensional oscillation of mass, and mass packs energy, and *c* emerges us as the property of spacetime that defines *how* exactly. When everything is said and done, this does not go beyond stating that mass is some scalar field. Now, a scalar field is, quite simply, some real *number* that we associate with a position in spacetime. The Higgs field is a scalar field but, of course, the theory behind it goes

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an inductor (*L*), and a capacitor (*C*). Writing the formula as $\omega_0^2 = C^{-1}/L$ introduces the concept of *elastance*, which is the equivalent of the mechanical stiffness (k) of a spring.

[13] The resistance in an electric circuit introduces a damping factor. When analyzing a mechanical spring, one may also want to introduce a drag coefficient. Both are usually defined as a fraction of the *inertia*, which is the mass for a spring and the inductance for an electric circuit. Hence, we would write the resistance for a spring as γm and as $R = \gamma L$ respectively.

[14] Photons are emitted by atomic oscillators: atoms going from one state (energy level) to another. Feynman (*Lectures*, I-33-3) shows us how to calculate the Q of these atomic oscillators: it's of the order of 10^8, which means the wave train will last about 10^{-8} seconds (to be precise, that is the time it takes for the radiation to die out by a factor 1/*e*). For example, for sodium light, the radiation will last about 3.2×10^{-8} seconds (this is the so-called decay time τ). Now, because the frequency of sodium light is some 500 THz (500×10^{12} oscillations per second), this makes for some 16 million oscillations. There is an interesting paradox here: the speed of light tells us that such wave train will have a length of about 9.6 m! How is that to be reconciled with the pointlike nature of a photon? The paradox can only be explained by relativistic length contraction: in an analysis like this, one need to distinguish the reference frame of the photon – riding along the wave as it is being emitted, so to speak – and our stationary reference frame, which is that of the emitting atom.

[15] This is a general result and is reflected in the K.E. $= T = (1/2)\cdot m\cdot\omega_0^2\cdot a^2\cdot\sin^2(\omega_0\cdot t + \Delta)$ and the P.E. $= U = k\cdot x^2/2 = (1/2)\cdot m\cdot\omega_0^2\cdot a^2\cdot\cos^2(\omega_0\cdot t + \Delta)$ formulas for the linear oscillator.

much beyond stating that we should think of mass as some scalar field. The fundamental question is: why and how does energy, or matter, *condense* into elementary particles? That is what the Higgs *mechanism* is about but, as this paper is exploratory only, we cannot even start explaining the basics of it.

What we *can* do, however, is look at the wave *equation* again (Schrödinger's equation), as we can now analyze it as an energy diffusion equation.

IV. Schrödinger's equation as an energy diffusion equation

The interpretation of Schrödinger's equation as a diffusion equation is straightforward. Feynman (*Lectures*, III-16-1) briefly summarizes it as follows:

> "We can think of Schrödinger's equation as describing the diffusion of the probability amplitude from one point to the next. […] But the imaginary coefficient in front of the derivative makes the behavior completely different from the ordinary diffusion such as you would have for a gas spreading out along a thin tube. Ordinary diffusion gives rise to real exponential solutions, whereas the solutions of Schrödinger's equation are complex waves."[16]

Let us review the basic math. For a particle moving in free space – with no external force fields acting on it – there is no potential (U = 0) and, therefore, the Uψ term disappears. Therefore, Schrödinger's equation reduces to:

$$\partial\psi(\mathbf{x}, t)/\partial t = i\cdot(1/2)\cdot(\hbar/m_{eff})\cdot\nabla^2\psi(\mathbf{x}, t)$$

The ubiquitous diffusion equation in physics is:

$$\partial\phi(\mathbf{x}, t)/\partial t = D\cdot\nabla^2\phi(\mathbf{x}, t)$$

The *structural* similarity is obvious. The key difference between both equations is that the wave equation gives us *two* equations for the price of one. Indeed, because ψ is a complex-valued function, with a *real* and an *imaginary* part, we get the following equations[17]:

1. $Re(\partial\psi/\partial t) = -(1/2)\cdot(\hbar/m_{eff})\cdot Im(\nabla^2\psi)$
2. $Im(\partial\psi/\partial t) = (1/2)\cdot(\hbar/m_{eff})\cdot Re(\nabla^2\psi)$

These equations make us think of the equations for an electromagnetic wave in free space (no stationary charges or currents):

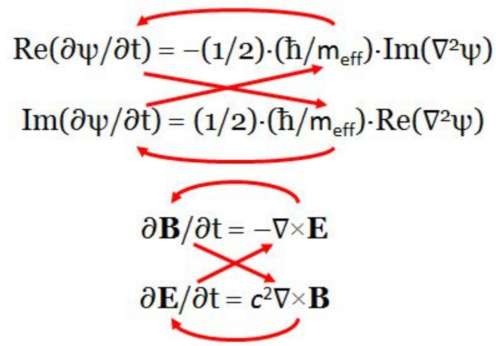
1. $\partial\mathbf{B}/\partial t = -\nabla\times\mathbf{E}$
2. $\partial\mathbf{E}/\partial t = c^2\nabla\times\mathbf{B}$

The above equations describe a *propagation* mechanism in spacetime, as illustrated below.

[16] Feynman further formalizes this in his *Lecture on Superconductivity* (Feynman, III-21-2), in which he refers to Schrödinger's equation as the "equation for continuity of probabilities". The analysis is centered around the *local* conservation of energy, which confirms the interpretation of Schrödinger's equation as an energy diffusion equation.

[17] The m_{eff} is the effective mass of the particle. It depends on the medium. For example, an electron traveling in a solid (a transistor, for example) will have a different effective mass than in an atom. In free space, we can drop the subscript.

Figure 4: Propagation mechanism



The Laplacian operator (∇^2), when operating on a *scalar* quantity, gives us a flux density, i.e. something expressed per square meter ($1/m^2$). In this case, it's operating on ψ(**x**, t), so what is the dimension of our wavefunction ψ(**x**, t)? To answer that question, we should analyze the diffusion constant in Schrödinger's equation, i.e. the $(1/2)\cdot(\hbar/m_{eff})$ factor:

1. As a *mathematical* constant of proportionality, it will *quantify* the relationship between both derivatives (i.e. the time derivative and the Laplacian);
2. As a *physical* constant, it will ensure the *physical dimensions* on both sides of the equation are compatible.

Now, the $\hbar/m_{eff}$ factor is expressed in $(N\cdot m\cdot s)/(N\cdot s^2/m) = m^2/s$. Hence, it does ensure the dimensions on both sides of the equation are, effectively, the same: ∂ψ/∂t is a time derivative and, therefore, its dimension is s^{-1} while, as mentioned above, the dimension of $\nabla^2\psi$ is m^{-2}. However, this does not solve our basic question: what is the dimension of the real and imaginary part of our wavefunction?

At this point, mainstream physicists will say: it doesn't have a physical dimension, and there is no geometric interpretation of Schrödinger's equation. One may argue, effectively, that its argument, (**p**·**x** – E·*t*)/ħ, is just a number and, therefore, that the real and imaginary part of ψ is also just some number.

To this, we may object that ħ may be looked as a *mathematical* scaling constant only. If we do that, the argument of ψ will, effectively, be expressed in *action* units, i.e. in N·m·s. It then does make sense to also associate a physical dimension with the real and imaginary part of ψ. What could it be?

We may have a closer look at Maxwell's equations for inspiration here. The electric field vector is expressed in *newton* (the unit of force) per unit of *charge* (*coulomb*). Now, there is something interesting here. The physical dimension of the magnetic field is N/C *divided* by m/s.[18] We may write **B** as the following vector cross-product: **B** = $(1/c)\cdot\boldsymbol{e_x}\times\mathbf{E}$, with $\boldsymbol{e_x}$ the unit vector pointing in the *x*-direction (i.e. the direction of propagation of the wave). Hence, we may associate the $(1/c)\cdot\boldsymbol{e_x}\times$ *operator*, which amounts to a rotation by 90 degrees, with the s/m dimension. Now, multiplication by *i* also amounts to a rotation by 90° degrees. Hence, we may boldly write: $\mathbf{B} = (1/c)\cdot\boldsymbol{e_x}\times\mathbf{E} = (1/c)\cdot i\cdot\mathbf{E}$. This allows us to also geometrically interpret Schrödinger's equation in the way we interpreted it above (see Figure 3).[19]

[18] The dimension of **B** is usually written as N/(m·A), using the SI unit for current, i.e. the *ampere* (A). However, 1 C = 1 A·s and, hence, 1 N/(m·A) = 1 (N/C)/(m/s).

[19] Of course, multiplication with *i* amounts to a *counter*clockwise rotation. Hence, multiplication by –*i* also amounts to a rotation by 90 degrees, but *clockwise*. Now, to uniquely identify the clockwise and counterclockwise directions, we need to establish the equivalent of the right-hand rule for a proper geometric interpretation of Schrödinger's equation in three-dimensional space: if we look at a clock from the back, then its hand will be

Still, we have not answered the question as to what the physical dimension of the real and imaginary part of our wavefunction should be. At this point, we may be inspired by the structural similarity between Newton's and Coulomb's force laws:

$$F = k_e \frac{q_1 \cdot q_2}{r^2}$$

$$F = G \frac{m_1 \cdot m_2}{r^2}$$

Hence, if the electric field vector **E** is expressed in force per unit *charge* (N/C), then we may want to think of associating the real part of our wavefunction with a force per unit *mass* (N/kg). We can, of course, do a substitution here, because the mass unit (1 kg) is equivalent to 1 N·s^2/m. Hence, our N/kg dimension becomes:

N/kg = N/(N·s^2/m)= m/s^2

What is this: m/s^2? Is *that* the dimension of the $a·cos\theta$ term in the $a·e^{-i\theta} = a·cos\theta - i·a·sin\theta$ wavefunction? My answer is: why not? Think of it: m/s^2 is the physical dimension of *acceleration*: the increase or decrease in velocity (m/s) per second. It ensures the wavefunction for *any* particle – matter-particles or particles with zero rest mass (photons) – and the associated wave *equation* (which has to be the same for all, as the spacetime we live in is *one*) are mutually consistent.

V. Energy densities and flows

Pursuing the geometric equivalence between the equations for an electromagnetic wave and Schrödinger's equation, we can now, perhaps, see if there is an equivalent for the energy density. For an electromagnetic wave, we know that the energy density is given by the following formula:

$$u = \frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E} + \frac{\epsilon_0 \cdot c^2}{2}\mathbf{B}\cdot\mathbf{B}$$

E and **B** are the electric and magnetic field vector respectively. The Poynting vector will give us the directional energy flux, i.e. the energy flow per unit area per unit time. We write:

$$\frac{\partial u}{\partial t} = -\nabla \cdot \boldsymbol{S}$$

Needless to say, the **∇**· operator is the divergence and, therefore, gives us the magnitude of a (vector) field's *source* or *sink* at a given point. To be precise, the divergence gives us the volume density of the outward *flux* of a vector field from an infinitesimal volume around a given point. In this case, it gives us the *volume density* of the flux of ***S***.

We can analyze the dimensions of the equation for the energy density as follows:

1. **E** is measured in *newton per coulomb*, so [**E**·**E**] = [E^2] = N^2/C^2.
2. **B** is measured in (N/C)/(m/s), so we get [**B**·**B**] = [B^2] = (N^2/C^2)·(s^2/m^2). However, the dimension of our c^2 factor is (m^2/s^2) and so we're also left with N^2/C^2.

moving *counter*clockwise. When writing **B** = (1/*c*)·*i*·**E**, we assume we are looking in the *negative* *x*-direction. If we are looking in the positive *x*-direction, we should write: **B** = –(1/*c*)·*i*·**E**. Of course, Nature does not care about our conventions. Hence, both should give the same results in calculations. We will show in a moment they do.

3. The ϵ_0 is the electric constant, aka as the vacuum permittivity. As a *physical* constant, it should ensure the dimensions on both sides of the equation work out, and they do: $[\varepsilon_0] = C^2/(N{\cdot}m^2)$ and, therefore, if we multiply that with N^2/C^2, we find that u is expressed in J/m^3.[20]

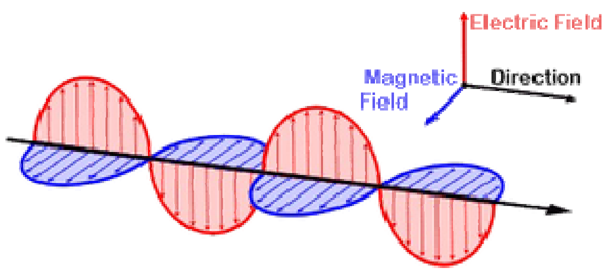
Replacing the *newton per coulomb* unit (N/C) by the *newton per kg* unit (N/kg) in the formulas above should give us the equivalent of the energy density for the wavefunction. We just need to substitute ϵ_0 for an equivalent constant. We will leave it to the reader to work that out.

If the energy densities can be calculated – which are also mass densities, obviously – then the probabilities should be proportional to them. We may to give it a try. Let us first see what we get for a photon, assuming the electromagnetic wave represents its wavefunction. Substituting **B** for $(1/c){\cdot}i{\cdot}\mathbf{E}$ or for $-(1/c){\cdot}i{\cdot}\mathbf{E}$ gives us the following result:

$$u = \frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E} + \frac{\epsilon_0\cdot c^2}{2}\mathbf{B}\cdot\mathbf{B} = \frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E} + \frac{\epsilon_0\cdot c^2}{2}\frac{i\cdot\mathbf{E}}{c}\frac{i\cdot\mathbf{E}}{c} = \frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E} - \frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E} = 0$$

Zero! An unexpected result! Or not? We have not stationary charges or no currents: only an electromagnetic wave in free space. Hence, the local energy conservation principle needs to be respected at all points in space and in time. The geometry makes sense of the result: for an electromagnetic wave, the magnitudes of **E** and **B** reach their maximum, minimum and zero point *simultaneously*, as shown below.[21] This is because their *phase* is the same.

Figure 5: Electromagnetic wave: **E** and **B**



Should we expect a similar result for the energy densities that we would associate with the real and imaginary part of the matter-wave? For the matter-wave, we have a phase difference between $a{\cdot}cos\theta$ and $a{\cdot}sin\theta$, which gives a different picture of the *propagation* of the wave (see Figure 3).[22] In fact, the geometry of the suggestion suggests some inherent spin, which is interesting. Making abstraction of any scaling constants, we can write:

$$u = a^2cos^2\theta + a^2sin^2\theta = a^2\,(cos^2\theta + sin^2\theta) = a^2$$

We get what we hoped to get: the absolute square of our amplitude is, effectively, an energy density !

$$|\psi|^2 = |a{\cdot}e^{-i{\cdot}E{\cdot}t/\hbar}|^2 = a^2 = u$$

[20] In fact, when multiplying $C^2/(N{\cdot}m^2)$ with N^2/C^2, we get N/m^2, but we can multiply this with 1 = m/m to get the desired result. It is significant that an energy density (*joule* per unit *volume*) can also be measured in *newton* (force per unit *area*.

[21] The illustration shows a linearly polarized wave, but the obtained result is general.

[22] The sine and cosine are essentially the same functions, except for the difference in the phase: $sin\theta = cos(\theta-\pi/2)$.

This is very deep. A photon has no rest mass, so it borrows and returns energy from empty space as it travels through it. In contrast, a matter-wave carries energy and, therefore, has some (*rest*) mass. It is therefore associated with an energy density, and this energy density gives us the probabilities. Of course, we need to fine-tune the analysis to account for the fact that we have a wave packet rather than a single wave, but that can easily be done.

As mentioned, the phase difference between the real and imaginary part of our wavefunction (a cosine and a sine function) appear to give some spin to our particle. We do not have this particularity for a photon. Of course, photons are bosons, i.e. spin-zero particles, while elementary matter-particles are fermions with spin-1/2. Hence, our geometric interpretation of the wavefunction suggests that, after all, there may be some more intuitive explanation of the fundamental dichotomy between bosons and fermions, which puzzled even Feynman found baffling: famously commented on:

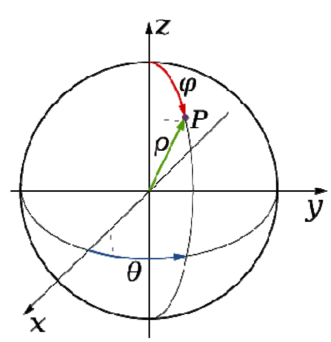
> "Why is it that particles with half-integral spin are Fermi particles, whereas particles with integral spin are Bose particles? We apologize for the fact that we cannot give you an elementary explanation. An explanation has been worked out by Pauli from complicated arguments of quantum field theory and relativity. He has shown that the two must necessarily go together, but we have not been able to find a way of reproducing his arguments on an elementary level. It appears to be one of the few places in physics where there is a rule which can be stated very simply, but for which no one has found a simple and easy explanation. The explanation is deep down in relativistic quantum mechanics. This probably means that we do not have a complete understanding of the fundamental principle involved." (Feynman, *Lectures*, III-4-1)

A geometric interpretation of the wavefunction may, perhaps, provide some better understanding of 'the fundamental principle involved.'

V. Concluding remarks

There are, of course, other ways to look at the matter. For example, we can imagine two-dimensional oscillations as *circular* rather than linear oscillations. Think of a tiny ball, whose center of mass stays where it is, as depicted below. Any rotation – around any axis – will be some combination of a rotation around the two other axes. Hence, we may want to think of a two-dimensional oscillation as an oscillation of a polar and azimuthal angle.

Figure 6: Two-dimensional *circular* movement



The point of this paper is not to make any definite statements. That would be foolish. Its objective is just to challenge the simplistic mainstream viewpoint on the *reality* of the wavefunction. Stating that it is a mathematical construct only without *physical significance* amounts to saying it has no meaning at all. That is, clearly, a non-sustainable proposition.

The interpretation that is offered here looks at amplitude waves as traveling fields. Their physical dimension may be expressed in force per mass unit, as opposed to electromagnetic waves, whose amplitudes are expressed in force per (electric) *charge* unit. Also, the amplitudes of matter-waves incorporate a phase factor, but this may actually explain the rather enigmatic dichotomy between fermions and bosons and is, therefore, an added bonus.

Appendix: The *de Broglie* relations and energy

The 1/2 factor in Schrödinger's equation is related to the concept of the *effective* mass (m_{eff}). It is easy to make the wrong calculations. For example, when playing with the famous *de Broglie* relations – aka as the matter-wave equations – one may be tempted to *derive* the following energy concept:

1. E = h·*f* and p = h/λ. Therefore, *f* = E/h and λ = p/h.
2. *v* = *f*·λ = (E/h)·(p/h) = E/p
3. p = m·*v*. Therefore, E = *v*·p = m·v^2

E = m·v^2? This *resembles* the E = mc^2 equation and, therefore, one may be enthused by the discovery, especially because the m·v^2 also pops up when working with the Least Action Principle in *classical* mechanics, which states that the path that is followed by a particle will minimize the following integral:

$$S = \int_{t_1}^{t_2} (\mathrm{KE} - \mathrm{PE})dt$$

Now, we can choose any reference point for the potential energy but, to reflect the energy conservation law, we can select a reference point that ensures the *sum* of the kinetic and the potential energy is zero *throughout* the time interval. If the force field is uniform, then the integrand will, effectively, be equal to KE − PE = *m*·v^2.[23]

However, that is *classical* mechanics and, therefore, not so relevant in the context of the *de Broglie* equations. The apparent paradox is solved by distinguishing between the *group* and the *phase* velocity of the matter wave.

References

This paper discusses general principles in physics only. Hence, references can be limited to references to physics textbooks only. For ease of reading, any reference to additional material has been limited to a more popular undergrad textbook that can be consulted online: Feynman's Lectures on Physics (http://www.feynmanlectures.caltech.edu). References are per volume, per chapter and per section. For example, Feynman III-19-3 refers to Volume III, Chapter 19, Section 3.

[23] We detailed the mathematical framework and detailed calculations in the following online article: https://readingfeynman.org/2017/09/15/the-principle-of-least-action-re-visited.