We set the following notation.

$K$ a global field
$K_v$ a local field, completion of $K$ at the place $v$ of $K$
$A_K$ the adele ring of $K$
$C_K$ the idele class group $GL_1(A_K)/K^*$
$\hat{C}_K$ the dual group of $C_K$.

We will summarize the spectral interpretation of critical zeros of $L(\chi, s)$ associated $\chi$ of $C_K$ by Alain Connes. Let $h$ be a test function. The Weil explicit formula says

$$\sum_v \int_{K_v} \frac{h(\mu^{-1})}{|1-\mu|} d^* \mu = \hat{h}(0) + \hat{h}(1) - \sum_{L(\chi, \rho) = 0} \hat{h}(\chi, \rho).$$

Suppose that there exists a representation $U$ of $C_K$, and that

$$\text{tr} U(h) = \sum_v \int_{K_v} \frac{h(\mu^{-1})}{|1-\mu|} d^* \mu$$

is satisfied. We see that

$$\text{tr} U(h) = \hat{h}(0) + \hat{h}(1) - \sum_{L(\chi, \rho) = 0} \hat{h}(\chi, \rho).$$
holds. We can say that critical zeros of $L(\chi, s)$ appear as the spectra of the operator $U$. It is just the spectral interpretation of critical zeros of $L(\chi, s)$.

Let

$$X = \Lambda_K/K^*.$$  

The left regular representation $U$ of $C_K$ on $L^2(\delta)(X)$ which is a weighted $L^2$ space can be used to accomplish our task. Namely, it holds that

$$\text{tr}U(h) = \hat{h}(0) + \hat{h}(1) - \sum_{L(\chi, \rho) = 0, \Re \rho = 1/2} \hat{h}(\chi, \rho) + \infty h(1).$$

However we will not try to treat the representation $(U, L^2(\delta)(X))$ directly. Instead of the representation $(U, L^2(\delta)(X))$, we will think of the operator $Q_KU$ where $U$ is the left regular representation of $C_K$ on $L^2(\delta)(X)$. Because, firstly there is a possibility of using some results to compute $\text{Trace} \ Q_KU$, secondly we can eliminate the parameter $\delta$ of $L^2(\delta)(X)$. Now, we can show that

$$\text{Trace} \ Q_KU(h) = \hat{h}(0) + \hat{h}(1) - \sum_{L(\chi, \rho) = 0, \Re \rho = 1/2} \hat{h}(\chi, \rho) + \infty h(1) \quad \Lambda \to \infty$$

for the function $h$ which belongs to Bruhat-Shwartz space $S(C_K)$ of functions on $C_K$.

We try to compute $\text{Trace} \ Q_KU(h)$. This has the relationship to the validity of the Riemann Hypothesis. Suppose that we can compute as follows;

$$\text{Trace} \ Q_KU(h) = 2h(1) \log' \Lambda + \sum_v \int_{K_v} h(\mu^{-1}) \frac{d^* \mu}{|1 - \mu|} + o(1) \quad \Lambda \to \infty$$

where $2 \log' \Lambda = \int_{\lambda \in C_K \cdot [-1, 1]} d^* \lambda$. We obtain a trace formula:

$$\hat{h}(0) + \hat{h}(1) - \sum_{L(\chi, \rho) = 0, \Re \rho = 1/2} \hat{h}(\chi, \rho) + \infty h(1) = 2h(1) \log' \Lambda + \sum_v \int_{K_v} h(\mu^{-1}) \frac{d^* \mu}{|1 - \mu|} + o(1) \quad \Lambda \to \infty.$$  

The left side is spectral and the right side is geometrical. From the Weil explicit formula, we have seen that

$$\sum_v \int_{K_v} h(\mu^{-1}) \frac{d^* \mu}{|1 - \mu|} = \hat{h}(0) + \hat{h}(1) - \sum_{L(\chi, \rho) = 0} \hat{h}(\chi, \rho).$$

Therefore, one obtains that
\[ \sum_{L(\chi, \rho) = 0} \hat{h}(\chi, \rho) = \sum_{L(\chi, \rho) = 0} \hat{h}(\chi, \rho). \]

It means the validity of the Riemann Hypothesis. Conversely, the validity of the Riemann Hypothesis implies that

\[ \text{Trace} Q_\Lambda U(h) = 2h(1) \log' \Lambda + \sum_v \int_{\nu} \frac{h(\mu^{-1})}{|1-\mu|} d^* \mu + o(1) \quad \Lambda \to \infty. \]
We try to characterize L-functions from the view of the representation theory.

We will begin with the local case. Denote the set of the irreducible representations of $K^*$ by $\text{Irr}(K^*)$. Let $(\pi_v, V_{\pi_v})$ be an irreducible representation of $K_v^*$. Put
\[ \pi_v(f) \nu = \int_{K_v^*} f(g)\pi_v(g) d^* g, \quad f \in S(K_v). \]
Suppose that $\text{tr}\pi_v(f)$ can be defined, namely $\pi_v(f)$ is a trace class operator. So we may think that there exists a character $\text{tr}\pi_v$ of $K_v^*$, and
\[ \text{tr}\pi_v(f) = \int_{K_v^*} f(g) \text{tr}\pi_v(g) d^* g. \]
Define the local zeta function as
\[ Z(s, \chi, \Phi) = \int_{K_v^*} \Phi(g)\chi(g)|g|^s d^* g. \]
Here $s \in \mathbb{C}$, $\chi$ is a character of $K_v^*$ and $\Phi \in S(K_v)$. The integral converges at $\text{Re}(s) > 0$. The L-factor $L(s, \chi)$ is defined as $Z(s, \chi, \Phi)/L(s, \chi)$ being entire. We will see that the local zeta function associated with $\pi_v$ can be
\[ Z(s, \text{tr}\pi_v, \Phi) = \int_{K_v^*} \Phi(g)\text{tr}\pi_v(g)|g|^s d^* g. \]
The L-factor $L(s, \pi_v)$ is defined as $Z(s, \text{tr}\pi_v, \Phi)/L(s, \pi_v)$ being entire.

Next, we will think of the global case. It is performed on the adele ring of $K$. Set
\[ \pi = \bigotimes_v \pi_v, \quad V_{\pi} = \bigotimes_v V_{\pi_v}. \]
We can obtain an irreducible representation $(\pi, V_{\pi})$ of $A_K^*$. Denote the set of the irreducible representations of $A_K^*$ by $\text{Irr}(A_K^*)$. Suppose that $\pi(f)$ where $f \in S(A_K)$ is a trace class operator. Then $\text{tr}\pi$ is given as a character of $A_K^*$. We also obtain the global zeta function
\[ Z(s, \text{tr}\pi, \Phi) = \prod_v Z(s, \text{tr}\pi_v, \Phi). \]
Here $\Phi \in S(A_K)$. We define the L-function associated with $\pi$ as follows;
\[ L(s, \pi) = \prod_v L(s, \pi_v). \]
Each $L$-factor $L(s, \pi)$ gives the Euler factor of $L(s, \pi)$, namely $L(s, \pi)$ has the Euler product. The $L(s, \pi)$ satisfies the functional equation which is given by the functional equation of the global zeta function. Thus, $L(s, \pi)$ is analytically continued to the function which is meromorphic in the whole plain $\mathbb{C}$.

We shall consider an irreducible representation $(\pi, V_\pi)$ of $C_K$. Let $\mathcal{H}_\pi$ be a suitable completion of $V_\pi$ with a certain inner product. One obtains a unitary representation $(\pi, \mathcal{H}_\pi)$, which is a left regular representation of $C_K$ on $\mathcal{H}_\pi$. We may say that if $\pi \in \text{Irr}(C_K)$ then $\pi \in \hat{C}_K$. Thus,

$$\mathcal{H} = \bigoplus_{\pi \in \text{Irr}(C_K)} \mathcal{H}_\pi, \quad \mathcal{H}_\pi = \{\xi \mid \xi(g^{-1}x) = \pi(g)\xi(x), \forall g \in C_K\}.$$  

We know that $\text{tr} \pi$ is a character of $C_K$. We frequently use $\chi$ to denote a character of $C_K$. Then, $\text{tr} \pi = \chi$. Correspondingly, $L(s, \pi) = L(s, \chi)$.

Lastly we will mention trace formulae. The trace formula which is given by a zeta function:

$$\underbrace{\cdots}_{\text{Zero points}} = \underbrace{\cdots}_{\text{Geometrical side}}$$

is a prototype. Selberg’s trace formula is that

$$\underbrace{\cdots}_{\text{Eigenvalues of Laplacian}} = \underbrace{\cdots}_{\text{Geometrical side}}.$$  

There exists an operator $M$ such that it is commutative with the Laplacian of $H$. The operator is the integral operator which has $k(z, w)$ as an integral kernel

$$M(f)(z) = \int_H k(z, w)f(w)\,d\mu(w).$$

The Selberg’s trace formula gives the explicit formula of Selberg’s zeta function. The trace formula given by Connes is the same type as Selberg’s. It is that

$$\underbrace{\cdots}_{\text{Characters}} = \underbrace{\cdots}_{\text{Geometrical side}}.$$  

Here $U(h): C_c^\infty(X) \to C_c^\infty(X)$

$$(U(h)\xi)(x) = \int_{C_K} h(g)(U(g)\xi)(x)d^*g.$$  

The operator $U(h)$ is the integral operator which has $k_h(x, y)$ as an integral kernel

$$(U(h)\xi)(x) = \int_{C_K} k_h(x, y)\xi(y)d^*y.$$
The space $S(A_K)_0$ is given as the codimension 2 subspace of $S(A_K)$ such that

$$f(0) = 0, \quad \int_X f(x) dx = 0.$$ 

Let $L^2(X)_0$ be the completion of $S(A_K)_0$. We obtain an exact sequence:

$$0 \to L^2(X)_0 \to L^2(X) \to \mathbb{C} \oplus \mathbb{C}(1) \to 0$$

where $\mathbb{C} \oplus \mathbb{C}(1) \cong L^2(X)/L^2(X)_0$.

[Remark] $\mathbb{C}$ is a trivial $C_K$ module:

$$T(g)\lambda = \lambda \quad g \in C_K, \lambda \in \mathbb{C}.$$ 

$\mathbb{C}(1)$ is Tate twist:

$$T(g)\lambda = |g|\lambda \quad g \in C_K, \lambda \in \mathbb{C}.$$ 

Here we have to give one’s attention to the space $X$. The space $X$ is a delicate quotient space. It must be non-compact. It must be also questionable to think that $X$ contains $C_K$ as a subspace. However, considering the construction of $L^2(C_K)$, if we restrict the function in $L^2(X)$ to $C_K$ then it can be a function on $C_K$. We can also obtain the following exact sequence:

$$0 \to L^2(X)_0 \overset{T}{\to} L^2(C_K) \to \mathcal{H} \to 0$$

where $\mathcal{H} \cong L^2(C_K)/\text{Im}(T)$. Let $U$ be a left regular representation of $C_K$ on $L^2(X, dx)$ and $V$ be a left regular representation of $C_K$ on $L^2(C_K, d'x)$. For $f(x) \in L^2(X, dx)$, let $(Tf)(a)$ be the restriction of $f(x)$ to $C_K$. Then,

$$(Tf)(a) = |a|^{1/2} f(a) \quad \forall a \in C_K.$$ 

Since $dx = |x|d'x$, we will understand that $(Tf)(a) \in L^2(C_K, d'x)$. Set

$$(U(g)f)(x) = f(g^{-1}x) \quad \forall g \in C_K, x \in X.$$ 

It turns out that

$$T(U(g)f)(a) = \text{the restriction of } f(g^{-1}x)$$

$$= |g|^{1/2}(V(g)Tf)(a) \quad \forall a, g \in C_K.$$
From this equation, it is that $|g|^{-1/2} T(U(g)f)(a) = V(g)(Tf)(a)$. For $(Tf)(a)$,

$$V(g)(Tf)(a) = \text{the restriction of } |g|^{-1/2} f(g^{-1}x) = |a|^{1/2} |g|^{-1/2} f(g^{-1}a).$$

From $f \in S(A_K)$, we will see that $|g|^{-1/2} f(x) \in L^2(X)_0$, and that $|g|^{-1/2} f(g^{-1}x) \in L^2(X)_0$. Thus $V(\text{Im}(T)) \subseteq \text{Im}(T)$, namely $\text{Im}(T)$ is an invariant subspace for $V$. Now, we have to turn one’s attention to using $L^2(C_K)$. Because $C_K$ is abelian locally compact, we can’t always decompose $L^2(C_K)$ in the direct sum of finite dimensional subspaces. This fact, $L^2(C_K)$ having no finite dimensional subrepresentation, is an obstacle to our attempt computing the trace of $U$.

“The second subtle point is that since $C_K$ is abelian and non compact, its regular representation does not contain any finite dimensional subrepresentation so that the Pólya-Hilbert space cannot be a subrepresentation (or unitary quotient) of $V$. There is an easy way out which is to replace $L^2(C_K)$ by $L^2_\delta(C_K)$ using the polynomial weight $(\log^2 |a|)^{\delta/2}$, i.e. the norm $\|\xi\|_{\delta}^2 = \int_{C_K} |\xi(a)|^2 (1 + \log^2 |a|)^{\delta/2} da$.” in A. Connes [2].

Because $L^2_\delta(C_K)$ is a weighted $L^2$ space, we can decompose it in the direct sum of finite dimensional subspaces. Let the Hilbert space $L^2_\delta(X)$ ($\delta > 1$) be the space of functions on $X$ with the square norm

$$\|f\|_{\delta}^2 = \int_X |f(x)|^2 (1 + (\log |x|)^2)^{\delta/2} dx.$$

The Hilbert space $L^2_\delta(C_K)$ is obtained from the space of functions with the square norm

$$\|f\|_{\delta}^2 = \int_{C_K} |f(g)|^2 (1 + (\log |g|)^2)^{\delta/2} d^* g$$

where we normalize the Haar measure of the multiplicative group $C_K$

$$\int_{|g| \in [1, \Lambda]} d^* g \sim \log \Lambda \quad \Lambda \to +\infty.$$

We understand that the representation $(V, L^2_\delta(C_K))$ isn’t unitary because of the suffix $(1 + (\log |g|)^2)^{\delta/2}$. However

$$\|V(a)\|_{\delta} = O((\log |a|)^{\delta/2}) \quad |a| \to \infty.$$

It is also satisfied that

$$\|V(a)\|_{\delta} = O((\log |a|)^{\delta/2}) \quad |a| \to 0.$$
[Remark] It holds that
\[ \|V(a)\|_6 \leq (c \cdot (1 + (\log|a|)^2) \delta/2)^{1/2}. \]
Here we may say that \( \|V(a)\|_6 \geq 0 \). We can compute as follows;
\[ \|V(a)\|_6^2 \leq c \cdot (1 + (\log|a|)^2)^{\delta/2} \]
moreover,
\[ \|V(a)\|_6^{4\delta} \leq c^{4\delta} \cdot (1 + (\log|a|)^2). \]
Thus,
\[ \frac{\|V(a)\|_6^{4\delta}}{(\log|a|)^2} \leq c^{4\delta} \cdot \frac{1 + (\log|a|)^2}{(\log|a|)^{\delta/2}}. \]
It turns out that
\[ \frac{\|V(a)\|_6^{4\delta}}{(\log|a|)^{\delta/2}} \leq c^{4\delta} \quad |a| \to \infty \quad \text{and} \quad \frac{\|V(a)\|_6^{4\delta}}{(\log|a|)^{\delta/2}} \leq c^{4\delta} \quad |a| \to 0. \]
We can show that
\[ \frac{\|V(a)\|_6^{4\delta}}{(\log|a|)^{\delta/2}} = \left( \frac{\|V(a)\|_6}{(\log|a|)^{\delta/2}} \right)^{4\delta}. \]
Therefore,
\[ \frac{\|V(a)\|_6}{(\log|a|)^{\delta/2}} \leq c \quad |a| \to \infty \quad \text{and} \quad \frac{\|V(a)\|_6}{(\log|a|)^{\delta/2}} \leq c \quad |a| \to 0. \]

We have a following decomposition:
\[ C_K \cong C_{K,1} \times N. \]
Here \( C_{K,1} \) is the maximal compact subgroup: \( \{ g \in C_K \mid |g| = 1 \} \) and \( N = \mathbb{R}^*_{>0} \). Let \( \chi_0 \) be a character of \( C_{K,1} \). We use \( \tilde{\chi}_0 \) to denote an extension of \( \chi_0 \) as a character of \( C_K \). Namely, \( \tilde{\chi}_0 (g) = \chi_0 (g); \forall g \in C_{K,1} \). Here \( \tilde{\chi}_0 \) has the form \( \tilde{\chi}_0 = \chi_0 \cdot \rho \cdot \rho \in i \mathbb{R} \).
Restrict \( V \) to \( C_{K,1} \), one decompose \( L^2_{-\delta} (C_K) \) in the direct sum of the finite dimensional subspaces,
\[ L^2_{-\delta, \chi_0} = \left\{ \xi \in L^2_{-\delta} (C_K) \mid \xi \xi^{-1} a (g) = \chi_0 (a) \xi (g) \quad \forall g \in C_K \quad \forall a \in C_{K,1} \right\}. \]
The dual space \( (L^2_{-\delta} (C_K))^* \) of \( L^2_{-\delta} (C_K) \) can be identified with \( L^2_{\delta} (C_K) \). It is also decomposed in the direct sum of the subspaces,
\[ L^2_{\delta, \chi_0} = \left\{ \xi \in L^2_{\delta} (C_K) \mid \xi \xi^{-1} a (g) = \chi_0 (a) \xi (g) \quad \forall g \in C_K \quad \forall a \in C_{K,1} \right\}. \]
Here, we use the transposed of \( V \)
\[(V^\tau(a)\eta)(x) = \eta(ax); \quad \eta(x) \in (L^2_\delta(C_K))^*.\]

The pairing between \(L^2_\delta(C_K)\) and its dual \((L^2_\delta(C_K))^* = L^2_{-\delta}(C_K)\) is given by

\[
\langle f, \eta \rangle = \int_{c_k} f(x)\eta(x)d^*x.
\]

We can obtain the following exact sequences:

\[0 \to L^2_\delta(X)_0 \to L^2_\delta(C_K) \to \mathcal{H} \to 0.\]

Let

\[
\text{Im}(T)^0 = \left\{ \eta \in (L^2_\delta(C_K))^* \mid \langle Tf, \eta \rangle = 0, \forall f \in S(A_K)_0 \right\}.
\]

It holds that

\[
\eta(x) \in \text{Im}(T)^0 \iff \int_{c_k} Tf(a)\eta(a)d^*a = 0, \forall f \in S(A_K)_0.
\]

For \(\eta(x) \in L^2_{-\delta, x_0}\), we may think that it has the form:

\[
\eta(x) = \tilde{\chi}_0(x)\Psi(|x|).
\]

Now

\[
\Psi(|x|) = \int_{-\infty}^{\infty} \hat{\Psi}(t)|x|^a dt
\]

where \(\hat{\Psi}(t) = \int_{c_k} \Psi(a)|a|^a d^*a\). Thus,

\[
\eta(x) = \int_{-\infty}^{\infty} \eta(x; t)dt; \quad \eta(x; t) = \tilde{\chi}_0(x)|x|^a \hat{\Psi}(t).
\]

Then,

\[
\eta(x) \in \text{Im}(T)^0 \iff \langle Tf, \eta \rangle = 0
\]

\[
\iff \int_{c_k} Tf(a)\int_{-\infty}^{\infty} \hat{\chi}_0(a)|a|^a \hat{\Psi}(t)dt d^*a = 0
\]

\[
\iff \int_{-\infty}^{\infty} \int_{c_k} Tf(a)\hat{\chi}_0(a)|a|^a \hat{\Psi}(t)d^*a dt = 0, \forall f \in S(A_K)_0.
\]

As the consequence of Tate’s work,
Lemma 2.1. For $\text{Re}(s) > 0$, and any character $\chi_0$ of $C_K$,

$$\int_{\mathcal{C}_K} Tf(a)\chi_0(a)|d|^{-\frac{1}{2}}d^*a = L(\chi_0, s)D'_s(f), \text{ } \forall f \in \mathcal{S}(A_K).$$

Here, $D'_s(f)$ is a holomorphic function of $s$ ($\text{Re}(s) > 0$).

From this lemma, we can say that

$$\eta(x) \in \text{Im}(T)^0 \iff L(\tilde{\chi}_0, 1/2 + \rho) = 0; \text{ } \rho \in i\mathbb{R}.$$ 

Here $\mathcal{H} \cong L^2(C_K)/\text{Im}(T)$. Think of the left regular representation $W$ of $C_K$ on $\mathcal{H}$: $(W, \mathcal{H})$, where one deduces $W$ from $V$. Restrict $W$ to $C_{K,1}$, one decompose $\mathcal{H}$ in the direct sum of the subspaces,

$$\mathcal{H} = \bigoplus_{\chi \in C_{K,1}} d(\chi)\mathcal{H}_{\chi} \oplus d(\chi_0)\mathcal{H}_{\chi_0}$$

where $\mathcal{H}_{\chi_0} = \{ \xi|W(a)\xi = \chi_0(a)\xi, \forall a \in C_{K,1}\}$ and we denote the dimension of $\mathcal{H}_{\chi_0}$ by $d(\chi_0)$. We will also consider its dual. We obtain the transposition $W^\tau$ of $C_K$ on $\mathcal{H}^\tau$: $(W^\tau, \mathcal{H}^\tau)$, where one deduces $W^\tau$ from $V^\tau$. Now, let $h$ be a test function on $C_K$ and set

$$W(h) = \int_{\mathcal{C}_K} h(g)W(g)\; d^*g.$$ 

Denote $h'$s Fourier transform by $\hat{h}$:

$$\hat{h}(\chi, z) = \int_{\mathcal{C}_K} h(\mu)\chi(\mu)|\mu|^z d^*\mu.$$ 

Recall

$$\mathcal{H}^\tau \cong (L^2(C_K)/\text{Im}(T))^\tau \cong \text{Im}(T)^0,$$

moreover

$$\text{tr}W = \text{tr}W^\tau.$$ 

We can compute

$$\int_{\mathcal{C}_K} h(g)(V^\tau(g)\eta)(x)\; d^*g = \int_{\mathcal{C}_K} h(g)(V^\tau(g)\int_{-\infty}^{\infty} \eta(\cdot; t)dt)(x)\; d^*g$$

$$= \int_{\mathcal{C}_K} \int_{-\infty}^{\infty} h(g)\tilde{\chi}_0(g)|g|^z\tilde{\chi}_0(x)|x|^\rho \hat{\Psi}(t)\; dt\; d^*g$$
\[
= \int_{-\infty}^{\infty} \hat{h}(\tilde{\chi}_0, it) \tilde{\chi}_0(x) |x|^{\alpha} \hat{\Psi}(t) dt,
\]
thus
\[
\langle Tf, (V^\tau(h)\eta)(x) \rangle = \int_{C_k} Tf(a) \int_{-\infty}^{\infty} \hat{h}(\tilde{\chi}_0, it) \tilde{\chi}_0(a) |a|^{\alpha} \hat{\Psi}(t) dt d^* a
= \int_{-\infty}^{\infty} \int_{C_k} Tf(a) \tilde{\chi}_0(a) |a|^{\alpha} \hat{h}(\tilde{\chi}_0, it) \hat{\Psi}(t) d^* a dt.
\]

If \( \eta(x) \in \text{Im}(T)^0 \) then \( \langle Tf, (V^\tau(h)\eta)(x) \rangle = 0 \), for all \( f \in S(\mathcal{A}_K)_0 \). Therefore, \( (V^\tau(h)\eta)(x) \in \text{Im}(T)^0 \). The above computation shows that
\[
\hat{h}(\tilde{\chi}_0, \rho) L(\tilde{\chi}_0, 1/2 + \rho) = 0; \ \rho \in i\mathbb{R}.
\]
So, we see that
\[
\text{tr} W(h) = \sum_{L(\tilde{\chi}_0, 1/2 + \rho) = 0} \hat{h}(\tilde{\chi}_0, \rho).
\]

Let \( \chi_0 \in \hat{C} \). Recall \( \tilde{\chi}_0 = \chi_0 | \cdot |^\rho (\rho \in \mathbb{C}) \). The action of \( C_k \) on \( \mathcal{H}_{\chi_0} \) can be \( W(g)\xi = \tilde{\chi}_0 (g)\xi \), and it turns out that \( W(g)\xi = |g|^\rho \xi; \ g \in N \). So it is satisfied that
\[
|g|^{\text{Re}(\rho)} \leq ||W(g)||_g, \ g \in C_k.
\]

Let \( W_{\chi_0} = W \mid _{\mathcal{H}_{\chi_0}} \) and \( e^t = g \ (g \in N) \). We will rewrite the action of \( N \) on \( \mathcal{H}_{\chi_0} \) as
\[
W_{\chi_0}(e^t): \mathbb{R} \rightarrow \mathcal{H}_{\chi_0}.
\]
The following things
\[\begin{array}{ll}
(a) & W_{\chi_0}(e^0) = 1, \\
(b) & W_{\chi_0}(e^{t+s}) = W_{\chi_0}(e^t)W_{\chi_0}(e^s)
\end{array}\]
are satisfied. Thus \( W_{\chi_0}(e^t) \) is a semi-group. From the theory of semi-group, we can say that
\[
W_{\chi_0}(e^t) = e^{D_{\chi_0} t}
\]
where
\[
D_{\chi_0} \xi = \lim_{t \to 0} \frac{W_{\chi_0}(e^t)\xi - W_{\chi_0}(e^0)\xi}{t}, \ \xi \in \mathcal{H}_{\chi_0}.
\]
The operator \( D_{\chi_0} \) has discrete spectra. We may think that the discrete spectrum is given by the element \( \xi \) which belongs to \( \text{Im}(T)^0 \).
Let \( \tilde{\chi}_0 \) be the unique extension of \( \chi_0 \in \hat{C}_{K,1} \) to \( C_K \) which is equal to 1 on \( N \). We see that \( \chi = \tilde{\chi}_0 | \cdot |^{i t_0} (t_0 \in \mathbb{R}) \) for \( \chi \in \hat{C}_K \). Then \( L(\chi, 1/2 + it) = L(\tilde{\chi}_0, 1/2 + i(t_0 + t)) \). Thus, as the extension of \( \chi_0 \), we will use the above unique extension \( \tilde{\chi}_0 \).

Theorem 2.1. \( \chi_0 \in \hat{C}_{K,1}, \delta > 1 \). Then \( D_{\chi_0} \) has discrete spectra, \( \text{sp} D_{\chi_0} \subset i\mathbb{R} \) is the set of imaginary parts of zeros of the \( L \) function with Grossencharacter \( \tilde{\chi}_0 \) which have real part equal to 1/2;

\[
\rho \in \text{sp} D \iff L(\tilde{\chi}_0, 1/2 + \rho) = 0 \text{ and } \rho \in i\mathbb{R}, \text{ where } \tilde{\chi}_0 \text{ is the unique extension of } \chi_0 \text{ to } C_K \text{ which is equal to 1 on } N.
\]

Moreover the multiplicity of \( \rho \) in \( \text{sp} D \) is equal to the largest integer of \( n < \frac{1 + \delta}{2} \), \( n \leq \text{multiplicity of } 1/2 + \rho \) as a zero of \( L \).

The action of \( N \) is that

\[
W_{\chi_0}(e^t)\xi = |e^t|^{\rho} \xi = e^{\rho t} \xi.
\]

Then,

\[
D_{\chi_0} \xi = \left. \frac{dW_{\chi_0}(e^t)\xi}{dt} \right|_{t=0} = \rho \xi.
\]

Therefore, \( \rho \) is the spectrum of \( D_{\chi_0} \). Consider

\[
\lim_{|t| \to \infty} \frac{|g|^\alpha}{|\log |g||} = \infty \quad (\alpha > 0) \quad \text{and} \quad \lim_{|t| \to \infty} \frac{|g|^\alpha}{|\log |g||} = \infty \quad (\alpha < 0).
\]

Because \( |g|^{\text{Re}(\rho)} \leq ||W(g)||_6 \); \( g \in C_K \), if \( \text{Re}(\rho) > 0 \) or \( \text{Re}(\rho) < 0 \) then each of them conflicts with

\[
||V(\alpha)||_6 = O((\log |\alpha|)^{\delta/2}) \quad |\alpha| \to \infty
\]

or

\[
||V(\alpha)||_6 = O((\log |\alpha|)^{\delta/2}) \quad |\alpha| \to 0
\]

Therefore, it is that \( \rho = it \) (\( t \in \mathbb{R} \)). Thus,

\[
\tilde{\chi}_0 = \chi_0 | \cdot |^{it} \quad t \in \mathbb{R}.
\]
We see that $D_{X_0}$ has a purely imaginary spectrum, so we obtain the following corollary.

**Corollary 2.2.** For any Shvarzt function $h \in S(C_K)$ the operator $\int_{C_K} h(g)W(g) \, d^* g$ in $H$ is of trace class, and its trace is given by

$$\text{tr} W(h) = \sum_{\mu \in \hat{H}, \text{pair}} \hat{h}(\hat{\chi}_0, \rho)$$

where the multiplicity is counted as in Theorem 2.1. and where Fourier transform $\hat{h}$ of $h$ is defined by $\hat{h}(\chi, z) = \int_{C_K} h(\mu) \chi(\mu) |\mu|^z d^* \mu$.

We can obtain the following exact sequences:

$$0 \to L^2_{\delta}(X)_0 \to L^2_{\delta}(X) \to \mathbb{C} \oplus \mathbb{C}(1) \to 0$$

and

$$0 \to L^2_{\delta}(X)_0 \to L^2_{\delta}(C_K) \to \mathcal{H} \to 0.$$  

We will compute $\text{tr} U(h)$ for $(U, L^2_{\delta}(X))$ from spectral side. From the above first sequence, considering *Lefchetz formula*, we will see that $$A = \text{tr} U|_{L^2_{\delta}(X)_0} - \text{tr} U|_{L^2_{\delta}(X)} + \text{tr} U|_{\mathbb{C} \oplus \mathbb{C}(1)}.$$  

From the second sequence, we will obtain $$A' = \text{tr} U|_{L^2_{\delta}(X)_0} - \text{tr} U|_{L^2_{\delta}(C_K)} + \text{tr} U|_{\mathcal{H}}.$$  

Therefore, it is satisfied that

$$\text{tr} U|_{L^2_{\delta}(X)} = \text{tr} U|_{\mathbb{C} \oplus \mathbb{C}(1)} - \text{tr} U|_{\mathcal{H}} + \text{tr} U|_{L^2_{\delta}(C_K)} + A' - A.$$  

We try to compute $\text{tr} U(h)$ spectrally. Here, $$U(h) = \int_{C_K} h(g)U(g) \, d^* g.$$  

The first term $\text{tr} U|_{\mathbb{C} \oplus \mathbb{C}(1)}$ gives $$\hat{h}(0) + \hat{h}(1).$$  

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Considering that $T(U(g)\xi)(a) = |g|^{1/2}(V(g)T\xi)(a)$,

$U\big|_{L^2_\delta(C_k)}$ is $(|\cdot|^{1/2}V, L^2_\delta(C_k))$ and $U\big|_{H}$ is $(|\cdot|^{1/2}V, \text{Im}(T)^0)$.

So we will understand that the second term gives

$$\sum_{\hat{\chi}_{0}, \rho} \hat{h}(\tilde{\chi}_{0}, \rho) .$$

Finally, the term $\text{tr}U\big|_{L^2_\delta(C_k)} + A' - A$ gives $\infty h(1)$. Thus,

$$\text{tr}U(h) = \hat{h}(0) + \hat{h}(1) - \sum_{\hat{\chi}_{0}, \rho} \hat{h}(\tilde{\chi}_{0}, \rho) + \infty h(1) .$$
We try to compute $\text{tr} U$ geometrically.

Let $S$ be a finite set of places of $K$ containing all infinite places. Set

$$A_S = \prod_{v \in S} K_v \times \prod_{v \notin S} R_v \quad \text{and} \quad J_S = \prod_{v \in S} K_v^* \times \prod_{v \notin S} R_v^*$$

where $R_v$ is the ring of integers of $K_v$. The $S$-units of $K$ is given by

$$O^*_S = J_S \cap K^*.$$ 

The idele class $C_K$ is embedded in $C_S = J_S/O^*_S$ and $X_S = A_S/O^*_S$ plays the same role as $X$. We will think of $L^2(X_S)$ which is obtained by a completion of $S(A_S)$. Let

$$R_A = \hat{P}_A P_A, \quad \Lambda \in \mathbb{R}_+.$$ 

Here $P_A$ is the orthogonal projection onto the subspace,

$$P_A = \left\{ \xi \in L^2(X_S) \mid \xi(x) = 0, \ \forall x, |x| > \Lambda \right\}$$

while $\hat{P}_A = F P_A F^{-1}$ where $F$ is the Fourier transform.

**Theorem 3.1.** For any $h \in S_c(C_S)$, one has

$$\text{Trace}(R_A U(h)) = 2 \log'(\Lambda) h(1) + \sum_{v \in S} \int_{K_v^*} \frac{h(\mu^{-1})}{|1 - \mu|} d^* \mu + o(1) \quad \Lambda \to \infty,$$

where $2 \log'(\Lambda) = \int_{\lambda \in C_S, \lambda \in [\Lambda^{-1}, \Lambda]} d^* \lambda$.

Let $\chi_0$ be a character of $C_{S,1}$ which is the subgroup: $\{ g \in C_S \mid |g| = 1 \}$. The Hilbert space $L^2(X_S)$ is decomposed in the subspace,

$$L^2_{\chi_0} = \left\{ \xi \in L^2(X_S) \mid \xi(a^{-1}x) = \chi_0(a) \xi(x) \quad \forall x \in X_S, a \in C_{S,1} \right\}.$$ 

Let $\mathcal{U}_S$ be the image in $C_S$ of the open subgroup $\prod R_v^*$. Fix a character $\chi$ of $\mathcal{U}_S$, and think of $\chi_0$ whose restriction to $\mathcal{U}_S$ is equal to $\chi$. Set
\[ L^2(X_S) = \left\{ \xi \in L^2(X_S) \mid \xi(a^{-1}x) = \chi(a)\xi(x) \quad \forall x \in X_S, \ a \in U_s \right\}. \]

We can find \( h_x \in S(C_S) \) such that
\[
\text{Supp}(h_x) = U_s \quad h_x(x) = \lambda \chi(x) \quad \forall x \in U_s
\]
where the constant \( \lambda \) is determined by corresponding normalization of the Haar measure on \( C_S \).

Let \( B_A = \text{Im}(P_A) \cap \text{Im}(\hat{P}_A) \) be the intersection of the ranges of the projection \( P_A \) and \( \hat{P}_A \). We will think of \( B_A^\chi \) which is the intersection of \( B_A \) with \( L^2(X_S) \). For each character \( \chi \) of \( U_s \), we can find a vector \( \eta_\chi \in L^2(X_S) \) such that
\[
U(g)(\eta_\chi) \in B_A \quad g \in C_S, \Lambda^{-1} \leq |g| \leq \Lambda.
\]
Then \( B_A^\chi \) is given as the linear span of \( U(g)(\eta_\chi) \):
\[
B_A^\chi = \sum_{g \in D_h, \mu \in (A^1, A)} \lambda_\mu U(g)(\eta_\chi) \quad D_S = C_S/U_s.
\]
Set
\[
(B_A^\chi)^0 = \text{The whole of sum}_{g \in D_h, \mu \in (A^1, A)} \lambda_\mu U(g)(\eta_\chi).
\]
It turns out that \((B_A^\chi)^0 \subseteq B_A^\chi \subseteq L^2(X_S)_\chi\). We may say that \((B_A^\chi)^0 \) is dense in \( B_A^\chi \).

So, from the compactness of \( \left\{ g \in C_S \mid \Lambda^{-1} \leq |g| \leq \Lambda \right\} \), we can consider that \( B_A^\chi \) is a vector space which has a countable basis at most. It must be hard to show that \( B_A^\chi = L^2(X_S)_\chi \) for sufficient large \( \Lambda \). We will replace \( R_A \) by the orthogonal projection \( Q_A \) on \( \text{Im}(P_A) \cap \text{Im}(\hat{P}_A) \). Suppose that \( B_A^\chi = L^2(X_S)_\chi \) for sufficient large \( \Lambda \).

Then we can identify tr\( R_A U \) with tr\( Q_A U \) of \((U, L^2(X_S))\). From the Theorem 3.1., we can show the following.

**Corollary.** Let \( Q_A \) be the orthogonal projection on the subspace of \( L^2(X_S) \) spanned by the \( f \in S(A_S) \), which vanish as well as Fourier transform for \(|x| > \Lambda \). Let \( h \in S(C_S) \) have compact support. Then when \( \Lambda \to \infty \), one has
\[
\text{Trace}(Q_A U(h)) = 2h(1) \log'(\Lambda) + \sum_{\nu \in \Lambda} \int_{\lambda \in C_S, \lambda \in (A^1, A)} \frac{h(\mu^{-1})}{|1 - \mu|} d^* \mu + o(1)
\]
where \( 2 \log'(\Lambda) = \int_{\lambda \in C_S, \lambda \in (A^1, A)} d^* \lambda \).
We can get from the above corollary an $S$-independent global formulation:

\[
\text{Trace} Q_A U(h) = 2h(1) \log \Lambda + \sum_{\nu} \int_{k, \nu} h(\mu^{-1}) d\mu + o(1) \quad \Lambda \to \infty
\]

where $Q_A U$ is a trace class operator for $(U, L^2(X))$.

In order to obtain the identification of $\text{tr} Q_A U$ with $\text{tr} R_A U$, we have to show that $B_A^X = L^2(X_S X)$ for sufficient large $A$. If $C_S$ is compact then the compactness must be sufficient for us to show the equation. If $C_K$ were compact, we could show the validity of the Riemann hypothesis.

So it must be interesting to think of the compactification of $C_K$. With this interest, we will examine the space $Y = A_K/K$. As the same way in the case of $X$, we can obtain $L^2(Y)$ and $L^2(Y)_0$. We will think of the case $K = Q$. It holds that

\[
A_Q = \prod_{\mu < \infty} Z_{\mu} \times [0, 1] + Q \quad \text{and} \quad A_Q^* = (\prod_{\mu < \infty} Z_{\mu}^\ast \times \mathbb{R}^*_0) \cdot Q^\ast.
\]

Thus, it turns out that

\[
Y = A_Q/Q \cong \prod_{\mu < \infty} Z_{\mu} \times [0, 1] \quad \text{and} \quad C_Q = A_Q^*/Q^\ast \cong \prod_{\mu < \infty} Z_{\mu}^\ast \times \mathbb{R}^*_0.
\]

Think of $r \mapsto 2/\pi \tan(r)^{-1}; r \in \mathbb{R}^*_0$, it must be allowed to say that

$\mathbb{R}^*_0$ is embedded in $[0, 1]$.

Thus,

$C_Q$ is embedded in $\prod_{\mu < \infty} Z_{\mu}^\ast \times [0, 1]$.

It may be allowed to say that

\[
Y = \{x \in Y \mid |x| < 1\} \cup \{x \in Y \mid |x| = 1\}
\]

and that $\{x \in Y \mid |x| = 1\}$ consists of the boundary of $Y$. Denote it by $\partial Y$. It must correspond to $\prod_{\mu < \infty} Z_{\mu}^\ast \times \{1\}$. Let

\[
C_Q = \prod_{\mu < \infty} Z_{\mu}^\ast \times (0, 1).
\]

We will think that $C_Q$ is the compactification of $C_Q$. We expect that $C_Q$ fills the same role of $C_Q$. 

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We can obtain an exact sequence:

\[ 0 \to L^2(Y)_0 \xrightarrow{T} L^2(C_0) \to \mathcal{H} \to 0 \]

where \( \mathcal{H} \cong L^2(C_0)/\text{Im}(T) \). Let \( U \) be a left regular representation of \( C_0 \) on \( L^2(Y, \, dx) \) and \( V \) be a left regular representation of \( C_0 \) on \( L^2(C_0, \, d^*x) \). One deduces the left regular representation \( W \) of \( C_0 \) on \( \mathcal{H} \) from \( V \). One may be allowed to say that \( C_0 \) is compact because it must be complete and totally bounded. So one can decompose \( L^2(C_0) \) in the direct sum of 1-dimensional subspaces,

\[ L^2_{\chi_0} = \{ \xi \in L^2(C_0) \mid \xi(a^{-1}g) = \chi_0(a)\xi(g) \quad \forall g, a \in C_0 \} \]

The dual space \( (L^2(C_0))^* \) of \( L^2(C_0) \) can be identified with \( L^2(C_0) \).

[Remark] The left regular representation \( U \) of \( C_0 \) on \( L^2(Y, \, dx) \) isn’t unitary. But the left regular representation \( T \) of \( Y \) on \( L^2(Y, \, dx) \):

\[ (T(g)\xi)(x) = \xi(-g+x) \quad g, x \in Y \]

is unitary. Because \( Y \) is abelian and compact, we obtain the following decomposition:

\[ L^2(Y) = \bigoplus_{\chi \in \hat{Y}} L^2_{\chi(Y)} \quad T_{\chi} = T|_{L^2_{\chi(Y)}} \]

where \( T_{\chi} \) is 1-dimensional representation.

Here \( Y \) is compact. Thus the following formula:

\[ \text{tr}U|_{L^2(Y)_0} = \text{tr}U|_{L^2(C_0)} - \text{tr}U|_{\mathcal{H}} + A \]

becomes meaningful.

Now, our problem is to compute \( \text{tr}U|_{L^2(Y)_0} \). Basically, we may think that this problem is how to construct \( L^2(Y)_0 \). Set

\[ \Delta = |x|^2 \frac{d^2}{dx^2} \]

which is a differential operator on \( Y \). We shall think of the eigenvalue problems:

\[ \Delta \xi - \lambda \xi = 0, \quad \xi(x) = 0 \quad \text{on} \, \partial Y \]

on the analogy of Sturm-Liouville problem. Recall that the action of \( C_0 \) on the functions on \( Y \) is
\[(U(g)\phi)(x) = \phi(g^{-1}x) \quad \forall g \in C_\Omega, x \in Y.\]

It turns out that \(U(g)\) and \(\Delta\) are commutative. Hence they share the same eigenspace. We try to construct the \(L^2(Y)_0\) space as the space of eigenfunctions of \(\Delta\).

[Remark] One computes

\[(U(g)|x|^2 \frac{d^2}{dx^2} \phi)(x) = (U(g) |x|^2 \phi'')(x) = |g^{-1}x|^2 \phi''(g^{-1}x).\]

It holds that \(dgx = \left|g\right| dx\), so

\[|x|^2 \frac{d^2}{dx^2} (U(g)\phi)(x) = |x|^2 \frac{d^2}{dx^2} \phi(g^{-1}x) = |x|^2 \frac{d^2 g^{-1}x}{d(g^{-1}x)} \frac{d^2}{d(g^{-1}x)} \phi(g^{-1}x)\]

\[= |g^{-1}x|^2 \phi''(g^{-1}x).\]

[Remark] The \(\Delta\) becomes a differential operator on \(X\). Since \(dg^{-1}x = \left|g^{-1}\right| dx\), if one restricts \(U\) to \(C_{K,1}\) then \(\left|d(g^{-1}x)\right| = |dx|\). Namely, \(|dx|\) is invariant under the action \(U(g)\); \(\forall g \in C_{K,1}\). Thus, the Laplacian \(\frac{d^2}{dx^2}\) is \(C_{K,1}\)–invariant.

If we can show that the Laplacian \(\frac{d^2}{dx^2}\) is \(C_K\)–invariant then we can say that \(U(g); \forall g \in C_K\) is isometry, namely unitary. On the other hand it does not always mean that \(U(g); \forall g \in C_K\) is unitary.

Considering \(\xi'' = \lambda \frac{\xi}{|x|^2}\), it turns out that

\[(\xi'\bar{\xi})' = \xi''\bar{\xi} + \xi'\bar{\xi}' = \lambda \frac{\xi\bar{\xi}}{|x|^2} + \xi'\bar{\xi}'.\]

One compute

\[\int_Y \xi'\bar{\xi}' dx + \lambda \int_Y \frac{\xi}{|x|^2} \bar{\xi}' dx = \int_Y (\xi'\bar{\xi})' dx.\]

We can write that \(\int_Y (\xi'\bar{\xi})' dx = \int_{\partial Y} \xi'\bar{\xi} dx\). From the boundary condition, it holds that

\[\int_{\partial Y} \xi'\bar{\xi} dx = 0.\]

Therefore, we obtain

\[\int_Y \xi'\bar{\xi}' dx = -\lambda \int_Y \frac{\xi}{|x|^2} \bar{\xi} dx.\]
Here $\int \xi' \bar{\xi'} dx, \int \frac{\xi \bar{\xi}}{|x|^\alpha} dx \geq 0$. Thus, $\lambda \leq 0$. Write a function $\xi(x)$ on $Y$

$$\xi(x) = \xi(ut) \quad u \in \prod_{p < \infty} Z_p, \ t \in [0, 1].$$

We will compute as follows.

(a) $\frac{\partial}{\partial u} x = t$.

(b) $\frac{\partial}{\partial u} \xi(x) = \frac{\partial x}{\partial u} \frac{\partial}{\partial x} \xi(x) = t \xi'(x)$.

[Remark] From the definition, the following things are satisfied.

(a) $dx = du dt$, so $\frac{dx}{du} = dt$.

(b) $\frac{d}{du} \xi(x) = \frac{dx}{du} \frac{d}{dx} \xi(x) = \xi'(x) dt$.

It holds that $\frac{\partial^2}{\partial u^2} = t^2 \frac{d^2}{dx^2}$, so we see that $|u|^2 \frac{\partial^2}{\partial u^2} = |u|^2 t^2 \frac{d^2}{dx^2} = |x|^2 \frac{d^2}{dx^2}$. Thus, we can identify $\Delta$ with $|u|^2 \frac{\partial^2}{\partial u^2}$. We will think of the eigenvalue problems:

$$|u|^2 \frac{\partial^2}{\partial u^2} \xi(x) - \lambda \xi(x) = 0.$$

Let

$$\eta(u) = \begin{cases} \xi(u0) & \cdots \ u \in \prod_{p < \infty} Z_p^* \\ \xi(u1) & \cdots \ \text{otherwise} \end{cases}.$$

Then we can interpret the eigenvalue problems as the following problem;

$$|u|^2 \frac{\partial^2}{\partial u^2} \eta(u) - \lambda \eta(u) = 0, \ \eta(u) = 0 \text{ on } \prod_{p < \infty} Z_p^*.$$

Here we will identify $\prod_{p < \infty} Z_p^*$ with $\partial Y$.  

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Here, \( |u|^2 \frac{\partial^2}{\partial u^2} \xi(x) = |u|^2 |t|^2 \xi''(ut) \). From the above definition of \( \eta(u) \), let

\[
\eta''(u) = |0|^2 \xi''(u0) \quad \forall u \in \prod_{p \in \mathbb{C}} \mathbb{Z}^*_p \quad \text{and} \quad \eta''(u) = |1|^2 \xi''(u1) \quad \forall u \in \prod_{p \in \mathbb{C}} \mathbb{Z}^*_p .
\]

Then we can say that \( |u|^2 \frac{\partial^2}{\partial u^2} \xi(x) \) gives \( |u|^2 \frac{\partial^2}{\partial u^2} \eta(u) \).

We can show that \( \lambda \leq 0 \). Here, think of the heat equation:

\[
\begin{cases}
\frac{\partial}{\partial t} \xi(ut) = |u|^2 \frac{\partial^2}{\partial u^2} \xi(ut) \\
\eta(u) = \xi(u0) = 0 \quad \text{on} \quad \prod_{p \in \mathbb{C}} \mathbb{Z}^*_p .
\end{cases}
\]

Let \( \eta_\lambda(u) \) be an eigenfunction of \( |u|^2 \frac{\partial^2}{\partial u^2} \) with eigenvalue \( \lambda \). Then, \( e^{-\lambda t} \eta_\lambda(u) \) is a particular solution. We obtain general solutions

\[
\xi(x) = \sum_\lambda c_\lambda \cdot e^{-\lambda t} \eta_\lambda(u).
\]

Here, \( c_\lambda \) is the constant. There exists some function, called heat kernel, \( p(t, \mu, \nu) \) on \( Y^2 \) and we can say that

\[
\xi(ut) = \int_Y p(t, u, v) \eta(v) dv .
\]

From the theory of semi-group, it holds that

\[
\sum_\lambda e^{-\lambda t} = \int_Y p(t, u, u) du .
\]

We can say that such a function \( \eta(u) \) associated with \(|x|^s\)

\[
| \cdot |^s(u) = \begin{cases}
|u0| \quad \ldots \quad u \in \prod_{p \in \mathbb{C}} \mathbb{Z}^*_p \\
|u|^s \quad \text{ohterwise}
\end{cases}
\]

is an eigenfunction of \( \Delta \) with eigenvalue \( \lambda = s(s-1) \leq 0 \). It must be allowed that

\( L^2(Y) \) is decomposed in the subspace \( \{ c | \cdot |^s \mid c \in \mathbb{C} \} \).
Here $|0|^s = 0$, so we can say that $| \cdot |^s \in L^2(Y)_0$. Moreover, since $L^2(Y)$ is a Hilbert space, \{ $| \cdot |^s$ \} is discrete. Now

$$(U(g)| \cdot |^s)(x) = |g^{-1}x|^s = |g^{-1}|^s |x|^s \quad \forall g \in C_Q, x \in Y.$$ \hspace{1cm} \text{(1)}

We shall think that $|g^{-1}|^s = |g|^s$ is extended as a quasi-character of $C_Q$. We may be allowed to think that the quasi-character $|g|^s$ is equivalent to a quasi-character $|g|^s$. We may say that

$$\text{tr}U|_{L^2(Y)_0} \text{ extends over } \{ |a|^s | s(s-1) = \lambda \}.$$ \hspace{1cm} \text{(2)}

Moreover,

$$\text{tr}U|_{L^2(C_Q)} \text{ extends over } \{ \chi(a)|a|^{1/2} | \chi \text{ is a character of } C_Q \}$$ \hspace{1cm} \text{(3)}

and

$$\text{tr}U|_{\mathcal{H}} \text{ extends over }$$

$$\{ \pi(a)|a|^{1/2} | \pi \text{ is the character of } C_Q \text{ which is given by } \eta(x) \in \text{Im}(T) \}.$$ \hspace{1cm} \text{(4)}

Recall $T(U(g)\xi)(a) = |g|^{1/2}(V(g)T\xi)(a)$,

$$T(U(g)| \cdot |^s)(a) = |g|^{1/2}(V(g)| \cdot |^s)(a)$$

$$= |g|^{1/2}(V(g)| |^{1/2+s})(a)$$

$$= |g|^{-s}|a|^{1/2+s}$$ \hspace{1cm} \text{(5)}

$|g|^{-s}$ being equivalent to $|g|^s$

$$= |g|^s|a|^{1/2+s}.$$ \hspace{1cm} \text{(6)}

Thus we see that $\text{tr}U|_{L^2(C_Q)}$ contains $\text{tr}U|_{L^2(Y)_0}$ and it also expands over such a quasi-character as $|a|^s$.

The compactness of $Y$ guarantees to compute

$$\text{tr}U(h) = \sum_{\rho} \int_{Q_s} \hat{h}(\mu^{-1}) \frac{d^s\mu}{|1-\mu|}.$$ \hspace{1cm} \text{(7)}

Thus we can say that

$$\sum_{L(\chi,\rho) = 0} \hat{h}(\chi,\rho) = \sum_{L(\chi,\rho) = 0} \hat{h}(\chi,\rho).$$ \hspace{1cm} \text{(8)}
In order to obtain an expected formula like Theorem 3.1., we need the evaluation of a certain error term. We shall compare $\langle U, L^2(Y) \rangle$ with $\langle U, L^2_\delta(X) \rangle$. For the latter, $U$ is also trace-class so that we formally get $\text{tr} U(h) = \sum \int_{\mathcal{D},} \frac{h(\mu^{-1})}{|1-\mu|} d^* \mu$. However, since $L^2_\delta(X)$ is a weighted space $L^2$, we can’t always obtain the expected formula. On the other hand, in the case of $L^2(Y)$, we can expect to obtain the desired formula.

On the other hand, the compactness must also guarantee $\lambda \leq -1/4$. So, we will see a certain relationship between the validity of the Riemann hypothesis and the fact that $\lambda \leq -1/4$. We shall suppose that the compactification of $C_\mathbb{Q}$ is equivalent to the fact that $\lambda \leq -1/4$ for the eigenvalue $\lambda$ of $\Delta$ on $X$. Then, we may say that the validity of the Riemann hypothesis is equivalent to showing $\lambda \leq -1/4$.  

[Remark]
We will think of the case $GL_2(\mathbb{Q}) \backslash GL_2(A_{\mathbb{Q}})$.

Let $(\pi, V)$ be an irreducible admissible infinite dimensional representation of $GL_2(A_{\mathbb{Q}})$ with central character $\omega$. Here, $\omega$ is a quasi-character of $GL_2(A_{\mathbb{Q}})$ defined by

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}\right) = \omega(a) id_V \quad a \in A_{\mathbb{Q}}^*.$$ 

Suppose that $W(\pi, \psi)$ is the \(\psi\)-Whittaker model. Let $\chi$ be a character of $A_{\mathbb{Q}}^*/\mathbb{Q}^*$. The Jacquet–Langlands zeta integrals are defined by

$$Z(s, W, \chi; g) = \int_{A_{\mathbb{Q}}^*} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \chi(a)|a|^{s-1/2} d^* a$$

and

$$Z^\vee(s, W, \chi; g) = \int_{A_{\mathbb{Q}}^*} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \chi(a)|a|^{s-1/2} \omega^{-1}(a) d^* a.$$ 

There exists $s_0 \in \mathbb{R}$ such that $Z(s, W, \chi; g)$ and $Z^\vee(s, W, \chi; g)$ absolutely converge whenever $\mathrm{Re}(s) > s_0$ for all $g \in GL_2(A_{\mathbb{Q}})$ and $W \in W(\pi, \psi)$. There exists a unique $L$-function $L(s, \pi \otimes \chi)(s) = \pi(g)\chi(\det g)$ such that

$$\phi(s, W, \chi; g) = Z(s, W, \chi; g) / L(s, \pi \otimes \chi)$$

is entire in $s$ for all $g \in GL_2(A_{\mathbb{Q}})$ and $W \in W(\pi, \psi)$. Therefore, we may say

$$Z(s, W, \chi; g) = 0 \iff L(s, \pi \otimes \chi) = 0.$$ 

Moreover, we will see that

$$Z(s, W, \chi; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = 0 \iff L(s, \pi \otimes \chi) = 0.$$ 

It must be instructive to compare the Jacquet–Langlands zeta integral with the Tate integral. The Tate integral is defined by

$$Z(s, \chi, \Phi) = \int_{A_{\mathbb{Q}}^*} \Phi(a) \chi(a)|a|^s d^* a$$

where $\chi$ is a character of $A_{\mathbb{Q}}^*$ and $\Phi \in S(A_{\mathbb{Q}})$. We will see that $\Phi(\chi)$ corresponds to $|x|^{-1/2} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) g)$. 

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We may say that \( W(g) \in L^2(A_{\mathbb{Q}}', d'x) \). Thus we see that
\[
|x|^{-1/2} W\left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) \in L^2(A_{\mathbb{Q}}', dx).
\]
The Jacquet-Langlands zeta integral is defined by
\[
Z(s, \chi, |x|^{-1/2} W\left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right)) = \int_{A_0} |a|^{-1/2} W\left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) \chi(a) |a| d'a.
\]

Considering \( \phi(s, W; g) = Z(s, W; g)/L(s, \pi) \), of which \( \chi \) is trivial, we will understand that the L-function \( L(s, \pi) \) is determined associated with \( |x|^{-1/2} W(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g) \). In \( GL_1 \) case, we may similarly think that there exists a unique L-function \( L(s, \pi \otimes \chi) \) which is determined associated with \( \pi \). In \( GL_2 \) case, we may similarly think that there exists a unique L-function \( L(s, \pi \otimes \chi) \) which is determined associated with \( \pi \). In \( GL_2 \) case, we may similarly think that there exists a unique L-function \( L(s, \pi \otimes \chi) \) which is determined associated with \( \pi \).

Set
\[
\phi^\vee(s, W, \chi^{-1}, g) = Z^\vee(s, W, \chi^{-1}, g)/L(s, \pi^\vee \otimes \chi^{-1}).
\]

Here \( \pi^\vee \) is the contragredient representation of \( \pi \) and \( \pi^\vee = \omega(\det)^{-1} \pi \). Then there exists a unique exponential function \( \epsilon(s, \pi, \chi, \psi) \) such that
\[
\phi^\vee(1-s, W, \chi^{-1}; g) = \epsilon(s, \pi, \chi, \psi) \phi(s, W, \chi; g).
\]

We shall think of the cuspidal automorphic representation of \( GL_2(A_{\mathbb{Q}}) \). We may think that a right regular representation \( (V, L^2(GL_2(\mathbb{Q}) \backslash GL_2(A_{\mathbb{Q}}), d'g)) \) is given.

**Theorem 4.1.** Let \( \pi \) be a cuspidal automorphic representation of \( GL_2(A_{\mathbb{Q}}) \). One obtains \( \pi \simeq \bigotimes_p \pi_p \). We will think of the case where \( \chi \) is trivial.

1. The L-function \( L(s, \pi) \) has the Euler product:
\[
L(s, \pi) = \prod_p L(s, \pi_p).
\]
2. There exists a exponential function \( \epsilon(s, \pi, \psi) \), and the functional equation:
\[
L(s, \pi) = \epsilon(s, \pi, \psi)L(1-s, \pi^\vee)
\]
is satisfied.
Proposition 4.2. Let $\pi$ be a cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$. It has its Whittaker model.

Recall the sequence

$$0 \to L^2(X)_0 \xrightarrow{\text{Tr}} L^2(C_\mathbb{Q}) \to \mathcal{H} \to 0.$$ 

We may say that

$$|x|^{-1/2} W(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g) \in L^2(X, dx)_0.$$ 

Think of the pairing $\langle T f, \eta \rangle$ for $f \in L^2(X, dx)_0$ and $\eta \in (L^2(C_\mathbb{Q}))^*$. Then,

$$\langle T f, \eta \rangle = \langle W(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g), \eta \rangle = \int_{-\infty}^{\infty} \int_{\mathbb{A}_\mathbb{Q}} W(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g) \chi(a)|a|^s \hat{\Psi}(t)d^* a \, dt.$$ 

This computation makes us to say that

$$\eta \in \text{Im}(T)^0 \iff Z(1/2+it, W, \chi; g) = 0 \iff L(1/2+it, \pi) = 0 \quad t \in \mathbb{R}.$$ 

Therefore, also in $GL_2$ case, we can give the same spectral interpretation of critical zeros of $L(s, \pi)$.

Expect that $\{ |x|^{-1/2} W(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g) \mid W \in W(\pi, \psi) \}$ are dense in $L^2(X, dx)_0$. Then we may say that $GL_2(\mathbb{Q})/GL_2(\mathbb{A}_\mathbb{Q})$ has no complementary series representation, so if $Z(s, W, \chi; g) = 0$ then $s = 1/2+it$. This must accomplish the spectral interpretation of critical zeros of $L(\chi, s)$, and we can confirm the Riemann hypothesis.
Reference


