



**Artifact 7.4:  $Y^-$  Transform Fields.** As the function quantity from the first to second horizon, a scalar field  $\phi^-$  forms and projects its potentials to its surrounding space, arisen by or acting on its opponent  $\phi^+$  through a duality of reciprocal interactions dominated by *Lorentz Generators*. Under the  $Y^-$  primary given by the generator of (7.1), the event processes institute the entangling fields:

$$\check{F}_{m\alpha}^{-n} = \frac{\hbar}{E_n^+} \phi_n^+ \dot{x}^\alpha J_{m\alpha}^- \partial_m \phi_n^- \quad \check{F}_{m\alpha}^{+n} = \frac{\hbar}{E_n^+} \phi_n^- \dot{x}_\alpha J_{m\alpha}^+ \partial^m \phi_n^+ \quad (7.8)$$

$$\check{F}_{m\alpha}^{-n} = \begin{pmatrix} \eta_0 & \beta_1 & \beta_2 & \beta_3 \\ -\beta_1 & \eta_1 & -e_3 & e_2 \\ -\beta_2 & e_3 & \eta_2 & -e_1 \\ -\beta_3 & -e_2 & e_1 & \eta_3 \end{pmatrix} = \eta_m + \begin{pmatrix} 0 & \mathbf{B}_q^- \\ -\mathbf{B}_q^- & \mathbf{b} \times \mathbf{E}_q^- \end{pmatrix}_\times \quad (7.9)$$

$$\eta_m = \check{F}_{mm}^{-n} \quad \beta_\alpha = \check{F}_{0\alpha}^{-n} \quad \varepsilon_{iam}^- e_i = \check{F}_{m\alpha}^{-n} \quad (7.10)$$

where  $\mathbf{b}$  is a base vector, symbol  $(\ )_\times$  indicates the off-diagonal elements of the tensor, and the *Levi-Civita* [5] connection  $\varepsilon_{iam}^- \in Y^-$  represents the left-hand chiral. At a constant speed, this  $Y^-$  Transform Tensor constructs a pair of its off-diagonal fields:  $\check{F}_{m\alpha}^{+n} = -\check{F}_{m\alpha}^{-n}$  and embeds a pair of the antisymmetric matrix as a foundational structure, giving rise to a foundation of the magnetic ( $\beta^\alpha \mapsto \mathbf{B}_q^+$ ) and electric ( $e^\nu \mapsto \mathbf{E}_q^+$ ) fields.

**Artifact 7.5:  $Y^+$  Transform Fields.** In the parallel fashion of (7.8), the event processes generate the reciprocal entanglements of the  $Y^+$  commutation of the scalar  $\phi^+$  and  $\phi^-$  fields, shown by the equations:

$$\hat{F}_{\nu\alpha}^{+n} = \frac{\hbar}{E_n^-} \phi_n^- \dot{x}_\alpha J_{\nu\alpha}^+ \partial^\nu \phi_n^+ \quad \hat{F}_{\nu\alpha}^{-n} = \frac{\hbar}{E_n^-} \phi_n^+ \dot{x}_\alpha J_{\nu\alpha}^- \partial_\nu \phi_n^- \quad (7.11)$$

$$\hat{F}_{\nu\alpha}^{+n} = \begin{pmatrix} \eta^0 & d^1 & d^2 & d^3 \\ -d^1 & \eta^1 & h^3 & -h^2 \\ -d^2 & -h^3 & \eta^2 & h^1 \\ -d^3 & h^2 & -h^1 & \eta^3 \end{pmatrix} = \eta^\nu + \begin{pmatrix} 0 & \mathbf{D}_q^+ \\ -\mathbf{D}_q^+ & \frac{\mathbf{u}}{c^2} \times \mathbf{H}_q^+ \end{pmatrix}_\times \quad (7.12)$$

$$\eta^\nu = \hat{F}_{\nu\nu}^{+n} \quad d^\alpha = \hat{F}_{0\alpha}^{+n} \quad \varepsilon_{\nu\mu}^+ h^\nu = c^2 \hat{F}_{\mu\alpha}^{+n} \quad (7.13)$$

where the *Levi-Civita* connection  $\varepsilon_{iam}^+$  represents the right-hand chiral. At a constant speed, this  $Y^+$  Transport Tensor constructs another pair of off-diagonal fields  $\hat{F}_{\nu\alpha}^{-n} = -\hat{F}_{\nu\alpha}^{+n}$ , giving rise to the displacement  $d_\alpha \mapsto \mathbf{D}_g^-$  and magnetizing  $h_\nu \mapsto \mathbf{H}_g^-$  fields.

**Artifact 7.6: Spiral Torque Generators.** Because of the  $Y^- Y^+$  commutation infrastructure of rising horizons, an event generates entanglements between the manifolds, and performs the operators of  $\partial^\mu$  and  $\partial_m$ , transports the motion vectors of  $\dot{x}^\alpha$  and  $\dot{x}_\alpha$ , and gives rise to the vector potentials of  $\dot{x}^\mu \partial^\mu \psi$  or  $\dot{x}_m \partial_m \psi$ . Parallel to the boost generators  $J_{\mu\alpha}^\mp$  of (7.8) and (7.11), *Spiral Torque*  $K_{\mu\alpha}^\pm$  generators naturally construct a pair of operational matrices that are also antisymmetric for elements in the 4x4 matrixes of the respective manifolds:

$$\check{T}_{m\alpha}^{-n} = \frac{\hbar}{E_n^+} \phi_n^+ \dot{x}^\alpha K_{m\alpha}^- \partial_m \phi_n^- \quad \check{T}_{m\alpha}^{+n} = \frac{\hbar}{E_n^+} \phi_n^- \dot{x}_\alpha K_{m\alpha}^+ \partial^m \phi_n^+ \quad (7.14)$$

$$\check{T}_{m\alpha}^{-n} = \begin{pmatrix} \xi_0 & \pi_1 & \pi_2 & \pi_3 \\ -\pi_1 & \xi_1 & -\vartheta_3 & \vartheta_2 \\ -\pi_2 & \vartheta_3 & \xi_2 & -\vartheta_1 \\ -\pi_3 & -\vartheta_2 & \vartheta_1 & \xi_3 \end{pmatrix} = \xi_m + \begin{pmatrix} 0 & \mathbf{B}_g^- \\ -\mathbf{B}_g^- & \mathbf{b} \times \mathbf{E}_g^- \end{pmatrix}_\times \quad (7.15)$$

$$\xi_m = \check{T}_{mm}^{-n} \quad \pi_\alpha = \check{T}_{0\alpha}^{-n} \quad \varepsilon_{iam}^- \vartheta_i = \check{T}_{m\alpha}^{-n} \quad (7.16)$$

$$\hat{T}_{m\alpha}^{+n} = \frac{\hbar}{E_n^-} \phi_n^- \dot{x}_\alpha K_{m\alpha}^+ \partial^m \phi_n^+ \quad \hat{T}_{m\alpha}^{-n} = \frac{\hbar}{E_n^-} \phi_n^+ \dot{x}^\alpha K_{m\alpha}^- \partial_m \phi_n^- \quad (7.17)$$

$$\hat{T}_{\nu\alpha}^{+n} = \begin{pmatrix} \xi^0 & \chi^1 & \chi^2 & \chi^3 \\ -\chi^1 & \xi^1 & \omega^3 & -\omega^2 \\ -\chi^2 & -\omega^3 & \xi^2 & \omega^1 \\ -\chi^3 & \omega^2 & -\omega^1 & \xi^3 \end{pmatrix} = \xi^\nu + \begin{pmatrix} 0 & \mathbf{D}_g^+ \\ -\mathbf{D}_g^+ & \frac{\mathbf{u}}{c^2} \times \mathbf{H}_g^+ \end{pmatrix}_\times \quad (7.18)$$

$$\xi^\nu = \hat{T}_{\nu\nu}^{+n} \quad \chi^\alpha = \hat{T}_{0\alpha}^{+n} \quad \varepsilon_{\nu\mu}^+ \omega^\nu = c^2 \hat{T}_{\mu\alpha}^{+n} \quad (7.19)$$

At a constant speed, the *Torsion Tensors* construct two pairs of the off-diagonal fields:  $\check{T}_{m\alpha}^{+n} = -\check{T}_{m\alpha}^{-n}$  and  $\hat{T}_{m\alpha}^{-n} = -\hat{T}_{m\alpha}^{+n}$  and embed the antisymmetric matrixes as a foundational structure giving rise to i) a pair of the virtual motion stress ( $\pi \mapsto \mathbf{B}_g^-$ ) and physical twist torsion

( $\vartheta \mapsto \mathbf{E}_g^-$ ) fields, and ii) another pair of the virtual displacement stress ( $\chi \mapsto \mathbf{D}_g^+$ ) and physical *polarizing* twist ( $\omega \mapsto \mathbf{H}_g^+$ ) fields.

## VIII. QUANTUM MECHANICS

At the first horizon, the individual behaviors of objects or particles are characterized by their timestate functions of  $\phi_n^+$  or  $\phi_n^-$  in the first  $W_a$  horizon. Due to the duality nature of virtual and physical coexistences, particle fields appear as quantization in mathematics.

Under a steady environment of the energy fluxions  $W_n^\pm$ , the (6.7) and (6.13) equations [1] can be reformulated into the compact forms for the  $Y^+$  supremacy of the entanglements: *the  $Y^+$  Quantum Equations*

$$\frac{-\hbar^2}{2E_n^+} \partial_\lambda \partial_\lambda \phi_n^+ - \frac{\hbar}{2} (\partial_\lambda - \check{\partial}^\lambda) \phi_n^+ + \frac{\hbar^2}{2E_n^+} \check{\partial}_\lambda (\partial_\lambda - \check{\partial}^\lambda) \phi_n^+ = E_n^- \phi_n^+ \quad (8.1)$$

$$\frac{\hbar^2}{2E_n^-} \check{\partial}^\lambda \check{\partial}_\lambda \phi_n^- - \frac{\hbar}{2} (\check{\partial}^\lambda - \partial_\lambda) \phi_n^- + \frac{\hbar^2}{2E_n^-} (\check{\partial}_\lambda - \partial_\lambda) \check{\partial}^\lambda \phi_n^- = E_n^+ \phi_n^- \quad (8.2)$$

$$\kappa_1 = \hbar c^2 / 2 \quad \kappa_2 = \pm (\hbar c)^2 / (2E_n^\mp) \quad W_n^\pm = c^2 E_n^\mp \quad (8.3)$$

It emanates that the bi-directional transformation has two rotations one with left-handed  $\phi_n^+ \mapsto \phi_n^L$  pointing from the  $Y^+$  source to the  $Y^-$  manifold, and the other with right-handed  $\phi_n^- \mapsto \phi_n^R$  reacting from the  $Y^-$  back to the  $Y^+$  manifold. Both fields are alternating into one another under a parity operation with relativistic preservation.

The  $Y^+$  entanglement represents the important principles of  $Y^+$  natural governances - **Law of Virtual Creation and Annihilation**:

1. The operational action  $\hat{\partial}^\lambda$  of virtual supremacy results in the physical effects as the parallel and reciprocal reactions or emanations  $\partial_\lambda$  in the physical world; and
2. The virtual world transports the effects  $\hat{\partial}_\lambda \hat{\partial}_\lambda$  emerging into or appearing as the creations of the physical world, even though the bi-directional transformations seem balanced between the commutative operations of  $\hat{\partial}_\lambda$  and  $\check{\partial}^\lambda$ ,
3. As a part of the reciprocal processes, the physical world transports the reactive effects  $\check{\partial}^\lambda \check{\partial}_\lambda$  concealing back or disappearing as annihilation processes of virtual world.

As one set of the universal laws, the events incepted in the virtual world not only generate its opponent reactions but also create the real-life objects in the physical world. The obvious examples are the formations of the elementary particles that i) the antiparticles in a virtual world generate the physical particles through their opponent duality of the event operations; ii) by carrying and transitioning the informational messages, the antiparticles grow into more real-life objects in a physical world through their event operations; and iii) recycling objects of a physical world as one of continuity processes for virtual-life streaming.

As a reciprocal process, another pair of the equations (6.12) and (6.8) [1] simultaneously formulates the following components for the  $Y^-$  supremacy of entanglements: *the  $Y^-$  Quantum Equations*

$$\frac{\hbar^2}{2E_n^-} \check{\partial}^\lambda \check{\partial}_\lambda \phi_n^- - \frac{\hbar}{2} \left(1 + \frac{\hbar}{E_n^-} \hat{\partial}^\lambda\right) (\check{\partial}_\lambda - \hat{\partial}^\lambda) \phi_n^- = \frac{W_n^-}{c^2} \phi_n^- \quad (8.4)$$

$$\frac{-\hbar^2}{2E_n^+} \hat{\partial}^\lambda \hat{\partial}_\lambda \phi_n^+ - \frac{\hbar}{2} \left(1 - \frac{\hbar}{E_n^+} \check{\partial}^\lambda\right) (\hat{\partial}^\lambda - \check{\partial}_\lambda) \phi_n^+ = \frac{W_n^+}{c^2} \phi_n^+ \quad (8.5)$$

The  $Y^-$  parallel entanglement represents the essential principles of  $Y^-$  natural behaviors - **Law of Physical Animation and Reproduction**:

1. The operational action  $\check{\partial}_\lambda$  of physical supremacy results in their conjugate or imaginary effects of animations because of the parallel reaction  $\hat{\partial}^\lambda$  in the virtual world;
2. Neither the actions nor reactions impose their final consequences  $\check{\partial}^\lambda \check{\partial}_\lambda$  on their opponents because of the parallel mirroring residuals for the horizon phenomena of reproductions  $\hat{\partial}^\lambda \hat{\partial}_\lambda$  during the symmetric fluxions;
3. There are one-way commutations of  $\check{\partial}^\lambda \check{\partial}_\lambda$  in transporting the events of the physical world into the virtual world asymmetrically. As a part of the reciprocal processes, the

virtual world replicates  $\hat{\partial}^\lambda$  the physical events during the mirroring  $\check{\partial}^\lambda \check{\partial}_\lambda$  processes in the virtual world.

As another set of laws, the events initiated in the physical world must leave a life copy of its mirrored images in the virtual world without the intrusive effects in the virtual world. In other words, the virtual world is aware of and immune to the physical world. In this perspective, continuity for a virtual-life steaming might become possible as a part of recycling or reciprocating a real-life in the physical world.

**Artifact 8.1: Lagrangian of Entanglements.** To seamlessly integrate with the classical dynamic equations, it is critical to interpret or promote the natural meanings of *Lagrangian* mechanics  $\mathcal{L}$  in forms of the dual manifolds. As a function of generalized information and formulation, *Lagrangians*  $\mathcal{L}$  can be redefined as a pair of continuities, entangling between the  $Y^-Y^+$  manifolds respectively:

$$\hat{\mathcal{L}}^+ = \frac{-\hbar^2}{2E_n^+ E_n^-} \varphi^- (\hat{\partial}_\lambda \hat{\partial}_\lambda + \check{\partial}^\lambda \check{\partial}^\lambda) \phi^+ \quad (8.6a)$$

$$\hat{\mathcal{L}}^- = \frac{-\hbar^2}{2E_n^+ E_n^-} \phi^+ (\check{\partial}^\lambda \check{\partial}^\lambda + \hat{\partial}_\lambda \hat{\partial}_\lambda) \varphi^- \quad (8.6b)$$

The formulae generalize the *Lagrangian* and state that the central quantity of *Lagrangian*, introduced in 1788, represents the bi-directional fluxions that sustain, stream, harmonize and balance the dual continuities of entanglements of the  $Y^-Y^+$  dynamic fields.

**Artifact 8.2: Mass-energy.** In mathematical formulations of entanglements, we redefine the energy-mass formations in forms of virtual complex as the following:

$$E_n^\mp = \pm imc^2 \quad : \quad \hbar\omega = mc^2 \quad (8.7)$$

Compliant with a duality of *Universal Topology*  $W = P \pm iV$ , it extends *Einstein* mass-energy equivalence, introduced in 1905 [10], into the virtual energy formulae as one of the essentials of the topological manifolds.

**Artifact 8.3: Dirac Equation.** At the intrinsic heterogeneous, one of the characteristics of spin is that the events in the  $Y^+$  or  $Y^-$  manifold transform into their opponent manifold in forms of bispinors of special relativity, reciprocally. Considering the first order  $\check{\partial}$  only, we have (8.1)-(8.2) in the simple components:

$$i \frac{\hbar}{2} (\hat{\partial}_\lambda - \check{\partial}^\lambda) \psi^\pm \pm i E_n^\mp \psi^\pm = 0 \quad : \quad \psi^\pm = \{\phi_n^+, \varphi_n^-\} \quad (8.8)$$

Because of the force transformational characteristics  $\hat{\partial}_\lambda = \check{x}_a J_{\mu a}^+ \partial^\mu$  and  $\check{\partial}^\lambda = \check{x}^\alpha J_{\nu \alpha}^- \partial_\nu$ , it can be reformulated into the compact equations:

$$2c\gamma^\mu \partial^\mu = \check{x}^\alpha J_{\mu a}^- \partial_\mu - \check{x}_a J_{\mu a}^+ \partial^\mu \quad : \quad \gamma_\mu \partial_\mu = -\gamma^\mu \partial^\mu \quad (8.9)$$

$$(i\hbar\gamma^\mu \partial^\mu + mc)\phi_n^+ = 0, \quad (i\hbar\gamma_\nu \partial_\nu - mc)\varphi_n^- = 0 \quad (8.10)$$

As a pair of entanglements, they philosophically extend to and are known as *Dirac Equation*, introduced in 1925 [7].

**Artifact 8.4: Spinor Fields.** From the *Spin Generators* (7.4)-(7.5), the respective transformations of spinors are given straightforwardly by the matrixes of spinorial  $\sigma_n$  quantities.

$$\phi_n^L = \exp\left\{\frac{1}{2}(\sigma_k \hat{\theta}_k + i\sigma_m \hat{\theta}_m)\right\} \phi_n^+ \quad : \quad (\phi_n^L)^{-1} \gamma_\mu \phi_n^L = \Lambda^- \gamma_\mu \quad (8.11)$$

$$\phi_n^R = \exp\left\{\frac{1}{2}(\sigma_k \check{\theta}_k - i\sigma_m \check{\theta}_m)\right\} \varphi_n^- \quad : \quad (\phi_n^R)^{-1} \gamma^\mu \phi_n^R = \Lambda^+ \gamma^\mu \quad (8.12)$$

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^\kappa = \begin{pmatrix} 0 & \sigma_\kappa \\ -\sigma_\kappa & 0 \end{pmatrix} \quad : \quad \gamma^\kappa \gamma^\kappa = I \quad (8.13)$$

where the matrix  $\gamma^\nu$  or  $\gamma_\nu$  is created by W. K. Clifford [8] in the 1870s. Each of the first terms is the transformation matrix of the two-dimensional world planes, respectively. Each of the second terms is an extension to the additional dimensions for the physical freedoms. The quantities are irreducible, preserve full parity invariant with respect to the physical change  $\hat{\theta}_i \rightarrow -\hat{\theta}_i$  for spin-up and spin-down positrons, which has the extra freedoms and extends the two degrees from a pair of each physical dimension of the world planes.

**Artifact 8.5: Weyl Equation.** In the limit as  $m \rightarrow 0$ , the *Dirac* equation (8.10) is reduced to the massless particles:

$$\sigma^\mu \partial_\mu \psi = 0, \quad \text{or} \quad I_2 \frac{1}{c} \frac{\partial \psi}{\partial t} + \sigma_x \frac{\partial \psi}{\partial x} + \sigma_y \frac{\partial \psi}{\partial y} + \sigma_z \frac{\partial \psi}{\partial z} = 0 \quad (8.14)$$

known as *Weyl* equation introduced in 1918 [12].

**Artifact 8.6: Schrödinger Equation.** For observations under an environment of  $W_n^\mp = \pm ic^2\{\hat{V}, \check{V}\}$ ,  $E_n^- = imc^2$  and  $\hat{\partial}^\lambda = -\sigma_3 \check{\partial}_\lambda$  at the constant transport speed  $c$ , the homogeneous fields are in a trace of diagonalized tensors. Applying to  $Y^-$  Quantum Fields (8.4) and (8.5), we obtain the following equation:

$$i\hbar \frac{\partial \phi_n^-}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \phi_n^-}{c^2 \partial t^2} - \frac{\hbar^2}{2m} \nabla^2 \phi_n^- = -\check{V}(\mathbf{r}, t_0) \phi_n^- \quad (8.15)$$

$$i\hbar \frac{\partial \varphi_n^+}{\partial t} - \frac{\hbar^2}{2m} \frac{\partial^2 \varphi_n^+}{c^2 \partial t^2} + \frac{\hbar^2}{2m} \nabla^2 \varphi_n^+ = \hat{V}(\mathbf{r}, t_0) \varphi_n^+ \quad (8.16)$$

where  $m$  is the rest mass. For the first order of time evolution, it emerges as *Schrödinger* equation, introduced in 1926 [6]:

$$i\hbar \frac{\partial \psi_n}{\partial t} = \hat{H} \psi_n \quad : \quad \hat{H} \equiv -\frac{\hbar^2}{2m} \nabla^2 + \hat{V}(\mathbf{r}, t_0) \quad (8.17)$$

where  $\hat{H}$  is the classical *Hamiltonian* operator, introduced in 1834 [4].

**Artifact 8.7: Photon.** Considering  $\hat{\partial}^\lambda = -\sigma_3 \check{\partial}_\lambda$  and  $W_n^\mp = c^2 E_n^\pm$ , we convert equations (8.4) and (8.5) into the expressions:

$$\hbar^2 \varphi_n^+ \check{\partial}_\lambda \check{\partial}_\lambda \phi_n^- - \hbar \varphi_n^+ (E_n^- + \hbar \hat{\partial}^\lambda) (I_2 + \sigma_3) \check{\partial}_\lambda \phi_n^- = 2\varphi_n^+ E_n^- E_n^+ \phi_n^- \quad (8.18)$$

$$\hbar^2 \phi_n^- \hat{\partial}_\lambda \hat{\partial}_\lambda \varphi_n^+ - \hbar \phi_n^- (E_n^+ - \hbar \check{\partial}^\lambda) (I_2 + \sigma_3) \hat{\partial}^\lambda \varphi_n^+ = 2\phi_n^- E_n^- E_n^+ \varphi_n^+ \quad (8.19)$$

For the boost transformations at constant speed  $c$ , it obeys time-invariance ( $\check{\mathcal{L}}_{Force}^{1H} \mapsto 0$ ) at the first horizon:

$$\check{\mathcal{L}}_{Force}^{1H} = \varphi_n^+ \frac{-\hbar^2}{2E_n^+} (I_2 + \sigma_3) \check{\partial}_\lambda \phi_n^- + \phi_n^- \frac{-\hbar^2}{2E_n^-} (I_2 + \sigma_3) \hat{\partial}^\lambda \varphi_n^+ \quad (8.20)$$

Adding (8.18) and (8.19) together, we derive a trace of the diagonalized tensors in forms of a wave equation at the event  $\lambda = t$  and  $\check{\mathcal{L}}_{Force}^{1H} = 0$ :

$$-\frac{1}{c^2} \frac{\partial^2 \Phi_n^-}{\partial t^2} + \nabla^2 \Phi_n^- = 4 \frac{E_n^- E_n^+}{(\hbar c)^2} \Phi_n^- \quad : \quad \Phi_n^- = \varphi_n^+ \phi_n^- \quad (8.21)$$

The energy area flow,  $4E_n^- E_n^+ / (\hbar c)^2$ , represents a pair of the irreducible units  $E_n^\mp$  that exist alternatively between physical-particle  $E_n^-$  and virtual-wave  $E_n^+$  states, but may or may not be at the same states  $E_n^\mp : mc^2 \rightleftharpoons \hbar\omega$ , where  $\hbar\omega$  is known as *Planck* matter-energy, introduced in 1900 [9]. Consequently, light consists of two units, a pair of *Photons*. For a total of mass-energy  $4m^2 c^4$ , the above equation presents a conservation of the photon energy-momentum and relativistic invariance.

**Artifact 8.8: Conservation of Energy-Momentum.** Since two photons have the mass-energy  $2mc^2$ , the equation (9.21) demonstrates empirical energy-momentum conservation in a complex formula:

$$(\mathbf{P} + i\bar{E})(\mathbf{P} - i\bar{E}) = 4E_n^+ E_n^- \quad \bar{E}^2 = \hat{\mathbf{p}}^2 c^2 + 4m^2 c^4 \quad : \quad \mathbf{P} = ic \hat{\mathbf{p}} \quad (8.22)$$

known as the relativistic invariance relating a pair of intrinsic masses at their energy  $\bar{E}$  and momentum  $\mathbf{P}$ . As a duality of alternating actions  $\bar{E} \propto \hbar\omega \rightleftharpoons mc^2$ , one operation  $\mathbf{P} + i\bar{E}$  is a process for physical reproduction or animation, while another  $\mathbf{P} - i\bar{E}$  is a reciprocal process for virtual annihilation or creation. Together, they comply with and are governed by *Universal Topology*:  $W = P \pm iV$ .

**Artifact 8.9: Conservation of Light.** The equations (8.10)-(8.21) state that, at a constant speed  $c$ , the light has the characteristics of:

#### Law of Conservation of Light

- 1) Light remains constant and conserves over time during its transportation.
- 2) Light has at least two photons for entanglement with zero net momentum.
- 3) Light consists of virtual energy duality as its irreducible unit: the photon.
- 4) Light transports and performs a duality of virtual waves and real objects.
- 5) A light energy of potential density neither can be created nor destroyed.
- 6) Light transforms from one form to another carrying potential messages.
- 7) Without an energy supply, no light can be delivered to its surroundings.
- 8) The net flow across a region is sunk to or drawn from physical resources.

**Artifacts 8.10: Photon.** Remarkably, an area energy fluxion of the potentials is equivalent to an entropy of the electromagnetic radiations. Applicable to the conservation (8.20), it also yields *Planck's* law in thermal equilibrium of entropy [4]:

$$S_A(\omega_c, T) = 4 \left( \frac{\omega_c^2}{4\pi^3 c^2} \right) = \eta_c \left( \frac{\omega_c}{c} \right)^2 \mapsto 4 \frac{E_c^- E_c^+}{(\hbar c)^2} \quad : \eta_c = \pi^{-3} \quad (8.23)$$

where the factor 4 is compensated to account for one black body with the dual states at minimum of two physical  $Y^-$  and one virtual  $Y^+$  quarks. The above equivalence results in a pair of the complex formulae:

$$E_c^\pm = \mp i \frac{1}{2} \hbar \omega_c \quad : \eta_c = \pi^{-3} \approx 33\% \quad (8.24)$$

The coupling constant at 33% implies that it is the triplet quarks that institute a pair of the photon energies  $\mp i \hbar \omega_c / 2$  for a black hole to emit lights by electromagnetic radiations.

## IX. QUANTUM ELECTRO- AND CHROMO-DYNAMICS.

By substituting (8.1) times  $\varphi_n^-$  and (8.2) times  $\phi_n^-$  into the *Lagrangians* (8.6), respectively, it comes out to *Quantum Electrodynamics (QED)* [13] that extends a pair of the first order *Dirac* equation (8.10) into the second orders in forms of *Lagrangians*:

$$\hat{\mathcal{L}}^+ = i \frac{c^2}{E_n^-} \varphi_n^- \left( i \frac{\hbar}{c} \gamma^\nu \partial_\nu + m \right) \phi_n^+ - \frac{\hbar}{E_n^-} \partial_\lambda \hat{F}_{\nu\mu}^{+\lambda} - \frac{1}{2} \hat{F}_{\nu\mu}^- \hat{F}_{\nu\mu}^+ \quad (9.1)$$

$$\hat{\mathcal{L}}^- = \frac{c^2}{i E_n^+} \phi_n^+ \left( i \frac{\hbar}{c} \gamma_\nu \partial_\nu - m \right) \varphi_n^- + \frac{\hbar}{2 E_n^+} \partial_\lambda \hat{F}_{\nu\mu}^- : \hat{F}_{\nu\mu}^- = - \hat{F}_{\nu\mu}^+ \quad (9.2)$$

where the term,  $\varphi_n^- \partial^\lambda \partial^\lambda \phi_n^+$ , is mapped to the electromagnetic fields  $\varphi_n^- \partial^\lambda \partial^\lambda \phi_n^+ = \varphi_n^- \partial^\lambda \phi_n^+ \varphi_n^- \partial^\lambda \phi_n^+ \mapsto \hat{F}_{\mu\nu}^- \hat{F}_{\mu\nu}^+ + \partial^\lambda \hat{F}_{\mu\nu}^-$ , where the horizon force  $\partial^\lambda \hat{F}_{\mu\nu}^-$  is consumed and balanced by an asymmetric force (20.3). As a pair of dynamics, it defines and generalizes a duality of *QED* for the entanglements among spins and electromagnetic fields.

Our primary goal is to bring together the original potentials and to acquire the root cause of the four known forces beyond the single variations of the *Lagrangian*. Apparently, the entangling states of the *Lagrangians* (9.1) and (9.2) establish the foundations to further extend the “pure” singlet into the field interactions (or equivalently “gauge” invariance) between doublets and among the triplets. Among the elementary particles, the four quantum fields (8.1)-(8.5) have embedded the ground foundation or intrinsic of field interactions of strong forces by coupling with the techniques of *Gauge* invariance.

**Artifact 9.1: Continuity of Electrodynamics.** Obeying the  $Y^- Y^+$  invariance for the off-diagonal elements (9.2)  $\mathcal{L}_\times^- = 0$ , we acquire the continuity equation:

$$ec \bar{\phi}_n \gamma_\nu \partial_\nu \varphi_n^- = \partial_\lambda \hat{F}_{\nu\mu}^- \quad : e \bar{\phi}_n = \frac{2c}{\hbar} \phi_n^+, \partial_\nu \mapsto c \partial_\nu \quad (9.3)$$

where  $\bar{\phi}_n$  is known as *Dirac* adjoint and  $e$  is a coupling constant of the bispinor field. Therefore, it gives rise to the *Dirac* equation interrupting with the charged particles by means of an exchange of photons.

**Artifact 9.2: Lorenz Gauge.** Imposing the vector potential field of  $\partial^\lambda A_\nu = \hat{F}_{\nu\mu}^-$ , the equation (9.3) becomes a wave function of the potential  $A_\nu$  field:

$$-\frac{1}{c^2} \frac{\partial^2 A_\nu}{\partial t^2} + \nabla^2 A_\nu = \frac{e}{c} \bar{\phi}_n \gamma^\nu \partial^\lambda \varphi_n^- \quad : \partial_\lambda \partial^\lambda A_\nu = \partial_\lambda \hat{F}_{\nu\mu}^- \quad (9.4)$$

known as *Lorenz* gauge [4] fixing the vector potential. It might be worthwhile to notice that the factor 2 in the adjoint potential  $\bar{\phi}_n$  implies there exist two fields ( $A_\nu \mapsto A_\nu^\pm$ ) for the entanglements, persistently.

**Artifact 9.3: Yang-Mills Theory.** To observe the  $Y^-$  reactions with the field entanglements  $\hat{F}_{\nu\mu}^- \hat{F}_{\nu\mu}^+$ , we impose the formulation of triplet quarks  $\hat{\mathcal{L}}^+ + 2\hat{\mathcal{L}}^-$  to the  $Y^+$  (9.1) and  $Y^-$  (9.2) for the dual streaming  $2\mathcal{L}_{QED} = \hat{\mathcal{L}}^+ + 2\hat{\mathcal{L}}^-$  and arrive at *Lagrangian QED* [14].

$$\mathcal{L}_{QED} = \bar{\psi}_n \left( i \frac{\hbar}{c} \gamma_\nu \partial_\nu - m \right) \varphi_n^- - \frac{1}{4} \hat{F}_{\nu\mu}^- \hat{F}_{\nu\mu}^+ \quad : \bar{\psi}_n = \frac{c^2}{2i E_n^+} \phi_n^+ \quad (9.5)$$

where the term  $\hat{F}_{\nu\mu}^- \hat{F}_{\nu\mu}^+ / 4$  is known as *Yang-Mills* actions, introduced in 1954. At the core of the quantum dynamics, it implies that a total of the three states exists among two  $\hat{\mathcal{L}}^-$  and one  $\hat{\mathcal{L}}^+$  dynamics to compose an

integrity of the dual  $\mathcal{L}_{QED}$  fields, revealing the particle physics of three natural “colors” [15] and representing an essential basis of the “global gauge” of the *Standard Model*, developed in the mid-1960-70s [16] and predicted various properties of weak neutral currents and the W and Z bosons with great accuracy.

**Artifact 9.4: Electroweak Fields.** Because of the linear functions between  $J_{ma}^\pm$  and  $K_{ma}^\pm$  tensors, the *Spiral Torques*  $K_{ma}^\pm = \Gamma_{ma}^\pm x_s$  of equations (3.5) and (3.7) can be straightforwardly extended into (9.5), shown by the following expressions:

$$\hat{\mathcal{L}}_{WF} = \bar{\psi}_n \left( i \hbar \gamma_\nu D_\nu - m \right) \varphi_n^- - \frac{1}{4} \hat{W}_{\nu\mu}^- \hat{W}_{\nu\mu}^+ - \frac{1}{4} \hat{F}_{\nu\mu}^- \hat{F}_{\nu\mu}^+ \quad (9.6)$$

$$\hat{W}_{\nu\alpha}^+ = \frac{\hbar}{E_n^+} \varphi_n^- \dot{x}_\alpha K_{\nu\alpha}^+ \partial^\nu \phi_n^+, \quad \hat{W}_{\nu\alpha}^- = \frac{\hbar}{E_n^-} \phi_n^+ \dot{x}^\alpha K_{\nu\alpha}^- \partial_\nu \varphi_n^- \quad (9.7)$$

where the *Torque*  $K_{ma}^\pm$  tensors generate the weak isospin field  $\hat{W}_{\nu\mu}^{\pm a}$ , while, simultaneously, *boost*  $J_{ma}^\pm$  tensors generate the hypercharge fields  $\hat{F}_{\nu\mu}^{\pm n}$ . Precisely, we extend the *Spiral Torque* fields  $\hat{F}_{\nu\mu}^{\pm n} \mapsto \hat{W}_{\nu\mu}^+ + \hat{F}_{\nu\mu}^+$  into (9.5) and map the operation  $\partial_\lambda \mapsto c D_\nu$  to extend the gauge fields that result in the above equations.

**Artifact 9.5: Standard Model.** Given the rise of the horizon from the scalar potentials (3.1)-(3.2) to the vector's (3.9)-(3.10) through the tangent transportations, the *Lagrangian* above is equivalently mapped by  $\partial_\lambda \mapsto c D_\nu$  and  $\hat{W}_{\nu\mu}^+ + \hat{F}_{\nu\mu}^+ \mapsto G_{\nu\mu}^a$  to represent the *Standard Model* associated with gauge transformation:

$$\hat{\mathcal{L}}_{SD} = \bar{\psi}_n \left( i \hbar \gamma_\nu D_\nu - m \right) \varphi_n^- - \frac{1}{4} G_{\nu\mu}^a G_{\nu\mu}^a + \hat{\mathcal{L}}_{CP} \quad (9.8)$$

$$D_\nu = \partial_\nu + i \frac{e}{c} A_\nu, \quad G_{\nu\mu}^a = \partial_\nu A_\mu^a - \partial_\mu A_\nu^a + g f^{abc} A_\nu^b A_\mu^c \quad (9.9)$$

where  $A_\nu$  are the gluon fields,  $f^{abc}$  are the structure constants of SU(3), and the variable  $g$  is subject to renormalization and corresponds to the quark coupling of the theory [17]. It illustrates that the carrier particles of a force can radiate further carrier particles during the rise of horizons. Further interactions are coupled with the strong forces:

$$\hat{\mathcal{L}}_{CP} = - \bar{\psi}_n \gamma^\mu \left( i g_s G_\mu^a T_{ij}^a \right) \varphi_n^- \mapsto \hat{\mathcal{L}}_{ST} \quad (9.10)$$

where  $g_s$  is the strong coupling constant,  $G_\mu^a$  is the 8-component SU(3) gauge field, and  $T_{ij}^a$  are the  $3 \times 3$  *Gell-Mann* matrices [18], introduced in 1962, as generators of the SU(3) color group.

**Artifact 9.6: Field Interactions.** Parallel to *Dirac* matrix (8.9) given by *Lorentz Generators* (7.4), *Spiral Torques*  $K_{ma}^\pm$  (3.5) and (3.7) create the generator  $\chi^\mu$  to couple with the gauge field  $B_\mu$  in the first horizon U(1).

$$2c \chi^\mu \partial^\mu = \dot{x}_\alpha K_{\mu\alpha}^+ \partial^\mu - \dot{x}^\alpha K_{\mu\alpha}^- \partial_\mu \quad : \chi_\mu = g' \frac{1}{2} \gamma^\mu Y_\nu B_\mu \quad (9.11)$$

where  $Y_\nu$  is the hypercharge. Therefore, the weak coupling  $\mathcal{L}_{CP}$  (9.8) is extendable to and known as classic *QCD*, discovered in 1973:

$$\hat{\mathcal{L}}_{CP} = - \bar{\psi}_n \gamma^\mu \left( g' \frac{1}{2} Y_\nu B_\mu + g \frac{1}{2} \sigma_\nu W_{\nu\mu} \right) \varphi_n^- \mapsto \mathcal{L}_{EW} \quad (9.12)$$

where gauge field  $W_\mu$  is a 3-component of the second horizon SU(2), Pauli matrices  $\sigma_\nu$  are the infinitesimal generators of SU(2) group for left-chiral fermions, and  $g$  is the SU(2) coupling constant.

**Artifact 9.7:  $Y^+$  Strong Interactions.** Giving rise to the horizon SU(2), the (8.20, 8-21) function as the classic *Lagrangian* and extend  $\bar{\varphi}_n^+ \mapsto \varphi_n^+ - \sqrt{\lambda_0} D^\nu \varphi_n^+ / m$  and  $\bar{\phi}_n^- \mapsto \phi_n^- + \sqrt{\lambda_0} D_\nu \phi_n^- / m$  to the form of:

$$\hat{\mathcal{L}}_{Force}^{2H} \propto -4 \frac{E_n^- E_n^+}{(\hbar c)^2} \bar{\varphi}_n^+ \bar{\phi}_n^- \mapsto \lambda_0 D^\nu \varphi_n^+ D_\nu \phi_n^- - m^2 \varphi_n^+ \phi_n^- \quad (9.13)$$

From spontaneous breaking of gauge symmetry  $D_\nu = \partial_\nu + i \sqrt{\lambda_2} \psi^-$  and  $D^\nu = \partial^\nu - i \sqrt{\lambda_2} \psi^+$ , the second horizon develops the next SU(3) forces:

$$\hat{\mathcal{L}}_{Force}^{3H} = \kappa_f \sum_n \left( \lambda_0 \partial^\nu \varphi_n^+ \partial_\nu \phi_n^- - m^2 \phi_n^2 + \lambda_2 \phi_n^2 \psi_n^2 \right) \quad (9.14)$$

where  $\kappa_f$  is a constant,  $\phi_n^2 = \varphi_n^+ \phi_n^-$  and  $\psi_n^2 = \psi_n^+ \psi_n^-$ . With the gauge invariance (9.9) and  $\phi_n \mapsto (v + \phi_n^a + i \phi_n^b) / \sqrt{2}$ , the strong force can be further developed into *Higgs* field, theorized in 1964 [19], and *Yukawa* interaction, introduced in 1935 [20].

In general, the weak and strong forces are characterizable and distinguishable under each scope of the horizons for its mathematical interpretations, respectively:

1. A weak force is among the dynamic interruptions under the  $Y^+$  supremacy:  $\{\phi_n^+, \varphi_n^-\}$ , dominated by the off-diagonal elements of the field tensors.
2. As a natural duality, a stronger force is among the dynamic interruptions under the  $Y^-$  supremacy:  $\{\phi_n^-, \varphi_n^+\}$ , dominated by the diagonal elements of the field tensors.
3. Apparently, given rise by the higher horizon (3.9)-(3.10) or classically known as *Gauge Theory*, the interactive forces define the simple symmetry group  $U(1) \times SU(2)$  in *Standard Model*.
4. Furthermore, an integrity of the strong nuclear forces is characterizable at the third horizon of the tangent vector interactions (9.14) or known as gauge  $SU(3)$ .

Finally, we have landed at the classic *QCD* and *Force* interactions, which unifies description of the known fundamental natural forces: electromagnetism, weak, strong and torque generators (graviton).

## X. ELECTROMAGNETISM

At the event  $\lambda = t$ , the formula (9.3) in the form of the  $Y^+$  vector (7.12) is balanced with the four-vector density currents  $\{\rho_q \mathbf{u}, \mathbf{J}_q\}$ , equivalent to  $\check{\partial}_\lambda \hat{F}_{\nu\mu}^{-n} = (\mathbf{u}\rho_q \quad \mathbf{J}_q)$ :

$$\check{\partial}_\lambda \hat{F}_{\nu\mu}^{-n} = - \left( ic \frac{\partial}{\partial x_0} \quad \mathbf{u}\nabla \right) \begin{pmatrix} 0 & \mathbf{D}_q^+ \\ -\mathbf{D}_q^+ & \frac{\mathbf{u}}{c^2} \times \mathbf{H}_q^+ \end{pmatrix} = \begin{pmatrix} \mathbf{u}\rho_q \\ \mathbf{J}_q \end{pmatrix} \quad (10.1a)$$

$$\rho_q \mathbf{u} = ec \bar{\phi}_n \gamma_0 \partial_\kappa \phi_n^- \quad \mathbf{J}_q = ec \bar{\phi}_n \gamma_r \partial_r \phi_n^+ \quad (10.2)$$

These formulae represent a set of the field equations:

$$(\mathbf{u}\nabla) \cdot \mathbf{D}_q^+ = \mathbf{u}\rho_q \quad ; \quad (10.3)$$

$$\frac{\mathbf{u} \cdot \mathbf{u}}{c^2} \nabla \times \mathbf{H}_q^+ - \frac{\partial \mathbf{D}_q^+}{\partial t} = \mathbf{J}_q + \mathbf{H}_q^+ \cdot \left( \frac{\mathbf{u}}{c} \nabla \right) \times \frac{\mathbf{u}}{c} \quad (10.4)$$

where the formula,  $\nabla \cdot (\mathbf{u} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{H})$ , is applied. Therefore, the charged particles under the  $Y^+$  supremacy generate *Electric Displacing*  $\mathbf{D}_q^+$  and *Magnetic Polarizing*  $\mathbf{H}_q^+$  fields, which are balanced by the physical resources  $\{\mathbf{u}\rho_q, \mathbf{J}_q\}$ .

Simultaneously at the  $Y^-$  supremacy, equation (8.5) contains only one off-diagonal matrix  $\check{\partial}^\lambda \check{\partial}_\lambda$ . It implies the off-diagonal elements be conserved to zero, and results in the continuity equation of

$$\check{\partial}_\lambda \left( \check{F}_{ma}^{+n} \right)_\times = 0 \quad (10.1b)$$

$$(\mathbf{u}\nabla) \cdot \mathbf{B}_q^- = 0 \quad (10.5)$$

$$\frac{\partial \mathbf{B}_q^-}{\partial t} + \left( \frac{\mathbf{u}}{c} \nabla \right) \times \mathbf{E}_q^- = 0 \quad (10.6)$$

The magnetic field  $\mathbf{B}_q^-$  in space is subject to time virtually associated with its physical opponent of electric field  $\mathbf{E}_q^-$  such that, together, they serves as commutative resources, entangling between the dual manifolds and balanced by massless waves at light speed.

**Artifact 10.1: Maxwell's Equations.** At the constant speed  $c$ , the above equations emerge in a set of classical equations:

$$\nabla \cdot \mathbf{B}_q = 0 \quad (10.7)$$

$$\nabla \cdot \mathbf{D}_q = \rho_q \quad (10.8)$$

$$\nabla \times \mathbf{E}_q + \frac{\partial \mathbf{B}_q}{\partial t} = 0 \quad (10.9)$$

$$\nabla \times \mathbf{H}_q - \frac{\partial \mathbf{D}_q}{\partial t} = \mathbf{J}_q \quad (10.10)$$

known as *Maxwell's Equations*, discovered in 1820s. Therefore, as the foundation, the quantum fields give rise to classical electromagnetism, describing how electric and magnetic fields are generated by charges,

currents, and interactions. One important consequence of the equations is that they demonstrate how fluctuating electric and magnetic fields propagate at the speed of light.

## CONCLUSION

Complying with classical and contemporary physics, this universal and unified theory demonstrates its holistic foundations applicable to the well-known natural intrinsics:

- 1) As evolutionary process, the theory acquires the empirical formulae of, but not limited to, *Lorentz* generators, *Pauli* spin matrices, torque gravitation, and transformational field structures.
- 2) *Einstein mass-energy* and *Lagrangian density*  $\mathcal{L}$  are refined philosophically as the entanglements of complex (8.7) and continuity (9.1)-(9.2) dynamically balanced between the manifolds.
- 3) **Law of Conservation of Light.** For the first time, the law of light is revealed in the comprehensive integrity and characteristics of photon beyond its well-known nature at a constant speed.
- 4) **Quantum Physics** is derived as the compliance to contemporary physics and particle physics, testified by the empirical theories of *Schrödinger* and *Dirac* equations, *Quantum Electrodynamics*, *Lorenz Gauge*, *Yang-Mills* actions, and *Quantum Chromodynamics*, including the weak and strong forces theorized by *Standard Model* and gauge invariance.
- 5) **Maxwell's Equations** is derived and unified by a set of generic field equations  $\check{\partial}_\lambda (\check{F}_{ma}^{+n})_\times = 0$  and  $\check{\partial}_\lambda \hat{F}_{\nu\mu}^{-n} = (\mathbf{u}\rho_q \quad \mathbf{J}_q)$ , rising from the quantum fields.

Consequently, this manuscript has testified to theoretical foundations of the *Universal Topology*, mathematical framework, event operations, and world equations [1] towards a unified physics.

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