THE KAKEYA TUBE CONJECTURE IMPLIES THE KAKEYA CONJECTURE

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Abstract. In this article we will give a proof that the Kakeya tube conjecture implies the Kakeya conjecture.

1. Introduction

We define the δ - tubes in standard way: for all $\delta > 0, \omega \in S^{n-1}$ and $a \in \mathbb{R}^n$, let

$$T_\omega^\delta(a) = \{x \in \mathbb{R} : |(x-a)\cdot\omega| \leq \frac{1}{2}, |proj_{\omega^\perp}(x-a)| \leq \delta\}.$$

In this paper any constant can depend on dimension n. A Kakeya set is a compact set that contains an unit line in every direction. We will give a proof that the result

$$\bigcup_{\omega\in\Omega} T_\omega \approx 1$$

for maximal set of δ - tubes implies the Kakeya conjecture.

Theorem 1 (Kakeya conjecture)**.** *Any Kakeya set has full Hausdorff dimension.*

2. The proof

For our definition of Hausdoff content see for example [6]. Let K be a Kakeya set, that is, a set that contains an unit line in every direction. let $\bigcup_{j=1}^\infty B_j$ be a cover of K with balls of diameters less than $1 > \beta > 0$. Let $n > n-\alpha > 0$ be such that

$$\sum_{j=1}^{\infty} r_j^{n-\alpha} < 1. \tag{1}$$

If the hausdorff content is zero that kind of cover exists. By compactness of the Kakeya set we can take a subcover with diameters such that $1 > \beta > r_j \geq \delta > 0$, where at least one $r_j \sim \delta$. Now, assume

$$\sum_{j=1}^{M} r_j^n \gtrsim |\bigcup_{j=1}^{M} B_j| \gtrsim |\bigcup_{i=1}^{N} T_i| \gtrapprox 1. \tag{2}$$

The second inequality above follows because the balls cover the middle lines of the tubes, so there exists a constant such that the second inequality above is valid. Using inequality (1) and (2) we obtain

$$C_{\alpha/k}\delta^{-\alpha/k}\sum_{j=1}^{M} r_j^n > \sum_{j=1}^{M} r_j^{n-\alpha}. \tag{3}$$

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Thus,

$$\sum_{j=1}^{M} r_j^n (C_{\alpha/k}\delta^{-\alpha/k} - r_j^{-\alpha}) > 0. \tag{4}$$

It follows that for the average value of a power of diameters it holds that

$$C_{\alpha/k}\delta^{-\alpha/k} > \frac{1}{M}\sum_{j=1}^{M} r_j^{-\alpha} \geq \frac{1}{M^{-\alpha}}(\sum_{j=1}^{M} r_j)^{-\alpha}, \tag{5}$$

where we used Jensen‘s inequality. Thus,

$$c_\alpha \frac{1}{M}\sum_{j=1}^{M} r_j > \delta^{1/k}. \tag{6}$$

From above it follows that

$$\frac{(c_\alpha)^n}{M}(\sum_{j=1}^{M} r_j^n) \geq (\frac{c_\alpha}{M})^n(\sum_{j=1}^{M} r_j)^n > \delta^{n/k},$$

where we used Jensen‘s inequality again. Thus, from above and inequality (1)

$$C_\alpha > M\delta^{n/k}.$$

It follows from above that

$$\delta^{-n/k} C_\alpha > M \tag{7}$$

We can do the steps (3), (4) and (5) again for $\epsilon = \alpha/2$ and obtain

$$C_{\alpha/2}\delta^{-\alpha/2} > \frac{1}{M}\sum_{j=1}^{M} r_j^{-\alpha}. \tag{8}$$

Let k and a small δ be such that

$$\delta^{-\alpha/3} > C_\alpha \delta^{-n/k}.$$

From above and inequalities (7) and (8) we obtain

$$C_{\alpha/2}\delta^{-\alpha/2} > \delta^{\alpha/3}\sum_{j=1}^{M} r_j^{-\alpha} > \delta^{\alpha/3}\delta^{-\alpha} = \delta^{-2/3\alpha}, \tag{9}$$

which is a contradiction when δ is small.

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