# RELATIVISTIC SPACETIME STRUCTURES OF A HYDROGEN ATOM

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Abstract: In our previous works on possible relationship between Schrödinger wavefunctions and spacetime structures of quantum particles we suggested that a Ricci scalar curvature for a particular quantum state of a quantum system could be constructed using the Schrödinger wavefunctions as mathematical objects. In this work we will extend our discussions by suggesting possible line elements that can be used to construct relativistic spacetime structures of a hydrogen atom. We also discuss the possibility to use a covariant Ricci flow as evolution equations to describe the dynamics of the hydrogen atom geometric structures.

In our previous works, we showed that by identifying geometrical objects with physical entities possible spacetime structures of both macroscopic and microscopic physical systems can be constructed purely geometrically by differential evolution equations [1,2]. In this work we will extend our discussions to the construction of the spacetime structures of a hydrogen atom using the contracted Bianchi identities and the Lie differentiation. It is shown in differential geometry that the Ricci tensor $R^{\alpha\beta}$ satisfies the contracted Bianchi identities

$$\nabla_\beta R^{\alpha\beta} = \frac{1}{2} g^{\alpha\beta} \nabla_\beta R \qquad (1)$$

where $R = g^{\alpha\beta} R_{\alpha\beta}$ is the Ricci scalar curvature. Even though Equation (1) is purely geometrical, it has a covariant form similar to the field equations for the electromagnetic tensor written in a covariant form $\nabla_\alpha F^{\alpha\beta} = \mu j^\beta$. If the quantity $\frac{1}{2} g^{\alpha\beta} \nabla_\beta R$ can be identified as a physical entity, such as a four-current of gravitational matter, then Equation (1) has the status of a dynamical law of a physical theory. In this case a four-current $j^\alpha = (\rho, \mathbf{j}_i)$ can be defined purely geometrical as

$$j^\alpha = \frac{1}{2} g^{\alpha\beta} \nabla_\beta R \qquad (2)$$

We showed that the purely geometrical four-current $j^\alpha$ given in Equation (2) can be used to construct spacetime structures for quantum particles by identifying the Ricci scalar curvature with the potential in classical physics. In this work, however, we will focus on the problem of how to construct possible structures for a hydrogen atom using Equation (1) for the case in which $\frac{1}{2} g^{\alpha\beta} \nabla_\beta R = 0$. Equation (1) reduces to the equation

$$\nabla_\beta R^{\alpha\beta} = 0 \qquad (3)$$

We will consider two different possible solutions that can be resulted from Equation (3). Firstly, since the metric tensor satisfies the identity $\nabla_\mu g^{\alpha\beta} \equiv 0$, Equation (3) implies

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \tag{4}$$

where $\Lambda$ is an undetermined constant. Using the identities $g^{\alpha\beta} g_{\alpha\beta} = 4$ and $g^{\alpha\beta} R_{\alpha\beta} = R$, we obtain $\Lambda = R/4$. Secondly, it is observed that within the subgroup of time-independent coordinate transformations, partial time-derivative of a covariant tensor is also a covariant tensor, and for the case of a covariant metric tensor $g_{\alpha\beta}$ we are able to derive the identity $\nabla_\gamma\left(\partial_t g_{\alpha\beta}\right) = \partial_t\left(\nabla_\gamma g_{\alpha\beta}\right) \equiv 0$ . In this case, Equation (3) also implies

$$\frac{\partial g_{\alpha\beta}}{c\partial t} = k_1 R_{\alpha\beta} \tag{5}$$

where $k_1$ is a scaling factor [3]. Equation (5) has the form of the Ricci flow [4]. Even though both Equations (4) and (5) are derived from Equation (3), they have different dynamical characteristics. While Equation (4) can be used to describe static physical structures, Equation (5) describes evolution processes which are intrinsic geometrical properties of physical systems. This can be seen by applying the Lie differentiation into relativistic dynamics as follows. It is shown in differential geometry that besides the covariant derivatives with respect to affine connections, Lie derivatives are also invariant under coordinate transformations on a differentiable manifold [5]. On a manifold $M$, we define a tangent vector field $dx^\mu/du$ on a congruence of curves which are given by $x^\mu = x^\mu(u)$. If a tangent vector field can be defined for every curve in the congruence then a vector field $X^\mu$ can be established over the whole manifold. On the other hand, due to the unique existence of solution for an ordinary differential equation, a congruence of curves can be obtained from a given non-zero vector field defined over a differentiable manifold and a tensor field $T^{\alpha\cdots}_{\beta\dots}(x)$ can then be differentiated by using the vector field $X^\mu$. We use the congruence of curves to drag the tensor at some point $P$, $T^{\alpha\cdots}_{\beta\dots}(P)$, along the curve passing through $P$ to some neighbouring point $Q$ and then compare the dragged-along tensor with the tensor already at $Q$, $T^{\alpha\cdots}_{\beta\dots}(Q)$. A derivative can be defined by subtracting the dragged-along tensor $T'^{\alpha\cdots}_{\beta\dots}(Q)$ and the tensor at $Q$, which is $T^{\alpha\cdots}_{\beta\dots}(Q)$, as follows

$$L_X T^{\alpha\cdots}_{\beta\dots}(Q) = \lim_{\delta u \to 0} \frac{T^{\alpha\cdots}_{\beta\dots}(Q) - T'^{\alpha\cdots}_{\beta\dots}(Q)}{\delta u} \tag{6}$$

Using Equation (6), the Lie derivative of a general tensor field $T^{\alpha\cdots}_{\beta\dots}$ can be obtained as

$$L_X T^{\alpha\cdots}_{\beta\dots} = X^\mu \partial_\mu T^{\alpha\cdots}_{\beta\dots} - T^{\mu\cdots}_{\beta\dots} \partial_\mu X^\alpha - \cdots T^{\alpha\cdots}_{\mu\dots} \partial_\beta X^\mu + \cdots \tag{7}$$

For the case of a covariant metric tensor $g_{\alpha\beta}$ we obtain

$$L_X g_{\alpha\beta} = X^\mu \partial_\mu g_{\alpha\beta} + g_{\mu\alpha} \partial_\beta X^\mu + g_{\mu\beta} \partial_\alpha X^\mu \tag{8}$$

Besides the important properties of being linear, satisfying the product rule for differentiation and commuting with contraction, the Lie differentiation with respect to a vector field $X^\mu$ also preserves the type of a tensor. Since the Lie derivative of a covariant metric tensor of second rank is also a covariant tensor of second rank we may propose the following tensor equation

$$L_X g_{\alpha\beta} = \kappa R_{\alpha\beta} \tag{9}$$

where $\kappa$ is a dimensional constant [6]. Using Equation (8), Equation (9) can also be written as

$$X^\mu \partial_\mu g_{\alpha\beta} + g_{\mu\alpha}\partial_\beta X^\mu + g_{\mu\beta}\partial_\alpha X^\mu = \kappa R_{\alpha\beta} \tag{10}$$

Mathematically, the covariant flow given in Equation (10) can be reduced to the Ricci flow given in Equation (5) if the vector field $X^\mu$ can be smoothly assigned values in the form $X^\mu = (X^0, 0,0,0)$, where $X^0$ is a constant temporal component of the vector field. In this case we obtain the form of the Ricci flow given in Equation (5) as

$$X^0 \partial_0 g_{\alpha\beta} = \kappa R_{\alpha\beta} \tag{11}$$

Physically, Equation (11) has the form of an evolution equation that can be explained in terms of relativistic dynamics using comoving synchronous coordinate systems as described in [7]. First we choose a homogeneous three-dimensional spatial manifold $S$. For physical analysis, this initial three-dimensional spatial manifold can be assumed to be formed by some form of a fluid substance. We then assign a coordinate time $t$ to all events on the manifold and set up a spatial coordinate system $x^\mu$, $\mu = 1,2,3$ on $S$. These spatial coordinates propagate off $S$ and throughout all spacetime by means of the world lines. The spatial coordinates are then considered to be comoving if they are assigned to events at which the world line intersects the hypersurface $S$. Because the hypersurfaces $S$ are given by the condition $t = constant$, the spatial basis vector $\partial/x^\mu$, $\mu = 1,2,3$ at any given event are tangent to the hypersurface and the temporal basis vector $\partial/\partial t$ is tangent to the world line. In this case, the temporal coordinate is the proper time of the world line and is the four-velocity of the motion of the fluid substance that form the three-dimensional spatial manifold. With this relativistic picture, solutions to Equations (4) and (5) are basically a description of the three-dimensional spatial structures of a physical system.

Now, we assume that Equation (4) describes the spacetime structure of a physical field, such as the gravitational field. If we consider a centrally symmetric field with a time-independent metric of the form

$$ds^2 = e^\psi c^2 dt^2 - e^\chi dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{12}$$

then the Schwarzschild solution can be found as [8]

$$ds^2 = \left(1 - \frac{C}{r} - \frac{\Lambda r^2}{3}\right) c^2 dt^2 - \left(1 - \frac{C}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{13}$$

where $C$ is an undetermined constant. If the constant $C$ is identified as $C = 2GM/c^2$ then the Schwarzschild solution can be used to describe a gravitational field. It is also observed that, as in the case of the electromagnetic field where the energy-momentum tensor is determined from the electromagnetic field tensor, with the Schwarzschild solution obtained and given in Equation (13), the energy-momentum tensor $T_{\alpha\beta}$ for the gravitational field can be established if we define it through Einstein's field equations by the relation $T_{\alpha\beta} = \frac{1}{\kappa}\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right)$. In this case, we obtain

$$T_{\alpha\beta} = -\frac{\Lambda}{\kappa}g_{\alpha\beta} \tag{14}$$

On the other hand, we assume that Equation (5) describes an evolution process that can be applied to the dynamics of a hydrogen atom. However, instead of Equation (5), we need an evolution equation that involves the Ricci scalar curvature so that we can identify it with the classical potential. By contracting Equation (5) with the contravariant metric $g^{\alpha\beta}$, we obtain

$$g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{c\partial t} = k_1 R \tag{15}$$

Consider an evolution process with a line element of the form

$$ds^2 = D(cdt)^2 - A(x,y,z,t)((dx)^2 + (dy)^2 + (dx)^2) \tag{16}$$

where $D$ is constant. This line element can be applied to either a macroscopic phenomenon such as the cosmological evolution or a microscopic phenomenon such as the evolution of a quantum particle. In our previous work [2], we already discussed the cosmological evolution using the line element given by Equation (16) therefore in this work we will only discuss the evolution of the spacetime structure of a hydrogen atom. If we apply Equation (3) in the appendix and the Ricci scalar curvature given in Equation (8) also in the appendix to Equation (15) we obtain the following evolution equation

$$-\frac{3}{c^2 D}\frac{\partial^2 A}{\partial t^2} + \frac{2}{A}\nabla^2 A + \frac{3}{ck}\frac{\partial A}{\partial t} + \frac{3}{2A^2}(\boldsymbol{\nabla} A)^2 = 0 \tag{17}$$

However, if we impose the conditions of isotropy and homogeneity on the three-dimensional spacelike hypersurface of the spacetime manifold then Equation (15) can be solved for the case when the Ricci scalar is constant. It is shown in differential geometry that a space of constant curvature is characterised by the equation

$$R_{\alpha\beta\mu\nu} = K\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right) \tag{18}$$

where $R_{\alpha\beta\mu\nu}$ is the Riemannian curvature tensor and $K$ is the Gaussian curvature. The Ricci curvature tensor $R_{\alpha\beta}$ and the Ricci scalar $R$ are found as

$$R_{\alpha\beta} = 2Kg_{\alpha\beta} \tag{19}$$

$$R = 6K \tag{20}$$

If the Ricci scalar $R$ is constant then from Equation (15) and Equation (3) in the appendix, we obtain the following solution for the quantity $A(x,y,z,t)$ for the line element given in Equation (16)

$$A(x,y,z,t) = A_0(x,y,z)e^{\frac{ck_1R}{3}t} \tag{21}$$

where $A_0$ is an initial spacelike hypersurface of spacetime manifold. Depending on the sign of the quantity $k_1R$, we have three different possibilities. If $k_1R = 0$ then spacetime is static and it reduces to a Minkowski spacetime if $A_0(x,y,z) = constant$. If $k_1R > 0$ then spacetime is expanding and it reduces to de Sitter universe if $A_0(x,y,z) = constant$. If $k_1R < 0$ then the spacelike hypersurface of the spacetime manifold will collapse.

Now, in order to incorporate the Schrödinger wavefunctions into the spacetime structures of a quantum system, we will follow the procedure that was used in Schrödinger's original works on the quantum structure of a hydrogen atom [9]. As shown in our previous works, there is a relationship between Schrödinger wavefunctions and the spacetime structures of a quantum system in the sense that Schrödinger wavefunctions are considered purely as mathematical objects that can be used for the construction of spacetime structures of the quantum states of a quantum system. Since Schrödinger's original works were on the time-independent quantum states of the hydrogen atom, we first recapture the main ideas of Schrödinger's method to obtain the time-independent wave equation for the hydrogen atom. Schrödinger commenced with the Hamilton-Jacobi equation, written in terms of the Cartesian coordinates $(x,y,z)$ as [9,10]

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial x}\right)^2 - 2m\left(E + \frac{kq^2}{r}\right) = 0 \tag{22}$$

However, in order to obtain a partial differential equation that would give rise to the required results, Schrödinger introduced a new function $\psi$, which is real, single-valued and twice differentiable, through the relation

$$S = \hbar \ln\psi \tag{23}$$

where the action $S$ is defined by

$$S = \int L dt \tag{24}$$

and $L$ is the Lagrangian defined by

$$L = T - V \tag{25}$$

with $T$ is the kinetic energy and $V$ is the potential energy. Now we show that Schrödinger wavefunction $\psi$ can be used to construct the spacetime structures of the quantum states of a hydrogen atom. By using the relations $L = dS/dt$, $dS/dt = \partial_t S + \sum_{\mu=1}^{3} \partial_\mu S(dx^\mu/dt)$, $T = m\sum_{\mu=1}^{3}(dx^\mu/dt)^2$ and $V = T - L$, we obtain

$$V = m\sum_{\mu=1}^{3}(dx^{\mu}/dt)^2 - \partial_t S + \sum_{\mu=1}^{3}\partial_\mu S(dx^{\mu}/dt) \tag{26}$$

In terms of the Schrödinger wavefunction $\psi$, Equation (26) can be rewritten as

$$V = \ m\sum_{\mu=1}^{3}(dx^{\mu}/dt)^2 - \hbar\frac{\partial_t\psi + \sum_{\mu=1}^{3}\partial_\mu\psi(dx^{\mu}/dt)}{\psi} \tag{27}$$

From the assumed relations $V = k_2\mathrm{R}$, we arrive at the following relation between the Schrödinger wavefunction $\psi$ and the Ricci scalar R

$$\mathrm{R} = \frac{1}{k_2}\left(m\sum_{\mu=1}^{3}(dx^{\mu}/dt)^2 - \hbar\frac{\partial_t\psi + \sum_{\mu=1}^{3}\partial_\mu\psi(dx^{\mu}/dt)}{\psi}\right) \tag{28}$$

If we use Equation (15) together with Equation (3) in the appendix and Equation (28) for the Ricci scalar curvature, we obtain the following evolution equation

$$\frac{\partial A}{c\partial t} = k\left(m\sum_{\mu=1}^{3}\left(\frac{dx^{\mu}}{dt}\right)^2 - \hbar\frac{\partial_t\psi + \sum_{\mu=1}^{3}\partial_\mu\psi\left(\frac{dx^{\mu}}{dt}\right)}{\psi}\right)A \tag{29}$$

where $k = k_1/k_2$. In order to solve Equation (29), the new function $\psi$ needs to be determined. In terms of the new function $\psi$ Equation (22) takes the form

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial x}\right)^2 - \frac{2m}{\hbar^2}\left(E + \frac{kq^2}{r}\right)\psi^2 = 0 \tag{30}$$

Then by applying the principle of least action $\delta\int Ldt = 0$, Schrödinger arrived at the required equation

$$\nabla^2\psi + \frac{2m}{\hbar^2}\left(E + \frac{kq^2}{r}\right)\psi = 0 \tag{31}$$

When the wavefunctions $\psi$ are found then they can be applied into Equation (29) to construct possible spacetime structures for the hydrogen atom. It should be mentioned here that we only use the Schrödinger wave equation given by Equation (31) to find mathematical objects to construct the geometrical structure of the three-dimensional spatial hypersurface of the total relativistic spacetime structure of the hydrogen atom. In terms of the spherical coordinates $(r, \theta, \phi)$, the Schrödinger wave equation given in Equation (31) becomes

$$\left(-\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\mathbf{L}^2}{\hbar^2 r^2}\right) - \frac{kq^2}{r} - E\right)\psi = 0 \tag{32}$$

where the orbital angular momentum $\mathbf{L}^2$ is defined by

$$\mathbf{L}^2 = -\hbar^2 \left( \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right) \tag{33}$$

The eigenfunctions $\psi_{nlm}(r, \theta, \phi)$ for the hydrogen atom, which are solutions to the Schrödinger wave equation given in Equation (31), can be found in the separable form as [11]

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r) Y_{lm}(\theta, \phi) \tag{34}$$

where $Y_{lm}(\theta, \phi)$ is the spherical harmonics and $R_{nl}(r)$ is a radial function. We obtain two separate equations and their solutions for the harmonics $Y_{lm}(\theta, \phi)$ and the radial function $R_{nl}(r)$ as follows

$$\mathbf{L}^2 Y_{lm}(\theta, \phi) = l(l+1)\hbar^2 Y_{lm}(\theta, \phi) \tag{35}$$

$$Y_{lm}(\theta, \phi) = (-1)^m \left( \frac{(2l+1)(l-m)!}{4\pi(l+m)!} \right)^{\frac{1}{2}} P_l^m(\cos\theta) e^{im\phi} \tag{36}$$

$$\left( -\frac{\hbar^2}{2m} \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) + \frac{l(l+1)\hbar^2}{2mr^2} - \frac{kq^2}{r} - E \right) R_{nl}(r) = 0 \tag{37}$$

$$R_{nl}(r) = -\left( \left( \frac{2}{na_0} \right)^3 \frac{(n-l-1)!}{2n\big((n+l)!\big)^3} \right)^{\frac{1}{2}} e^{-\frac{\rho}{2}} \rho^l L_{n+l}^{2l+1}(\rho) \tag{38}$$

where $\rho = 2r/na_0$ with $a_0 = \hbar^2/mkq^2$. The energy eigenvalues $E_n$ for the bound states are also obtained as

$$E_n = -\frac{mk^2q^4}{2\hbar^2} \frac{1}{n^2} \tag{39}$$

The first few normalised wavefunctions for the hydrogen atom, their corresponding Ricci scalar curvatures and evolution equations are given below

$$\psi_{100}(r, \theta, \phi) = \frac{1}{\sqrt{\pi}} \left( \frac{1}{a_0} \right)^{\frac{3}{2}} e^{-\frac{r}{a_0}} \tag{40}$$

$$\mathrm{R} = \frac{1}{k_2} \left( m \left( \left( \frac{dr}{dt} \right)^2 + r^2 \sin^2\phi \left( \frac{d\theta}{dt} \right)^2 + r^2 \left( \frac{d\phi}{dt} \right)^2 \right) + \frac{\hbar}{a_0} \frac{dr}{dt} \right) \tag{41}$$

$$\frac{\partial A}{c\partial t} = k \left( m \left( \left( \frac{dr}{dt} \right)^2 + r^2 \sin^2\phi \left( \frac{d\theta}{dt} \right)^2 + r^2 \left( \frac{d\phi}{dt} \right)^2 \right) + \frac{\hbar}{a_0} \frac{dr}{dt} \right) A \tag{42}$$

$$\psi_{200}(r, \theta, \phi) = \frac{1}{4\sqrt{2\pi}} \left( \frac{1}{a_0} \right)^{\frac{3}{2}} \left( 2 - \frac{r}{a_0} \right) e^{-\frac{r}{2a_0}} \tag{43}$$

$$R=\frac{1}{k_2}\left(m\left(\left(\frac{dr}{dt}\right)^2+r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2\right)+\frac{3\hbar}{a_0}\frac{dr}{dt}\right) \tag{44}$$

$$\frac{\partial A}{c\partial t}=k\left(m\left(\left(\frac{dr}{dt}\right)^2+r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2\right)+\frac{3\hbar}{a_0}\frac{dr}{dt}\right)A \tag{45}$$

$$\psi_{300}(r,\theta,\phi)=\frac{1}{81\sqrt{3\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}}\left(27-\frac{18r}{a_0}+\frac{2r^2}{a_0^2}\right)e^{-\frac{r}{3a_0}} \tag{46}$$

$$R=\frac{1}{k_2}\left(m\left(\left(\frac{dr}{dt}\right)^2+r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2\right)+\frac{\hbar(54-4r)}{a_0}\frac{dr}{dt}\right) \tag{47}$$

$$\frac{\partial A}{c\partial t}=k\left(m\left(\left(\frac{dr}{dt}\right)^2+r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2\right)+\frac{\hbar(54-4r)}{a_0}\frac{dr}{dt}\right)A \tag{48}$$

The evolution of the spacetime structures of a hydrogen atom can also be investigated if instead we apply the Ricci scalar curvature given in Equation (28) together with the scalar curvature given by Equation (8) in the appendix. In this case we obtain the following evolution equation

$$-\frac{3}{c^2DA}\frac{\partial^2A}{\partial t^2}+\frac{2}{A^2}\nabla^2A+\frac{3}{2A^3}(\boldsymbol{\nabla}A)^2=\frac{1}{k_2}\left(m\sum_{\mu=1}^{3}\left(\frac{dx^\mu}{dt}\right)^2-\hbar\frac{\partial_t\psi+\sum_{\mu=1}^{3}\partial_\mu\psi\left(\frac{dx^\mu}{dt}\right)}{\psi}\right) \tag{49}$$

In terms of spherical coordinates, the Ricci scalar given in Equation (28) and the evolution equation given in Equation (49) take the following respective forms

$$R=\frac{1}{k_2}\left(m\left(\left(\frac{dr}{dt}\right)^2+r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2\right)-\hbar\frac{\partial_t\psi+\partial_r\psi\frac{dr}{dt}+\partial_\theta\psi\frac{d\theta}{dt}+\partial_\phi\psi\frac{d\phi}{dt}}{\psi}\right) \tag{50}$$

$$-\frac{3}{c^2DA}\frac{\partial^2 A}{\partial t^2}+\frac{2}{A^2}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right)-\frac{\mathbf{L}^2 A}{\hbar^2 r^2}\right)+\frac{3}{2A^3}\left(\left(\frac{\partial A}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial A}{\partial \theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial A}{\partial \phi}\right)^2\right)$$
$$=\frac{1}{k_2}\left(m\left(\left(\frac{dr}{dt}\right)^2+r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2\right)-\hbar\frac{\partial_t\psi+\partial_r\psi\frac{dr}{dt}+\partial_\theta\psi\frac{d\theta}{dt}+\partial_\phi\psi\frac{d\phi}{dt}}{\psi}\right) \quad (51)$$

The first few normalised wavefunctions for the hydrogen atom, their corresponding Ricci scalar curvatures and evolution equations are given below

$$\psi_{100}(r,\theta,\phi)=\frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}}e^{-\frac{r}{a_0}} \quad (52)$$

$$\mathrm{R}=\frac{1}{k_2}\left(m\left(\left(\frac{dr}{dt}\right)^2+r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2\right)+\frac{\hbar}{a_0}\frac{dr}{dt}\right) \quad (53)$$

$$-\frac{3}{c^2DA}\frac{\partial^2 A}{\partial t^2}+\frac{2}{A^2}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right)-\frac{\mathbf{L}^2 A}{\hbar^2 r^2}\right)+\frac{3}{2A^3}\left(\left(\frac{\partial A}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial A}{\partial \theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial A}{\partial \phi}\right)^2\right)$$
$$=\frac{1}{k_2}\left(m\left(\left(\frac{dr}{dt}\right)^2+r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2\right)+\frac{\hbar}{a_0}\frac{dr}{dt}\right) \quad (54)$$

$$\psi_{200}(r,\theta,\phi)=\frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}}\left(2-\frac{r}{a_0}\right)e^{-\frac{r}{2a_0}} \quad (55)$$

$$\mathrm{R}=\frac{1}{k_2}\left(m\left(\left(\frac{dr}{dt}\right)^2+r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2\right)+\frac{3\hbar}{a_0}\frac{dr}{dt}\right) \quad (56)$$

$$-\frac{3}{c^2DA}\frac{\partial^2 A}{\partial t^2}+\frac{2}{A^2}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right)-\frac{\mathbf{L}^2 A}{\hbar^2 r^2}\right)+\frac{3}{2A^3}\left(\left(\frac{\partial A}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial A}{\partial \theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial A}{\partial \phi}\right)^2\right)$$
$$=\frac{1}{k_2}\left(m\left(\left(\frac{dr}{dt}\right)^2+r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2\right)+\frac{3\hbar}{a_0}\frac{dr}{dt}\right) \quad (57)$$

$$\psi_{300}(r,\theta,\phi)=\frac{1}{81\sqrt{3\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}}\left(27-\frac{18r}{a_0}+\frac{2r^2}{a_0^2}\right)e^{-\frac{r}{3a_0}} \quad (58)$$

$$\mathrm{R}=\frac{1}{k_2}\left(m\left(\left(\frac{dr}{dt}\right)^2+r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2\right)+\frac{\hbar(54-4r)}{a_0}\frac{dr}{dt}\right) \quad (59)$$

$$-\frac{3}{c^2DA}\frac{\partial^2 A}{\partial t^2}+\frac{2}{A^2}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right)-\frac{\mathbf{L}^2A}{\hbar^2r^2}\right)+\frac{3}{2A^3}\left(\left(\frac{\partial A}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial A}{\partial \theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial A}{\partial \phi}\right)^2\right)$$
$$=\frac{1}{k_2}\left(m\left(\left(\frac{dr}{dt}\right)^2+r^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2\right)+\frac{\hbar(54-4r)}{a_0}\frac{dr}{dt}\right) \quad (61)$$

Appendix

With the line element given in Equations (16), we have

$$g_{00}=D,\quad g_{ii}=-A \quad \text{where} \quad i=1,2,3 \quad (1)$$

$$g^{00}=\frac{1}{D},\quad g^{ii}=-\frac{1}{A} \quad \text{where} \quad i=1,2,3 \quad (2)$$

$$g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{c\partial t}=\frac{3}{A}\frac{\partial A}{c\partial t} \quad (3)$$

Using the affine connection defined in terms of the metric tensor

$$\Gamma^{\lambda}_{\mu\nu}=\frac{1}{2}g^{\lambda\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^{\mu}}+\frac{\partial g_{\sigma\mu}}{\partial x^{\nu}}-\frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right) \quad (4)$$

the non-zero components of the affine connection are found as

$$\Gamma^1_{01}=\Gamma^1_{10}=\frac{1}{2cA}\frac{\partial A}{\partial t},\quad \Gamma^2_{02}=\Gamma^2_{20}=\frac{1}{2cA}\frac{\partial A}{\partial t},\quad \Gamma^3_{03}=\Gamma^3_{30}=\frac{1}{2cA}\frac{\partial A}{\partial t}$$

$$\Gamma^0_{11}=\frac{1}{2cD}\frac{\partial A}{\partial t},\quad \Gamma^1_{11}=\frac{1}{2A}\frac{\partial A}{\partial x},\quad \Gamma^2_{11}=-\frac{1}{2A}\frac{\partial A}{\partial y},\quad \Gamma^3_{11}=-\frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^1_{12}=\Gamma^1_{21}=\frac{1}{2A}\frac{\partial A}{\partial y},\quad \Gamma^2_{12}=\Gamma^2_{21}=\frac{1}{2A}\frac{\partial A}{\partial x},\quad \Gamma^1_{13}=\Gamma^1_{31}=\frac{1}{2A}\frac{\partial A}{\partial z},\quad \Gamma^3_{13}=\Gamma^3_{31}=\frac{1}{2A}\frac{\partial A}{\partial x}$$

$$\Gamma^0_{22}=\frac{1}{2cD}\frac{\partial A}{\partial t},\quad \Gamma^1_{22}=\frac{1}{2A}\frac{\partial A}{\partial x},\quad \Gamma^2_{22}=\frac{1}{2A}\frac{\partial A}{\partial y},\quad \Gamma^3_{22}=-\frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^0_{33}=\frac{1}{2cD}\frac{\partial A}{\partial t},\quad \Gamma^1_{33}=-\frac{1}{2A}\frac{\partial A}{\partial x},\quad \Gamma^2_{33}=-\frac{1}{2A}\frac{\partial A}{\partial y},\quad \Gamma^3_{33}=\frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^2_{23}=\Gamma^2_{32}=\frac{1}{2A}\frac{\partial A}{\partial z},\quad \Gamma^3_{23}=\Gamma^3_{32}=\frac{1}{2A}\frac{\partial A}{\partial y} \quad (5)$$

With the components of the affine connection given in Equation (5), using the Ricci curvature tensor defined in terms of the affine connection

$$R_{\mu\nu}=\frac{\partial \Gamma^{\sigma}_{\mu\nu}}{\partial x^{\sigma}}-\frac{\partial \Gamma^{\sigma}_{\mu\sigma}}{\partial x^{\nu}}+\Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma}-\Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu} \quad (6)$$

we obtain the following non-zero components of the Ricci curvature tensor

$$R_{11} = \frac{1}{2c^2D}\frac{\partial^2 A}{\partial t^2} - \frac{1}{A}\frac{\partial^2 A}{\partial x^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial y^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial z^2} + \frac{3}{4c^2AD}\left(\frac{\partial A}{\partial t}\right)^2 + \frac{1}{A^2}\left(\frac{\partial A}{\partial x}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial y}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{22} = \frac{1}{2c^2D}\frac{\partial^2 A}{\partial t^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial x^2} - \frac{1}{A}\frac{\partial^2 A}{\partial y^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial z^2} + \frac{3}{4c^2AD}\left(\frac{\partial A}{\partial t}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial x}\right)^2 + \frac{1}{A^2}\left(\frac{\partial A}{\partial y}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{33} = \frac{1}{2c^2D}\frac{\partial^2 A}{\partial t^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial x^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial y^2} - \frac{1}{A}\frac{\partial^2 A}{\partial z^2} + \frac{3}{4c^2AD}\left(\frac{\partial A}{\partial t}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial x}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial y}\right)^2 + \frac{1}{A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{00} = -\frac{3}{2c^2A}\frac{\partial^2 A}{\partial t^2} + \frac{3}{4c^2A^2}\left(\frac{\partial A}{\partial t}\right)^2 \qquad (7)$$

Using the relation $R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}$ we arrive at

$$R = -\frac{3}{c^2DA}\frac{\partial^2 A}{\partial t^2} + \frac{2}{A^2}\nabla^2 A + \frac{3}{2A^3}(\boldsymbol{\nabla} A)^2 \qquad (8)$$

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