ON THE KAKEYA CONJECTURE

J.ASPEGREN

ABSTRACT. In this article we will prove a result that implies the Kakeya conjecture.

1. INTRODUCTION

We define the δ - tubes in standard way: for all $\delta > 0, \omega \in S^{n-1}$ and $a \in \mathbb{R}^n$, let

$$T_\omega^\delta(a) = \{x \in \mathbb{R} : |(x-a)\cdot\omega| \leq \frac{1}{2}, |proj_{\omega^\perp}(x-a)| \leq \delta\}.$$

Moreover, let $f \in L^1_{loc}(\mathbb{R}^n)$. Define the Kakeya maximal function $f_\delta^* : S^{n-1} \to \mathbb{R}$ via

$$f_\delta^*(\omega) = \sup_{a\in\mathbb{R}^n} \frac{1}{T_\omega^\delta(a)} \int_{T_\omega^\delta(a)} |f(y)|\mathrm{d}y.$$

In this paper any constant can depend on dimension $n \geq 2$. In study of the Kakeya maximal function conjecture we are aiming at the following bounds

$$||f_\delta^*||_p \leq C_\epsilon \delta^{-n/p+1+\epsilon}, \tag{1}$$

for all $\epsilon > 0$. A bound of the form (1) follows from a bound of the form

$$|| \sum_{\omega\in\Omega} 1_{T_\omega(a_\omega)}||_{p/(p-1)} \leq C_\epsilon \delta^{-n/p+1-\epsilon}, \tag{2}$$

for all $\epsilon > 0$, and for any set of δ-separated of δ - tubes. See for example [12] or [6]. As usuall we define that "$A \lessapprox B$ iff for all $\epsilon > 0$ and for all $\delta > 0$, it holds that $A \lesssim C_\epsilon \delta^{-\epsilon} B$.

An another formulation for the Kakeya maximal function conjecture (2) is the following. Let Ω be any set of δ - separated δ -tubes. Let A be any measurable set, let $\gamma < 1$, and let for all $T_\omega \in \Omega$, $|A \cap T_\omega| \geq \gamma|T_\omega|$. Then it holds that

$$\delta^{n-1}\#(\Omega)\gamma^d \lessapprox \delta^{d-n}|A|.$$

See [14].
We will prove the the following theorem:

Theorem 1. *Let Ω be a maximal set of δ - tubes and $\gamma \approx 1$. Moreover, let for all $|A \cap T_\omega| \geq \gamma|T_\omega|$, then, it holds that*

$$\gamma \lessapprox |\bigcup_{\omega\in\Omega} T_\omega \cap A|.$$

According to Tao [14] above implies the Kakeya conjecture:

Corollary 1 (Kakeya conjecture)**.** *Any Kakeya set has full Hausdorff dimension.*

1991 *Mathematics Subject Classification.* 42B37,28A75.
Key words and phrases. Kakeya conjectures.

2. Previously known results

We will use the following bound for the pairwise intersections of δ - tubes:

Lemma 1 (Corbòda). *For any pair of directions $\omega_i, \omega_j \in S^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n$, we have*

$$|T^\delta_{\omega_i}(a) \cap T^\delta_{\omega_j}(b)| \lesssim \frac{\delta^n}{|\omega_i - \omega_j|}.$$

A proof can be found for example in [6].

For any (spherical) cap $\Omega \subset S^{n-1}, |\Omega| \gtrsim \delta^{n-1}, \delta > 0$, deïňĄne its δ-entropy $N_\delta(\Omega)$ as the maximum possible cardinality for an $\delta-$separated subset of Ω.

Lemma 2. *In the notation just defined*

$$1 \leq N_\delta(\Omega) \sim \frac{|\Omega|}{\delta^{n-1}}.$$

Again, a proof can essentially be found in [6]

3. A proof of the theorem

In this section we will prove 1. Consider the integral

$$\int \sum_\Omega 1_{T_\omega} 1_A = \sum_\Omega |T_\omega \cap A|.$$

Split the domain of integration via dyadic decomposition:

$$E_{2^k} := \{x | 2^k \leq \sum_\Omega 1_{T_\omega}(x) \leq 2^{k+1}\}.$$

Define

$$B_{2^k} := \{x | 2^k \leq \sum_\Omega 1_{T_\omega}(x) 1_A(x) 1_{E_{2^k}}(x) \leq 2^{k+1}\}. \tag{3}$$

Now,

$$B_{2^k} = E_{2^k} \cap A.$$

Integrating inequality

$$2^k \leq \sum_\Omega 1_{T_\omega}(x) 1_A(x) \leq 2^{k+1}$$

over the domain $E_{2^k} \cap A$ we obtain

$$2^k |E_{2^k} \cap A| \leq \sum_\Omega |T_\omega \cap E_{2^k} \cap A| \leq 2^{k+1} |E_{2^k} \cap A|. \tag{4}$$

Let $\#(\Omega) = N$. Now, $k \in [0, .., C \log N]$. Notice that there exists k such that

$$\gamma \lesssim \sum_\Omega |T_\omega \cap A| \lesssim \log N \sum_\Omega |T_\omega \cap E_{2^k} \cap A| \sim \log N 2^k |E_{2^k} \cap A|. \tag{5}$$

Now, consider the terms $|T_\omega \cap E_{2^k} \cap A|$ in the above sum. We wan't to prove that we can essentially take them to be $\approx \delta^{n-1}$. Split the sum in two parts where $|T_{\omega'} \cap E_{2^k} \cap A| \geq \frac{\gamma c}{\log N} \delta^{n-1}$ and $|T_{\omega''} \cap E_{2^k}| < \frac{\gamma c}{\log N} \delta^{n-1}$,

$$\gamma \lesssim \log N \sum_{\omega' \in \Omega'} |T_{\omega'} \cap E_{2^k}| + \log N \sum_{\omega'' \in \Omega''} |T_{\omega''} \cap E_{2^k}|.$$

It‘s clear that because the number of terms in the sums is $\lesssim \delta^{n-1}$ the last sum above is neglible. Next, we wan‘t to prove that if $|T_\omega \cap E_{2^k} \cap A| \approx \delta^{n-1}$, then $k \approx 1$. Now, $|T_\omega \cap E_{2^k} \cap A| \approx \delta^{n-1}$, is a subset of an intersection of $\sim 2^k$ δ- tubes. Let‘s suppose that $2^k \gtrsim \delta^{-\beta}, 0 < \beta \leq n-1$. First, let‘s suppose that some tube $T_{\omega'}$ intersecting T_ω has it‘s direction outside of a cap of side $\sim \delta^{n-1+\beta}$ on the unit sphere. Then the angle between T_ω and $T_{\omega'}$ is greater than $\sim \delta^{1+\beta/(n-1)}$. Thus by lemma 1 the intersection $|T_\omega \cap E_{2^k} \cap A| \leq |T_\omega \cap T_{\omega'}|$ is less than $\sim \delta^{n-1-\beta/(n-1)}$, which is a contradiction. Thus, we can suppose that the directions in the intersection $E_{2^k} \cap T_\omega \cap A$ belong to a cap of size $\sim \delta^{n-1+\beta}$. If we δ - separate the cap via lemma 2 we get that the cap can contain at most ≈ 1 tube-directions, which is a contradiction. Thus, $2^k \approx 1$. From inequality (5) we have that

$$\gamma \lesssim \sum_{\Omega} |T_\omega \cap A| \lesssim \log N \sum_{\Omega} |T_\omega \cap E_{2^k} \cap A| \sim \log N 2^k |E_{2^k} \cap A| \lessapprox |E_{2^k} \cap A| \leq |\bigcup_{\omega \in \Omega} T_\omega \cap A|.$$

Thus,

$$\gamma \lessapprox |\bigcup_{\omega \in \Omega} T_\omega \cap A|, \tag{6}$$

and we have proved the theorem 1.

References

[1] J. Bourgain, *Besicovitch Type Maximal Operators and Applications to Fourier Anal- ysis, Geometric and Functional Analysis 1* (1991), 147-187.

[2] J. Bourgain, *Harmonic analysis and combinatorics: How much may they contribute to each other?*,IMU/Amer. Math. Soc. (2000), 13âĂŞ32

[3] A. Còrdoba, *The Kakeya Maximal Function and the Spherical Summation Multipli- ers*, American Journal of Mathematics 99 (1977), 1-22.

[4] R.O. Davies, *Some Remarks on the Kakeya Problem*, Proc. Camb. Phil. Soc. 69 (1971), 417-421.

[5] Z. Dvir, *On the Size of Kakeya Sets in Finite Fields*, J. Amer. Math. Soc. 22 (2009), 1093-1097.

[6] E.Kroc, *The Kakeya problem*, available at http://ekroc.weebly.com/uploads/2/1/6/3/21633182/mscessay-final.pdf

[7] N.H. Katz, I. Laba and T. Tao, *An improved bound on the Minkowski dimension of Besicovitch sets in R3* , Annals of Math. (2) 152, 2 (2000), 383âĂŞ446.

[8] N.H. Katz, T. Tao, *New bounds for Kakeya problems*, J. Anal. Math. 87 (2002), 231-263.

[9] I. Laba, T. Tao, *An improved bound for the Minkowski dimension of Besicovitch sets in medium dimension*, Geometric and Functional Analysis 11 (2001), 773âĂŞ806.

[10] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press (1993)

[11] T. Tao, *Lecture Notes*, available at math.ucla.edu/ tao/254b.1.99s/ (1999)

[12] T. Tao, *The Bochner-Riesz Conjecture Implies the Restriction Conjecture*, Duke Math. J. 96 (1999), 363-375.

[13] T.Tao, *From rotating needles to stability of waves: emerging connections between com- binatorics, analysis, and pde*,Notices Amer. Math. Soc., 48(3),(2001),294âĂŞ303.

[14] T. Tao *https://terrytao.wordpress.com/2009/05/15/the-two-ends-reduction-for-the-kakeya-maximal-conjecture/*

[15] T. Wolff, *An Improved Bound for Kakeya Type Maximal Functions*, Rev. Mat. Iberoamericana 11 (1995), 651-674.

[16] T. Wolff, *Recent work connected with the Kakeya problem* Prospects in mathematics (Princeton, NJ, 1996), (1999),129âĂŞ162.

E-mail address: jaspegren@outlook.com