General Relativity and Dark Energy are hidden in the Klein-Gordon equation in curved spacetime

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Abstract

By maintaining the form of the Klein-Gordon equation in curved 4-dimensional spacetime, it can be symmetrized into symmetric and antisymmetric (2,0) tensors with the wave function Ψ having spins $0, 1, 2, 1/2$ and $3/2$. Using a decomposition of symmetric tensors and two fundamental principles of general relativity, a new tensor describing the energy-momentum of the gravitational field and dark energy appears naturally alongside the Einstein tensor. The tensors that constitute the modified Einstein equation add to zero. General relativity and dark energy are hidden in the symmetric part of the Klein-Gordon equation for each spin. As quantum field theory in curved spacetime is built from the free field solutions of the Klein-Gordon equation, general relativity and dark energy are hidden in the formalism of quantum field theory. The metric as a field variable describing gravitons vanishes from the massless spin-2 KG equation in the low-energy long-range to particle regimes of spacetime. Massless spin-2 gravitons in those regimes do not exist. Unlike the other three known fundamental forces in nature, no particle exchange is required to explain the force of gravity in those regimes. It is shown that the cosmological constant must be zero. Dark energy effectively replaces the cosmological constant and describes a cyclic universe which developed after the Big Bang. Dark energy is interpreted to be the repulsive part of the trace of the tensor describing the energy-momentum of the gravitational field and dark energy. The dark energy density provides a natural explanation of why the vacuum energy density is so small, and why it dominates the present epoch of the universe.

Keywords
quantum mechanics; Klein-Gordon; general relativity; dark energy; dark matter, quantum field theory, quantum gravity

1. Introduction

It has been nearly a century since Schrödinger [1] wrote down his equation describing quantum mechanics and over a century since Einstein [2] formulated general relativity (GR). And yet today, there is still not a full understanding of the relationship between the two fundamental theories of physics. The quantization of gravity has been the major approach to unite the two theories, with string theory and loop quantum gravity the two mainstream theories of quantum gravity. However, those and other theories of quantum gravity have well documented successes and failures [3, 4]. Rather than trying to force quantum theory on general relativity, or vice versa, this article investigates if a connection between quantum theory and general relativity exists naturally.

The Klein-Gordon (KG) equation for a free field with a particular spin in Minkowski spacetime is fundamental to the formulation of quantum field theory (QFT). In curved spacetime, the KG equation is assumed to be the rudimentary equation for the development of quantum theory. It is an asymmetric wave equation which can be symmetrized into symmetric and antisymmetric rank 2 tensors for bosons with spins 0,1,2 (spinor-tensors for fermions with spins $1/2, 3/2$). As GR involves symmetric tensors, the

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only possibility to associate GR directly with quantum theory is through the symmetric part of the KG equation.

It is proven in section 3 that GR indeed resides in the symmetric part of the KG equation for spins $0, 1, 2, 1/2$ and $3/2$. In fact, GR is hidden in the KG equation as the tensors that constitute the Einstein equation and an additional tensor add to zero. The new symmetric tensor, $\Phi_{\alpha\beta}$, appears naturally alongside the Einstein tensor in the derivation of a modified equation of GR and represents the energy-momentum of the gravitational field and dark energy. In this article, dark energy is understood to be repulsive (positive) part of the trace of $\Phi_{\alpha\beta}$.

In section 4, some interesting results appear from the study of spin-2 massless "particles" described by the KG equation. If gravitons are the exchange particles of gravity, they must be massless because gravity is a long-range force. However, it is shown that *massless* spin-2 "particles" cannot be described with the metric in the low-energy long-range to particle regimes of spacetime; and massless spin-2 "particles" do not couple to a non-zero energy-momentum tensor. Massless gravitons therefore do not describe the force of gravity in those regimes. This result should not be viewed controversially. It is well known that GR, as a classical field theory, does not require particle exchange to describe the effective force of gravity; that is nicely done by the curvature of spacetime. Furthermore, there is nothing in the formalism of QFT that requires GR to be quantized. That was noted by Feynman who said: [5] "It is still possible that quantum theory does not absolutely guarantee that gravity *has* to be quantized". Gravity, unlike the other three known forces in nature, does not require the exchange of particles to describe its long-range force behavior. This explains why gravity is so much weaker than the long-range electromagnetic force which involves the photon as the exchange particle. However, when the mass of the spin-2 particle is non-zero, massive gravitons exist in the high-energy short-range regime near the Planck length.

Section 5 discusses the conservation equation for the energy-momentum tensor $T^{\alpha\beta} = \tilde{T}^{\alpha\beta} - \frac{c^4}{8\pi G}\Phi^{\alpha\beta}$ where $\tilde{T^{\alpha\beta}}$ is the matter energy-momentum tensor describing both baryonic and dark matter. Some properties of $\Phi_{\alpha\beta}$ are presented to show why it represents the energy-momentum of the gravitational field and dark energy. In particular, when $\tilde{T^{\alpha\beta}} = 0$, $G^{\alpha\beta} = -\Phi^{\alpha\beta}$ and the gravitational field is endowed with an intrinsic energy-momentum. The scalar Φ, being the trace of $\Phi_{\alpha\beta}$, is globally conserved: $\int g^{\alpha\beta}\Phi_{\alpha\beta}\sqrt{-g}d^4x = 0$. $\Phi > 0$ is gravitationally repulsive and represents dark energy.

In section 6, another interesting result is apparent from the calculation of the coupling interaction of the metric with the energy-momentum tensor; the cosmological constant Λ must vanish and is replaced with dark energy.

As shown in section 7, the cosmological constant Λ is not required to explain any epoch of the universe. Φ can accelerate or decelerate the universe but has a net zero effect on the entire universe. Dark energy describes the inflation of the universe immediately after the Big Bang when no baryonic or dark matter was present. The dark energy density then tends to the present value of the vacuum energy density. A cyclic universe is born with maximum and minimum values of the cosmological scale factor in the Friedmann-Robertson-Walker (FRW) metric. Dark energy explains the small value of the vacuum energy density and why it now dominates the expansion and acceleration of the present universe.

The qualitative cyclic behaviour of this description of the universe *after* the Big Bang is very similar to that described by Loop Quantum Cosmology (LQC). One essential difference between the theories is that LQC removes the classical singularity of the Big Bang. In the Planck regime, the quantum effects of geometry in LQC [6] produce a repulsive force that overrules the enormous gravitational attraction and causes a quantum bounce which replaces the Big Bang and the Big Crunch. Classical GR for a closed universe is applicable far from the Planck regime and a cyclic universe is described with no singularites.

However, other cyclic models of the universe described in the literature are quite different. Steinhardt and Turok [7, 8] proposed a cosmological model involving a cyclic collision of branes creating an endless sequence of cosmic epochs; each beginning with a bang and ending in a crunch. There is no inflationary epoch but rather each cycle includes a period of slow accelerated expansion followed by a slow contraction. Baum and Frampton [9] described a cyclic model involving dark energy in a brane world which leads to a turnaround at a time just before the Big Rip. Many independent similarly small contracting universes are then spawned. Penrose proposed Conformal Cyclic Cosmology [10] where GR was reformulated in a conformally invariant manner assuming all matter, including electrons, decay to massless radiation. With conformal invariance both in the remote future and at the Big Bang origin, the remote future of one phase of the universe becomes the Big Bang of the next.

Dark energy described in this article is not a quintessential theory which invokes negative pressures to explain the acceleration of the observable universe [11, 12]. It is not a phantom or quintom theory although it does share the result of a cyclic universe with the quintom model [13] (and further references therein).

As discussed in section 8, quantum field theory in curved spacetime is formulated from free field solutions to the KG equation. GR resides beautifully within QFT.

2. Decomposition of the Klein-Gordon Equation in curved spacetime

Minkowski spacetime is described with the metric $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$, $x_0 = ct$ and $a = 0,1,2,3$. We unconventionally choose not to set $c = 1$. The passage from special relativity (SR) to GR requires the equations in GR to be the same in all coordinate systems. The principle of minimal coupling is employed whereby inertial coordinates x^a are replaced by general coordinates x^μ; η_{ab} is replaced by the Lorentzian symmetric metric $g_{\alpha\beta}$; covariant derivatives replace partial derivatives; and the volume element d^4x is replaced by $\sqrt{-g}d^4x$. Curved spacetime is described by the 4-dimensional time oriented Lorentzian manifold with metric, $(M, g_{\alpha\beta})$.

The KG equation of SR is $\partial_a\partial^a\Psi = k^2\Psi$ where $k = \frac{m_0 c}{\hbar}$. In the transition to curved spacetime, it becomes

$$\nabla_\mu\nabla^\mu\Psi = k^2\Psi. \tag{1}$$

This is an asymmetric wave equation with Ψ representing spins 0,1,2,1/2 and 3/2. The KG equation can be constructed from the field vector A^β with real components and the (2,0) tensor $\mathit{\Psi}^{\alpha\beta} = \nabla^\alpha A^\beta$, for all spins. $\mathit{\Psi}^{\alpha\beta}$ can then be symmetrized according to

$$\begin{aligned}\mathit{\Psi}^{\alpha\beta} &= \frac{1}{2}(\nabla^\alpha A^\beta + \nabla^\beta A^\alpha) + \frac{1}{2}(\nabla^\alpha A^\beta - \nabla^\beta A^\alpha)\\ &:= \frac{1}{2}\tilde{\mathit{\Psi}}^{\alpha\beta} + \frac{1}{2}K^{\alpha\beta}.\end{aligned} \tag{2}$$

The symmetric tensor $\tilde{\mathit{\Psi}}_{\alpha\beta}$ is the Lie derivative of the metric along the regular vector A^β. The trivial solution of $\tilde{\mathit{\Psi}}_{\alpha\beta}$ is the Killing equation $\nabla_\alpha A_\beta + \nabla_\beta A_\alpha = 0$. This has the solution $\nabla_\alpha A_\beta = \nabla_\beta A_\alpha = 0$ which generates the constraint

$$\nabla_\alpha A^\alpha = 0. \tag{3}$$

This is well known as the Lorentz guage which is commonly adopted to remove spin-0 characteristics from spin-1 equations. However, there is no need to formally adopt (3) in this analysis because it is inherent in the formalism of (2) as a constraint. $\mathit{\Psi}^{\alpha\beta}$ is then traceless with respect to the metric: $g_{\alpha\beta}\nabla^\alpha A^\beta = 0$. This requires $\tilde{\mathit{\Psi}}^{\alpha\beta}$ to be traceless because $K^{\alpha\beta}$ is antisymmetric. Any spin-0 characteristics of the spin-1 and spin-2 KG equations (and their related spin-1/2 and spin-3/2 equations, respectively) are eliminated.

Thus, the decomposition of the KG equation is focused on the symmetric tensor $\tilde{\mathit{\Psi}}^{\alpha\beta}$ which satisfies condition (2) with property (3) for all of the noted spins. As will become evident, it is important to preserve the form of (1) in curved spacetime.

2.1. Orthogonal Decomposition of Symmetric Tensors

The orthogonal decomposition of symmetric tensors on Riemannian manifolds has been documented in the literature [14, 15, 16, 17]. Of importance to this article is the orthogonal decomposition of symmetric tensors on a Lorentzian manifold. Ma and Wang [17] extended their results on a Riemannian manifold to the decomposition of an arbitrary symmetric tensor $w_{\alpha\beta}$ on a four dimensional Minkowskian manifold into

$$w_{\alpha\beta} = v_{\alpha\beta} + \nabla_\alpha\partial_\beta\phi \tag{4}$$

where $v_{\alpha\beta} = v_{\beta\alpha}$, $\nabla^\alpha v_{\alpha\beta} = 0$, ϕ is a scalar and the first Betti number $b_1(M) = 0$. However, a more general decomposition of symmetric tensors on a time oriented Lorentzian manifold is required.

Theorem 2.1. *Orthogonal Decomposition Theorem (ODT): An arbitrary (0,2) symmetric tensor* $w_{\alpha\beta}$ *in the symmetric tensor bundle* S^2T^*M *on an n-dimensional time oriented Lorentzian manifold* $(M, g_{\alpha\beta})$ *can be orthogonally decomposed as*

$$w_{\alpha\beta} = v_{\alpha\beta} + \Phi_{\alpha\beta} \tag{5}$$

where $\nabla^\alpha v_{\alpha\beta} = 0$ *and* $\Phi_{\alpha\beta} = \frac{1}{2}\pounds_X g_{\alpha\beta} + \pounds_X u_\alpha u_\beta$ *with X a regular vector along an integral curve in M and u a timelike unit vector field collinear with X.*

Proof. Let the Lorentzian manifold $(M, g_{\alpha\beta})$ be paracompact, or compact and orientable with vanishing Euler-Poincaré characteristic; and let M be endowed with a Riemannian metric $g^+_{\alpha\beta}$. A regular vector field X exists as does a timelike unit vector u collinear with X. The Lorentzian manifold $(M, g_{\alpha\beta})$ is associated with an adapted Riemannian metric $g^+_{\alpha\beta}$ by setting [18, 19]

$$g_{\alpha\beta} = g^+_{\alpha\beta} - 2u_\alpha u_\beta. \tag{6}$$

Let $w_{\alpha\beta}$ and $v_{\alpha\beta}$ belong to S^2T^*M, the tensor bundle of symmetric $(0,2)$ tensors on the Lorentzian manifold $(M, g_{\alpha\beta})$. In the Riemannian open subset of S^2T^*M which contains $g^+_{\alpha\beta}$, an arbitrary $(0,2)$ symmetric tensor $w_{\alpha\beta}$ can be orthogonally decomposed by the Berger-Ebin theorem [15] according to

$$w_{\alpha\beta} = v_{\alpha\beta} + \frac{1}{2}\pounds_X g^+_{\alpha\beta} \tag{7}$$

where $\nabla^{+\alpha} v_{\alpha\beta} = 0$. Given the unit vector field u collinear with X, the Riemann connection ∇^+ on M is the same as the Lorentzian connection ∇ on M because both connections have the same geodesics with the same parameterization and both connections are torsionless [20, 21]. Hence,

$$\begin{aligned} w_{\alpha\beta} &= v_{\alpha\beta} + \frac{1}{2}\pounds_X g_{\alpha\beta} + \pounds_X u_\alpha u_\beta \\ &= v_{\alpha\beta} + \Phi_{\alpha\beta} \end{aligned} \tag{8}$$

where $\nabla^\alpha v_{\alpha\beta} = 0$ and

$$\Phi_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha) + u^\lambda(u_\alpha \nabla_\beta X_\lambda + u_\beta \nabla_\alpha X_\lambda). \tag{9}$$

■

3. General relativity and dark energy are hidden in the Klein-Gordon equation in curved spacetime

Theorem 3.1. *The equation* $-\frac{8\pi G}{c^4}\tilde{T}_{\alpha\beta} + R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} + \Phi_{\alpha\beta} = 0$ *describing general relativity with a baryonic and dark matter energy-momentum tensor* $\tilde{T}_{\alpha\beta}$*, and the energy-momentum of the gravitational field including dark energy with the tensor* $\Phi_{\alpha\beta}$*, is contained in the symmetric part of the Klein-Gordon equation* $\nabla_\mu\nabla^\mu\Psi = k^2\Psi$ *in an open subset of a 4-dimensional time oriented Lorentzian manifold, with* Ψ *having spins 0,1,2,1/2 and 3/2.*

Proof. $\tilde{\Psi}_{\alpha\beta}$, defined in (2), is a general symmetric tensor which can be decomposed into a set of physically relevant symmetric tensors by using two fundamental principles of GR and the Orthogonal Decomposition Theorem (5).

Firstly, the field equations contained in $\tilde{\Psi}_{\alpha\beta}$ which are sought to describe general relativity, dark matter and dark energy must be derivable from an action functional. It is well known that the variation with respect to the metric of an action functional describing all matter fields generates a symmetric energy-momentum tensor $\tilde{T}_{\alpha\beta}$. It must represent both baryonic matter and dark matter if total matter, and the radiation associated with it, are to be described. $\tilde{\Psi}_{\alpha\beta}$ must then be expressed as

$$\tilde{\Psi}_{\alpha\beta} = \frac{a}{c}\tilde{T}_{\alpha\beta} + b w_{\alpha\beta} \tag{10}$$

where $w_{\alpha\beta}$ is an unknown symmetric tensor independent of $\tilde{T}_{\alpha\beta}$; and a and b are arbitrary constants. Using the ODT, $w_{\alpha\beta}$ can then be orthogonally decomposed into

$$w_{\alpha\beta} = v_{\alpha\beta} + \Phi_{\alpha\beta} \tag{11}$$

where $\Phi_{\alpha\beta}$ is given by (9) and $\nabla^{\alpha} v_{\alpha\beta} = 0$.

Secondly, Einstein concluded that the metric should describe both the geometry of spacetime and the gravitational field. He postulated the totality of the matter energy-momentum tensor and the energy-momentum of the gravitational field, to be the source of the gravitational field. Adhering to this philosophy, the energy-momentum tensor $T_{\alpha\beta}$ describing total matter and the energy-momentum of the gravitational field, including dark energy, is postulated to be the source of the gravitational field.

$\Phi_{\alpha\beta}$ does not represent total matter or the radiation associated with it, which is entirely described by $\tilde{T}_{\alpha\beta}$; and it is not divergenceless. $\Phi_{\alpha\beta}$ is therefore the sole candidate to describe the energy-momentum of the gravitational field and dark energy. Thus, $T_{\alpha\beta} = \tilde{T}_{\alpha\beta} + \frac{bc}{a}\Phi_{\alpha\beta}$.

It was proven by Lovelock [22] that the only tensors in four dimensional spacetime which are symmetric, divergence free, and a concomitant of the metric tensor together with its first two derivatives are the metric and the Einstein tensor $G_{\alpha\beta}$. $v_{\alpha\beta}$ must therefore consist of the Lovelock tensors and a residual symmetric divergenceless tensor $h_{\alpha\beta}$ representing the sum of any other possible (0,2) symmetric divergenceless tensors. $h_{\alpha\beta}$ does not contain $g_{\alpha\beta}$ or $G_{\alpha\beta}$ and is *not* the linearized gravitational metric.

With $v_{\alpha\beta} = H_{\alpha\beta} + h_{\alpha\beta}$, $\tilde{\Psi}_{\alpha\beta}$ is then formally decomposed as

$$\tilde{\Psi}_{\alpha\beta} = \frac{a}{c} T_{\alpha\beta} + bH_{\alpha\beta} + bh_{\alpha\beta} \tag{12}$$

with $\nabla_{\alpha} H^{\alpha\beta} = 0$ and $\nabla_{\alpha} h^{\alpha\beta} = 0$ where $H_{\alpha\beta}$ contains the Lovelock tensors: $H_{\alpha\beta} := G_{\alpha\beta} + \Lambda g_{\alpha\beta} + \chi g_{\alpha\beta}$. Λ *is an integration constant* (in hindsight identified as the cosmological constant) and the Einstein tensor is defined by $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$. χ is a scalar with the property $\partial_{\alpha}\chi = 0$. It is introduced to facilitate the metric as a field variable when $\tilde{\Psi}_{\alpha\beta}$ is the symmetric spin-2 wave function. The traceless condition

$$g^{\alpha\beta}(\chi g_{\alpha\beta} + h_{\alpha\beta}) = 0 \tag{13}$$

is imposed on $h_{\alpha\beta}$ and the χ term. This requires $\chi = -\frac{1}{4} h$. $\tilde{\Psi}_{\alpha\beta}$ can now be written as

$$\tilde{\Psi}_{\alpha\beta} = \frac{a}{c} \tilde{T}_{\alpha\beta} + b(G_{\alpha\beta} + \Lambda g_{\alpha\beta} + \Phi_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{4} h g_{\alpha\beta}). \tag{14}$$

A modified Einstein equation of GR which involves $\Phi_{\alpha\beta}$ can now be obtained. With $\tilde{\Psi}_{\alpha\beta}$ being traceless and $\mid g^{\alpha\beta} \mid \neq 0$, the only solution to

$$\mid g^{\alpha\beta} \tilde{\Psi_{\alpha\beta}} \mid = 0 \tag{15}$$

is $\tilde{\Psi}_{\alpha\beta} = \frac{a}{c}\tilde{T}_{\alpha\beta} + b(G_{\alpha\beta} + \Lambda g_{\alpha\beta} + \Phi_{\alpha\beta}) = 0$. By setting $a = -\frac{1}{2}$ and $b = \frac{c^3}{16\pi G}$, we obtain the modified Einstein equation of general relativity with cosmological constant Λ and the gravitational energy-momentum dark energy term $\Phi_{\alpha\beta}$

$$-\frac{8\pi G}{c^4} \tilde{T}_{\alpha\beta} + R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} + \Phi_{\alpha\beta} = 0. \tag{16}$$

This equation can be derived from the action functional

$$\begin{aligned} S &= S^F + bS^{EH} + bS^{DE} \\ &= \int L^F(A^{\beta}, \nabla^{\alpha} A^{\beta}, ..., g^{\alpha\beta}) \sqrt{-g} d^4x + b \int (R - 2\Lambda) \sqrt{-g} d^4x - b \int \Phi_{\alpha\beta} g^{\alpha\beta} \sqrt{-g} d^4x \end{aligned} \tag{17}$$

where S^F and L^F refer to the action and Lagrangian for all types of matter fields including those of dark matter, S^{EH} is the Einstein-Hilbert action for general relativity and S^{DE} is the action for the energy-momentum of the gravitational field and dark energy. From the variation of S^F with respect to $g^{\alpha\beta}$

$$\begin{aligned} \delta S^F &= \int (\sqrt{-g} \delta L^F + L^F \delta \sqrt{-g}) d^4x \\ &= \int (\frac{\delta L^F}{\delta g^{\alpha\beta}} - \frac{1}{2} L^F g_{\alpha\beta}) \delta g^{\alpha\beta} \sqrt{-g} d^4x \end{aligned} \tag{18}$$

the energy-momentum tensor $\tilde{T}_{\alpha\beta}$ is defined as

$$\tilde{T}_{\alpha\beta} = -2c(\frac{\delta L^F}{\delta g^{\alpha\beta}} - \frac{1}{2} L^F g_{\alpha\beta}). \tag{19}$$

The variation of S with respect to $g^{\alpha\beta}$ is

$$\begin{aligned}
\delta S &= -\frac{1}{2c}\int \tilde{T}_{\alpha\beta}\delta g^{\alpha\beta}\sqrt{-g}d^4x + b\ \delta\int (R-2\Lambda)\sqrt{-g}d^4x - b\ \delta\int \nabla_\alpha X_\beta (g^{\alpha\beta} + 2u^\alpha u^\beta)\sqrt{-g}d^4x \\
&= \int [-\frac{1}{2c}\tilde{T}_{\alpha\beta} + b(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R) + b\Lambda g_{\alpha\beta} + b(\nabla_\alpha X_\beta - \frac{1}{2}g_{\alpha\beta}\nabla^\lambda X_\lambda + 2u^\lambda u_\beta \nabla_\lambda X_\alpha - u_\alpha u_\beta \nabla_\mu X_\nu g^{\mu\nu} \\
&+ \frac{1}{2}\nabla^\lambda X_\lambda g_{\alpha\beta} + u^\mu u^\nu \nabla_\mu X_\nu g_{\alpha\beta})]\delta g^{\alpha\beta}\sqrt{-g}\ d^4x \\
&= \int [-\frac{1}{2c}\tilde{T}_{\alpha\beta} + b(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R) + b\Lambda g_{\alpha\beta} + b(\nabla_\alpha X_\beta + 2u^\lambda u_\beta \nabla_\alpha X_\lambda \\
&+ \nabla_\mu X_\nu(-u_\alpha u_\beta g^{\mu\nu} + u^\mu u^\nu g_{\alpha\beta}))]\delta g^{\alpha\beta}\sqrt{-g}\ d^4x
\end{aligned} \tag{20}$$

after calculating $\delta\Gamma^\lambda_{\alpha\beta}$ induced by the variations in the metric, and integrating by parts several times. The last term vanishes which follows by writing the tensor in brackets, $-u_\alpha u_\beta g^{\mu\nu} + u^\mu u^\nu g_{\alpha\beta}$, as its equivalent, $\frac{1}{2}(g^{+\mu\nu}g_{\alpha\beta} - g^+_{\alpha\beta}g^{\mu\nu})$; and choosing an orthonormal frame at a point $x \in M$ for g^+ with one axis of u along the x^0 axis of g^+; and noting $g^{00} = -g^{+00}, g_{00} = -g^+_{00}$ with all other components of the metric g equal to those of the metric g^+. As all indicies are summed, the second last term $b(\nabla_\alpha X_\beta + 2u^\lambda u_\beta \nabla_\alpha X_\lambda)$ can be expressed as $b(\frac{1}{2}(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha) + u^\lambda(u_\alpha \nabla_\beta X_\lambda + u_\beta \nabla_\alpha X_\lambda)) = b\Phi_{\alpha\beta}$. With $\delta S = 0$ and arbitrary variations $\delta g^{\alpha\beta}$, we have

$$-\frac{1}{2c}\tilde{T}_{\alpha\beta} + b(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R) + b\Lambda g_{\alpha\beta} + b\Phi_{\alpha\beta} = 0. \tag{21}$$

Setting $b = \frac{c^3}{16\pi G}$ as above yields the modifed Einstein equation described in (16).

Equation (14) simplifies to

$$\tilde{\Psi}_{\alpha\beta} = \frac{c^3}{16\pi G}(h_{\alpha\beta} - \frac{1}{4}hg_{\alpha\beta}) \tag{22}$$

and we see that *GR and dark energy are hidden in* $\tilde{\Psi}_{\alpha\beta}$. With $\partial_\mu h = 0$, it follows that

$$\nabla_\alpha \tilde{\Psi}^{\alpha\beta} = 0. \tag{23}$$

Equations (22) and (23) apply to all of the noted spins. The relationship of $\tilde{\Psi}^{\alpha\beta}$ to each spin is now investigated.

3.1. Spin-1 Klein-Gordon equation

With Ψ a vector with real components A^β, (1) yields the covariant equation

$$\nabla_\alpha \nabla^\alpha A^\beta = k^2 A^\beta. \tag{24}$$

In general, this equation has both spin-0 and spin-1 characteristics; but the constraint (3) ensures (24) describes a vector boson. In curved spacetime, covariant derivatives generally do not commute, so the ansatz

$$\nabla_\alpha K^{\alpha\beta} = k^2 A^\beta \tag{25}$$

is not equivalent to $\nabla_\alpha \nabla^\alpha A^\beta = k^2 A^\beta$ and $\nabla_\alpha A^\alpha = 0$, contrary to the situation in Minkowski spacetime. Rather,

$$\nabla_\alpha K^{\alpha\beta} = k^2 A^\beta - \nabla_\alpha \nabla^\beta A^\alpha. \tag{26}$$

Since the traditional Proca equation in curved spacetime is described [23, 36] by (25), the KG equation must be expressed with a new term as

$$\nabla_\alpha \nabla^\alpha A^\beta = k^2 A^\beta + \nabla_\alpha \nabla^\beta A^\alpha. \tag{27}$$

This method restricts the left side of (25) to that of an antisymmetric tensor and modifies the form of the KG equation.

Contrary to this approach, we choose to retain the form of the KG equation given by (24) and abandon the *sole* reliance on the antisymmetric tensor $K^{\alpha\beta}$. The KG equation can be preserved and still involve the useful tensor $K^{\alpha\beta}$ by using the second rank tensor $\Psi^{\alpha\beta} = \nabla^\alpha A^\beta$ and expressing the spin-1 KG equation as

$$\nabla_\alpha(\tilde{\Psi}^{\alpha\beta} + K^{\alpha\beta}) = 2k^2 A^\beta. \tag{28}$$

It follows that *GR and dark energy are hidden within the symmetric part of the spin 1 KG equation in curved spacetime.*

Remark 1. *The quantum mechanics of the spin-1 particle resides solely in the antisymmetric part of the KG equation.*

The Proca equation

$$\nabla_\alpha K^{\alpha\beta} = 2k^2 A^\beta \tag{29}$$

follows from (2) and (23). The factor of two is the result of preserving the form of the KG equation to facilitate its symmetrization. The Proca equation can be derived from the action

$$S^P = \int (-\frac{1}{4}K^{\alpha\beta}K_{\alpha\beta} - k^2 A_\alpha A^\alpha)\sqrt{-g}d^4x. \tag{30}$$

With the commutation relation

$$\begin{aligned}[\nabla_\alpha, \nabla_\lambda]A^\alpha &= \nabla_\alpha\nabla_\lambda A^\alpha \\ &= A^\sigma R^\alpha{}_{\sigma\alpha\lambda} \\ &= A^\sigma R_{\sigma\lambda}\end{aligned} \tag{31}$$

we have

$$\nabla_\alpha K^{\alpha\beta} = k^2 A^\beta - A^\sigma R^\beta_\sigma. \tag{32}$$

From (29) it follows that

$$k^2 A^\beta = -A^\sigma R^\beta_\sigma. \tag{33}$$

We see how the geometry of spacetime is related to the mass of the vector components of the spin-1 KG equation.

3.2. Spin-0 Klein-Gordon equation

When Ψ is a scalar, (1) yields the spin-0 Klein-Gordon equation

$$\nabla_\alpha\nabla^\alpha\varphi = k^2\varphi. \tag{34}$$

With

$$\varphi = A^\beta A_\beta \tag{35}$$

and $\Psi^{\alpha\beta} = \nabla^\alpha A^\beta$, we have

$$\begin{aligned}\nabla_\alpha\nabla^\alpha\varphi &= 2\nabla_\alpha(A_\beta\Psi^{\alpha\beta}) \\ &= A_\beta\nabla_\alpha(\tilde{\Psi}^{\alpha\beta} + K^{\alpha\beta}) + \frac{1}{2}(\tilde{\Psi}_{\alpha\beta}\tilde{\Psi}^{\alpha\beta} + K_{\alpha\beta}K^{\alpha\beta})\end{aligned} \tag{36}$$

where $\tilde{\Psi}^{\alpha\beta}K_{\alpha\beta} = 0$ because the tensors in the product have opposite symmetries. This proves that *GR and dark energy are hidden within the symmetric part of the spin-0 KG equation in curved spacetime.*

3.3. Spin-1/2 Klein-Gordon equation

The spin-1/2 KG equation in curved spacetime expressed in terms of its spinor indices is

$$\nabla_\alpha \nabla^\alpha \Psi^{A\dot{A}} = k^2 \Psi^{A\dot{A}}. \tag{37}$$

A two index spinor $\varphi^{A\dot{B}}$ can be expressed in terms of an associated tensor A^β according to [24, 25]

$$\varphi^{A\dot{B}} = \sigma_\beta^{A\dot{B}} A^\beta. \tag{38}$$

The Hermitian connecting quantities $\sigma_\beta^{A\dot{B}}$ transform as a spacetime vector on the index β and as spinors on the index $A = 1, 2$ and conjugate index $\dot{B} = 1, 2$. Covariant derivatives of spinors are introduced in the same formalism as that for tensors by adopting the spinor affinities $\Gamma^A_{\alpha B}$ and defining

$$\nabla_\alpha \Psi_A = \partial_\alpha \Psi_A - \Gamma^B_{\alpha A} \Psi_B, \;\; \nabla_\alpha \Psi^A = \partial_\alpha \Psi^A + \Gamma^A_{\alpha B} \Psi^B \tag{39}$$

for the spinors Ψ_A and Ψ^A respectively. The covariant derivative of a mixed index spinor-tensor is defined as

$$\nabla_\alpha \Psi^{\beta A} = \partial_\alpha \Psi^{\beta A} + \Gamma^\beta_{\alpha\kappa} \Psi^{\kappa A} + \Gamma^A_{\alpha B} \Psi^{\beta B} \tag{40}$$

and the covariant derivative of the connection quantities is postulated to vanish

$$\nabla_\kappa \sigma^\alpha_{A\dot{B}} = 0. \tag{41}$$

This means

$$\partial_\kappa \sigma^\alpha_{A\dot{B}} + \sigma^\beta_{A\dot{B}} \Gamma^\alpha_{\kappa\beta} - \sigma^\alpha_{C\dot{B}} \Gamma^C_{\kappa A} - \sigma^\alpha_{A\dot{C}} \Gamma^{\dot{C}}_{\kappa\dot{B}} = 0. \tag{42}$$

The spinor-tensor covariant derivative is equivalent to the tensor covariant derivative if and only if this postulate holds. We can prove that from the covariant derivatives of a vector A^α and its spinor equivalent as follows:

$$\begin{aligned} \nabla_\kappa A^\alpha &= \sigma^\alpha_{A\dot{B}} \nabla_\kappa A^{A\dot{B}} \\ &= \sigma^\alpha_{A\dot{B}} (\partial_\kappa A^{A\dot{B}} + \Gamma^A_{\kappa C} A^{C\dot{B}} + \Gamma^{\dot{B}}_{\kappa\dot{C}} A^{A\dot{C}}) \\ &= \partial_\kappa A^\alpha + \Gamma^\alpha_{\kappa\beta} A^\beta - A^{A\dot{B}} (\partial_\kappa \sigma^\alpha_{A\dot{B}} + \sigma^\beta_{A\dot{B}} \Gamma^\alpha_{\kappa\beta} - \sigma^\alpha_{C\dot{B}} \Gamma^C_{\kappa A} - \sigma^\alpha_{A\dot{C}} \Gamma^{\dot{C}}_{\kappa\dot{B}}) \\ &= A^\alpha{}_{;\kappa} \end{aligned} \tag{43}$$

Equation (37) is then equivalent to

$$\sigma_\beta^{A\dot{A}} \nabla_\alpha \nabla^\alpha A^\beta = k^2 \Psi^{A\dot{A}} \tag{44}$$

using (38). This can be rewritten in terms of $\Psi^{\alpha\beta} = \nabla^\alpha A^\beta$ and symmetrized as in (2) to give

$$\sigma_\beta^{A\dot{A}} \nabla_\alpha (\tilde{\Psi}^{\alpha\beta} + K^{\alpha\beta}) = 2k^2 \sigma_\beta^{A\dot{A}} A^\beta. \tag{45}$$

We see that *GR and dark energy are hidden within the symmetric part of the spin-1/2 KG equation in curved spacetime.*

Remark 2. *The quantum mechanics of the spin-1/2 particle resides solely in the antisymmetric part of the spin-1/2 KG equation. Solutions of the Dirac equations are solutions of their parent spin-1/2 KG equation.*

From (23), equation (45) does not involve the symmetric part of $\Psi^{\alpha\beta}$. The Dirac equations in curved spacetime

$$(\gamma^\nu \nabla_\nu - k)\Psi^A = 0, \;\; (\gamma^\nu \nabla_\nu + k)\Psi^{\dot{A}} = 0 \tag{46}$$

are taken to be factorizations of the spin-1/2 KG equation. The gamma matrices γ^μ in curved spacetime are assumed to satisfy the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{47}$$

The product of the Dirac factorizations

$$(\gamma^\mu \nabla_\mu + k)(\gamma^\nu \nabla_\nu - k)\Psi^{A\dot{A}} = 0 \tag{48}$$

must yield the spin-1/2 KG equation. Using

$$\nabla_\mu \gamma^\nu = 0 \tag{49}$$

we have

$$\gamma^\mu \gamma^\nu \nabla_\mu \nabla_\nu \Psi^{A\dot{A}} = k^2 \Psi^{A\dot{A}} \tag{50}$$

which is expressed in the literature [26, 27] (with the metric having a +2 signature) as

$$\nabla_\mu \nabla^\mu \Psi^{A\dot{A}} = (k^2 + \frac{1}{4}R)\Psi^{A\dot{A}}. \tag{51}$$

With only the algebra of (47), the spin-1/2 KG equation in curved spacetime is not precisely recoverable due to the additional $\frac{1}{4}R$ term. This term can be eliminated by inserting the scalar $\Omega = \frac{1}{2}\sqrt{R}$ into (46) to obtain the modified Dirac equations

$$(\gamma^\nu \nabla_\nu + \Omega - k)\Psi^A = 0, \;\; (\gamma^\nu \nabla_\nu + \Omega + k)\Psi^{\dot{A}} = 0. \tag{52}$$

By defining the algebras

$$\{\gamma^\mu \nabla_\mu, \Omega\} = 0 \tag{53}$$

and

$$\frac{1}{2}\{\Omega, \Omega\} = \frac{R}{4}, \tag{54}$$

the product of the modified Dirac equations in curved spacetime yields their parent spin-1/2 KG equation (37).

3.4. Spin-2 and spin-3/2 Klein-Gordon equations

The KG equation for a spin-2 field is

$$\nabla_\mu \nabla^\mu \Psi^{\alpha\beta} = k^2 \Psi^{\alpha\beta}. \tag{55}$$

The spin-2 wave function $\Psi^{\alpha\beta}$ is unique in that it and the metric are second rank tensors. The symmetric part of the wave function must therefore involve the metric as a field variable to enable gravity to be described by the spin-2 KG equation. The decomposition (14) automatically incorporates the metric as a field variable in the symmetric part of the spin-2 wave function

$$\tilde{\Psi}^{\alpha\beta} = \frac{c^3}{16\pi G}(h^{\alpha\beta} - \frac{1}{4}hg^{\alpha\beta}). \tag{56}$$

$\tilde{\Psi}^{\alpha\beta}$ is symmetric and divergenceless. It is also traceless which ensures any possible spin-0 terms are eliminated and has 5 degrees of freedom if $k \neq 0$. $\tilde{\Psi}^{\alpha\beta}$ must satisfy

$$\nabla_\mu \nabla^\mu h^{\alpha\beta} = k^2(h^{\alpha\beta} - \frac{1}{4}hg^{\alpha\beta}) \tag{57}$$

with $h = h^{\alpha\beta}g_{\alpha\beta}$ a solution to $\partial_\mu h = 0$ and the wave equation

$$\nabla_\mu \nabla^\mu h = 0. \tag{58}$$

If $k \neq 0$, massive gravitons are described in the short-range high-energy Planck regime by (57).

The action

$$S^G = \frac{1}{2}\int (\nabla_\mu \Psi_{\alpha\beta} \nabla^\mu \Psi^{\alpha\beta} + k^2 \Psi_{\alpha\beta}\Psi^{\alpha\beta})\sqrt{-g}d^4x \tag{59}$$

can be used to derive the spin-2 KG equation (55). Because $\tilde{\Psi}^{\alpha\beta}$ contains the modified Einstein equation, *GR and dark energy are hidden within the symmetric part of the KG equation for a spin-2 field in curved spaetime.*

Spin-3/2. A spin-3/2 field can be described by the vector-spinor wave equation

$$\nabla_\mu \nabla^\mu \Psi^{\alpha A\dot{A}} = k^2 \Psi^{\alpha A\dot{A}}. \tag{60}$$

Using (44), this equation can be written as

$$\sigma_\beta^{A\dot{A}}(\nabla_\mu \nabla^\mu - k^2)\Psi^{\alpha\beta} = 0. \tag{61}$$

It follows from the arguments above that *GR and dark energy are hidden within the KG equation for a spin-*3/2 *field in curved spacetime.*

Some properties of $\Phi_{\alpha\beta}$ are discussed in sections 5 and 7 which validate why it describes the energy-momentum of the gravitational field and dark energy. ■

4. Massless spin-2 "particles"

Gravitons are taken to be massless particles because of the $\frac{1}{r^2}$ long-range force behaviour of gravity. They have spin-2 so that they can couple to the energy-momentum tensor. However, if $k = 0$, the metric as a field variable vanishes from the spin-2 KG equation leaving

$$\nabla_\mu \nabla^\mu h^{\alpha\beta} = 0 \tag{62}$$

as the equation describing a massless spin-2 "particle". Because $h^{\alpha\beta}$ does not depend on the metric, this equation cannot describe a massless spin-2 graviton in the low-energy long-range to particle regimes of spacetime. Furthermore, massless spin-2 "particles" cannot couple to a non-zero energy-momentum tensor as force mediators for gravity in these regimes. If we calculate the interaction of the matter energy-momentum tensor with $h^{\alpha\beta}$, we obtain

$$\begin{aligned} S_h^{int} &= -\frac{1}{2c}\int \tilde{T}_{\alpha\beta} h^{\alpha\beta}\sqrt{-g}d^4x \\ &= -\frac{c^3}{16\pi G}\int R_{\alpha\beta} h^{\alpha\beta}\sqrt{-g}d^4x \end{aligned} \tag{63}$$

since $h^{\alpha\beta}$ is divergenceless and traceless. The variation of S_h^{int} with respect to $h^{\alpha\beta}$ requires $R_{\alpha\beta}$ to vanish because by definition, $h^{\alpha\beta}$ does not contain the metric or the Einstein tensor; and conversely. Then $S_h^{int} = -\frac{c^3}{16\pi G}\int \Phi_{\alpha\beta} h^{\alpha\beta}\sqrt{-g}d^4x$ and its variation with respect to $h^{\alpha\beta}$ requires $\Phi_{\alpha\beta}$ to vanish since $\Phi_{\alpha\beta}$ does not contain $h_{\alpha\beta}$. It follows that $\tilde{T}_{\alpha\beta}$ vanishes and there is no coupling to the energy momentum tensor. Massless spin 2 "particles" are therefore confined to the vacuum in accordance with $R_{\alpha\beta} = 0$.

The hierarchy problem of particle physics can be stated as the question: why is the force of gravity so much weaker than the other three known forces in nature? In the case of electrodynamics, if both gravity and electrodynamics have long-range massless force mediators, why is the electromagnetic force 10^{40} times stronger than that of gravity? The electroweak force is 10^{24} times stronger than gravity. And as the name suggests, the strong nuclear force presents the largest disparity to gravity at nuclear dimensions.

At the basis of this problem is the notion that gravity *must* be quantized in the particle and long-range regimes. However, the spin-2 KG equation in 4-dimensional spacetime excludes massless gravitons as force mediators of gravity in these regimes. This starkly contrasts the spin-1 KG equation for a massless photon which mediates the electromagnetic field; similarly for the electroweak force and the massive spin-1 W and Z bosons, and the spin-1 massless gluons mediating the strong nuclear force. Gravity has no massless particles that act as force mediators in the particle and long-range regimes. Thus, the hierarchy problem is at least substantially if not fully explained without the need of extra spatial dimensions inherent in string theory; or any other theory that involves massless gravitons.

5. Conserved Energy-Momentum Tensor and Energy-Momentum of the Gravitational Field

The invariance of the action functional describing gravity, total matter and dark energy fields under the symmetry of diffeomorphisms demands a symmetric divergenceless energy-momentum tensor $T^{\alpha\beta} = \tilde{T}^{\alpha\beta} - \frac{c^4}{8\pi G}\Phi^{\alpha\beta}$. This follows from an analysis of each term in the action functional S defined in (17). The action S^{EH} is independently invariant under a diffeomorphism. Variation of the action S^F with respect to the metric contains only $\tilde{T}^{\alpha\beta}$ because the variations of S^F with respect to each field and its derivatives vanish with the corresponding Euler-Lagrange equations. Variation of S^{DE} with respect to the metric involves only $\Phi^{\alpha\beta}$. Therefore, we can write

$$\int (-\frac{1}{2c}\tilde{T}^{\alpha\beta} + b\Phi^{\alpha\beta})\delta g_{\alpha\beta}\sqrt{-g}d^4x = 0. \tag{64}$$

As the Lie derivative along a vector X^β generates the infinitesimal change in a tensor under a diffeomorphism, $\delta g_{\alpha\beta} = \nabla_\alpha X_\beta + \nabla_\beta X_\alpha$. Integrating by parts then gives

$$\int \nabla_\alpha(-\frac{1}{2c}\tilde{T}^{\alpha\beta} + b\Phi^{\alpha\beta})X_\beta\sqrt{-g}d^4x = 0 \tag{65}$$

which requires

$$\nabla_\alpha T^{\alpha\beta} = 0 \tag{66}$$

for diffeomorphisms generated by arbitrary vector fields X^β. Equation (66) is the local description of the conservation of energy and momentum in a modified theory of GR which includes dark energy as described by (16).

Since a Lorentzian manifold is locally flat at every point in spacetime, $\Phi_{\alpha\beta}$, as defined in terms of the Lie derivatives

$$\Phi_{\alpha\beta} = \frac{1}{2}\pounds_X g_{\alpha\beta} + \pounds_X u_\alpha u_\beta, \tag{67}$$

vanishes locally. This yields the Killing equation

$$\partial_\alpha X_\beta + \partial_\beta X_\alpha = 0 \tag{68}$$

which gives the 10 generators of the Poincaré group. Equation (66), with its uniquely intrinsic local group generator, is fully consistent with the Einstein Equivalence Principle.

If the total matter tensor $\tilde{T_{\alpha\beta}} = 0$, $\Phi_{\alpha\beta}$ becomes the source of the geometry of spacetime:

$$G_{\alpha\beta} = -\Phi_{\alpha\beta}. \tag{69}$$

The gravitational field has an intrinsic energy-momentum which is attributed to $\Phi_{\alpha\beta}$. Being independent of $\tilde{T_{\alpha\beta}}$, it provides the additional energy and momentum from the gravitational field and dark energy necessary to complete the source $T_{\alpha\beta}$ of the geometry of spacetime.

It is straightforward to calculate the coupling of the gravitational field with its energy-momentum dark energy tensor

$$\int g^{\alpha\beta}\Phi_{\alpha\beta}\sqrt{-g}d^4x = \int \Phi\sqrt{-g}d^4x = 0 \tag{70}$$

where $\Phi = \nabla_\alpha X_\beta(g^{\alpha\beta} + 2u^\alpha u^\beta)$. This means Φ has local positive and negative values, all of which add to zero when integated over the entire spacetime; Φ is globally conserved. The trivial value of Φ is the Killing equation (68).

$\Phi_{\alpha\beta}$ is a symmetric tensor which includes repulsive components attributed to dark energy. This follows from the vierbein formalism that links local Lorentz transformations to each point in curved spacetime. A symmetric second rank tensor in curved spacetime can then be decomposed under the Lorentz group into five spin-2 subspaces, three vector spin-1 subspaces and two spin-0 subspaces. As demonstrated in section VII, the spin-1 vector subspaces can represent a gravitational repulsion; and the scalar spin-0 subspace can be a gravitationally repulsive negative energy density. Dark energy, a scalar, is apparently not so mysterious afterall. It can be interpreted as the repulsive part of the trace of the tensor describing the energy-momentum of the gravitational field and dark energy, $\Phi > 0$.

It is then appropriate to consider the gravitationally attractive condition $\Phi < 0$ to be associated with dark matter. Thus, in this article, dark matter is accomodated by $\tilde{T_{\alpha\beta}}$ or $\Phi < 0$ or both. An investigation of the properties of $\Phi < 0$ to describe dark matter is a topic for future consideration.

Ma and Wang [28] obtained a similar result to (16) and did consider $\Phi < 0$ as a description of dark matter, but with an entirely different $\mathit{\Phi}_{\alpha\beta}$. They postulated $\nabla_\alpha(\mathit{\Phi}^{\alpha\beta} + \frac{8\pi G}{c^4}T^{\alpha\beta}_{matter}) = 0$ with $\mathit{\Phi}_{\alpha\beta} = \nabla_\alpha\partial_\beta\phi$ for some scalar ϕ by using the decomposition (4). $\frac{c^4}{8\pi G}\Phi$, with $\Phi = g^{\alpha\beta}\mathit{\Phi}_{\alpha\beta}$, was identified as the potential energy density caused by the non-uniform distribution of matter in the universe. The negative part of Φ represented dark matter and the positive part represented dark energy.

6. Cosmological Constant

The metric cannot appear as a field variable alongside the cosmological constant. Λ is an integration constant and must vanish.

Theorem 6.1. *The cosmological constant* Λ *must vanish and is dynamically replaced by the trace of* $\mathit{\Phi}_{\alpha\beta}$*.*

Proof. The action S_g^{int} for the coupling of the energy-momentum tensor with the metric is:

$$\begin{aligned} S_g^{int} &= -\frac{1}{2c}\int T_{\alpha\beta}g^{\alpha\beta}\sqrt{-g}d^4x \\ &= \frac{c^3}{16\pi G}\int (R - 4\Lambda)\sqrt{-g}d^4x. \end{aligned} \tag{71}$$

Since the energy-momentum tensor, for baryonic matter, dark matter, dark energy and the energy-momentum of the gravitational field itself, is the source of the gravitational field and the geometry of spacetime, $S_g^{int} \equiv S^{EH}$. This requires the integration constant Λ to vanish.

Using (70), S^{EH} with $\Lambda = 0$ can then be written as

$$S^{EH} = \frac{c^3}{16\pi G}\int (R - \Phi)\sqrt{-g}d^4x \tag{72}$$

which generates the modified Einstein equation with no cosmological constant. If $\Phi = 2\Lambda$ *locally*, the Einstein equation with the cosmological constant is obtained. The trace of the tensor describing the energy-momentum of the gravitational field and dark energy, dynamically replaces the cosmological constant but must obey the global equation (70).

■

7. Energy-momentum of the gravitational field in the FRW metric

Some properties of $\mathit{\Phi}_{\alpha\beta}$ in the Friedmann-Robertson-Walker metric are now investigated. The FRW metric is typically used to describe a spatially maximal symmetric universe according to the cosmological principle [29] whereby the universe is homogeneous and isotropic when measured on a large scale. This metric is given by

$$ds^2 = -c^2dt^2 + a(t)^2[\frac{1}{1-\kappa r^2}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2)] \tag{73}$$

where $a(t)$ is the cosmological scale factor and κ is a constant used to describe a particular spatial geometry. The connection components of the FRW metric are

$$\Gamma^i_{j0} = \frac{\dot{a}}{ca}\delta^i_j, \;\; \Gamma^0_{ij} = \frac{\dot{a}}{ca}g_{ij}, \;\; \Gamma^\mu_{00} = 0 \tag{74}$$

where $j = 1, 2, 3$. The Ricci tensor components are

$$R_{00} = -3\frac{\ddot{a}}{ac^2}, \;\; R_{ij} = (\frac{\ddot{a}}{ac^2} + 2\frac{\dot{a}^2}{a^2c^2} + 2\frac{\kappa}{a^2})g_{ij} \tag{75}$$

and the Ricci scalar is

$$R = \frac{6}{a^2c^2}(a\ddot{a} + \dot{a}^2 + \kappa c^2). \tag{76}$$

It was proven in [29] that a maximally spatial form invariant symmetric second rank tensor $B_{\alpha\beta}$ has components in the form

$$B_{00} = \varrho(t), \;\; B_{ij} = p(t)g_{ij} \tag{77}$$

where $\varrho(t)$ and $p(t)$ are arbitrary functions of time. We therefore set,

$$\tilde{T}_{00} = c^2\varrho, \;\; \tilde{T}_{ij} = pg_{ij}, \;\; \tilde{T}^{\mu}_{\mu} = -c^2\varrho + 3p \tag{78}$$

where $\varrho(t)$ and $p(t)$ are designated as the mass density and pressure functions, respectively, of combined baryonic matter and dark matter. Similarly,

$$\mathit{\Phi}_{00} = \Lambda_d, \;\; \mathit{\Phi}_{ij} = \frac{P_d}{c^2}g_{ij}, \;\; \mathit{\Phi}^{\mu}_{\mu} = -\Lambda_d + 3\frac{P_d}{c^2} \tag{79}$$

where $\Lambda_d(t)$ and $P_d(t)$ refer to the energy density and pressure, respectively, of the tensor describing the energy-momentum of the gravitational field and dark energy.

To obtain the Friedmann equations, we use the trace of the modified Einstein equation

$$-\frac{8\pi G}{c^4}\tilde{T} - R + \Phi = 0 \tag{80}$$

to rewrite the modified Einstein equation as

$$R_{\alpha\beta} = \frac{8\pi G}{c^4}(\tilde{T}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\tilde{T}) + \frac{1}{2}g_{\alpha\beta}\Phi - \mathit{\Phi}_{\alpha\beta} \tag{81}$$

from which we obtain

$$3\frac{\ddot{a}}{a} = -4\pi G(\varrho + \frac{3p}{c^2}) + \frac{1}{2}c^2\Lambda_d + \frac{3}{2}P_d \tag{82}$$

from the R_{00} component. The R_{11} component gives

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2\kappa c^2}{a^2} = \frac{c^2}{2}(-\Lambda_d + \frac{P_d}{c^2}) + 4\pi G(\varrho - \frac{p}{c^2}) \tag{83}$$

and the conservation law for $T^{\alpha\beta}$ yields

$$\dot{\varrho} - \frac{c^2}{8\pi G}\dot{\Lambda}_d = -3\frac{\dot{a}}{a}(\varrho + \frac{p}{c^2} - \frac{c^2}{8\pi G}(\Lambda_d + \frac{P_d}{c^2})). \tag{84}$$

Inserting (82) into (83) produces a simpler equation

$$\frac{\dot{a}^2}{a^2} + \frac{\kappa c^2}{a^2} = \frac{8\pi G}{3}\varrho - \frac{1}{3}c^2\Lambda_d. \tag{85}$$

Equations (82) and (85) are the Friedmann equations modified with $\mathit{\Phi}_{\alpha\beta}$.

From (82) we immediately see that $\Phi = \Lambda_d + \frac{3}{c^2}P_d > 0$ tends to accelerate the universe; while baryonic matter and dark matter, when both have positive mass density and pressure, tend to decelerate the universe. $\Phi > 0$ is a gravitationally repulsive condition. This confirms the conjecture that dark energy is the repulsive part of the trace of the tensor describing the energy-momentum of the gravitational field and dark energy. Hence, Λ_d is called the dark energy density and P_d the dark energy pressure. However, Φ is constrained by (70) and must have positive and negative values. Φ can accelerate or decelerate the universe but has a net zero effect on it. $\mathit{\Phi}_{\alpha\beta}$ and therefore Φ, provide the flexibility to describe various eras in the evolution of the universe. The cosmological constant Λ, on the other hand, can be expressed as a fixed negative energy density which would have tended to accelerate the universe during *all* epochs.

One of the recent challenges in cosmology has been to find a natural mechanism that describes a small but positive vacuum energy density to explain the observed acceleration of the present universe. Dark energy provides a natural explanation of this challenge without the need of a cosmological constant.

After the discovery in 1929 by Hubble [30] that the universe was expanding, Λ was not required to obtain a solution to the Einstein equations with a positive mass density. Since the cosmological constant was vastly smaller than any value predicted by particle theory, most particle theorists simply assumed, that for some unknown reason, this quantity was zero [31]. This was widely believed to be true until the discovery of the presently accelerating universe in 1998-99 [32, 33]. Λ was then considered to be associated with the dark energy conumdrum. However, it is just an integration constant in the Einstein equation and *must* vanish as proven in theorem 6.1. By restricting the dark energy variables to the constant values $\Lambda_d = -\Lambda$ and $P_d = c^2\Lambda$ in (82) and (85), the Friedmann equations with the cosmological constant Λ are recovered in accordance with theorem 6.1.

The Friedmann equations are now considered with $\kappa = 1$ describing a closed universe:

$$\dot{a}^2 = \frac{8\pi G}{3}\varrho a^2 - \frac{c^2}{3}\Lambda_d a^2 - c^2 \tag{86}$$

and

$$\ddot{a} = -\frac{4\pi G}{3}a(\varrho + \frac{3p}{c^2}) + \frac{ac^2}{6}(\Lambda_d + \frac{3}{c^2}P_d). \tag{87}$$

To avoid confusion with Λ, we will denote the constant vacuum energy density measured in the present epoch as Λ_v with the property $\Lambda_v > 0$. By defining

$$\tilde{\varrho} = 8\pi G\varrho - c^2\Lambda_d \tag{88}$$

and

$$\tilde{p} = -\frac{4\pi G}{c^2}p + \frac{1}{2}P_d, \tag{89}$$

these equations can be simplified to

$$\dot{a}^2 = \frac{\tilde{\varrho}a^2}{3} - c^2, \tag{90}$$

and

$$\ddot{a} = a(-\frac{\tilde{\varrho}}{6} + \tilde{p}) \tag{91}$$

with the continuity equation

$$\dot{\tilde{\varrho}} = -3\frac{\dot{a}}{a}(\tilde{\varrho} - 2\tilde{p}). \tag{92}$$

After the Big Bang, the qualitative cyclic properties of the universe associated with this model are demonstrated from a traditional analysis of these equations. Unless otherwise stated, $\varrho > 0$ and $p > 0$. Equation (90) requires $\tilde{\varrho} > 0$. After the Big Bang, the universe violently expands with $\dot{a} > 0$. If P_d turns off or turns negative while the universe is still expanding, $\tilde{p}$ turns negative. From the continuity equation (92) under these conditions,

$$\frac{d}{dt}(\tilde{\varrho}a^3) = 6a^2\dot{a}\tilde{p} \leqq 0 \tag{93}$$

and if $a \to \infty$, $\tilde{\varrho}a^2 \to 0$. It follows from (90) that $\dot{a}^2 \to -c^2 < 0$ which is contradictory. The universe cannot expand indefinitely and has a maximum value a_{max}. The universe then decelerates toward zero, but if $P_d > \frac{8\pi Gp}{c^2}$ while the universe is contracting, the inequality (93) holds and $\tilde{\varrho}a^2 \to 0$ as $a \to 0$. From (90), it follows that as $a \to 0$, $\dot{a}^2 \to -c^2$. This is again a contradiction which means there must be a minimum value a_{min}. The dark energy pressure smoothly controls the maximum and minimum values that the cosmological scale factor can have. This model of the universe starts with the Big Bang and then cycles to eternity. It does not suffer the catastrophes of the Big Crunch or the Big Rip.

It is interesting to further explore how the energy-momentum of the gravitational field can describe critical features of a Big Bang universe. Immediately after the event of the Big Bang, the universe violently expands. For a very short time, there is no matter; $\varrho = 0$ and $p = 0$. If we set $\varrho = 0$ in (86), the inequality $\Lambda_d < -\frac{3}{a^2}$ must hold. As the universe expands, let $-\Lambda_d$ approach Λ_v. With $a > 0$, this suggests the expression

$$\Lambda_d = -\frac{3}{a^2} - \Lambda_v \tag{94}$$

which satisfies the condition $\Lambda_d < -\frac{3}{a^2}$ and which tends to $-\Lambda_v$ as a increases. Λ_d is asymptotic to a constant vacuum energy density $-\Lambda_v = -1.3 \times 10^{-52} m^{-2}$. The expansion of the universe is then described by $\dot{a}^2 = \frac{1}{3} a^2 c^2 \Lambda_v$. Dark energy generates Λ_v during this epoch of the universe. Using (84), the pressure density of dark energy is $P_d = \frac{c^2}{a^2} + \Lambda_v c^2$ and the acceleration of the universe is $\ddot{a} = \frac{1}{3} a c^2 \Lambda_v$.

With matter and dark matter appearing after the initial inflation, Λ_d plays an important role in the description of the evolving universe. With the ansatz (94) and ϱ constant, the expansion becomes

$$\dot{a}^2 = \frac{8\pi G}{3} \varrho a^2 + \frac{c^2}{3} \Lambda_v a^2 \tag{95}$$

and the acceleration is

$$\ddot{a} = \frac{8\pi G}{3} \varrho a + \frac{c^2}{3} \Lambda_v a \tag{96}$$

which shows why Λ_v is important in the present era. However, as a approaches a_{max}, equation (95) cannot satisfy $\dot{a} = 0$. The ansatz (94) must morph into another form such as $\Lambda_d = -\frac{3}{a^2} + \Lambda_v$ which can satisfy the extremum conditions $\Lambda_d < \frac{8\pi G \varrho}{c^2}$ at a_{max} and a_{min}. This suggests the dark energy density should be expressed as

$$\Lambda_d = -\frac{3}{a^2} - \vartheta \Lambda_v \tag{97}$$

where $\vartheta = \pm 1$. $\vartheta = 1$ during an expansion phase until just before a_{max} is reached and it flips its sign to -1. A contraction phase begins during which $\vartheta = -1$ until the minimum extremum is completed. It flips its sign to 1 and another expansion phase begins. The expansion and acceleration equations for constant total matter then become

$$\dot{a}^2 = \frac{8\pi G}{3} \varrho a^2 + \frac{c^2}{3} \vartheta \Lambda_v a^2 \tag{98}$$

and

$$\ddot{a} = \frac{8\pi G}{3} \varrho a + \frac{c^2}{3} \vartheta \Lambda_v a \tag{99}$$

repectively. If $\dot{\varrho} \neq 0$, the continuity equation (84) governs the dark energy pressure P_d, given an ansatz for Λ_d.

Although recent data and analysis [34] suggests the observable universe is flat, the data represents a small fraction of the entire universe. If the entire universe has a positive curvature, a measurement of it will appear to be nearly flat if data from large enough distances is not available. Therefore, at this time, the conjecture of a flat universe which expands forever based on observational evidence is no more likely than the cyclic universe described.

Dark energy thus provides a natural explanation of why the vacuum energy density is so small, and why it dominates the present and future epochs of the universe.

8. Quantum field theory in curved spacetime

The important part of this section is to determine how QFT in curved spacetime is formulated from the free field solutions to the KG equation for a particular spin.

8.1. Canonical QFT in Curved Spacetime

As discussed in [35] and summarized nicely by Mostepanenko [36], canonical quantization in curved spacetime for spin-0, 1 and 1/2 fields involves constructing a complete orthonormal set $\{\Psi_\alpha^{(+)}; \Psi_\alpha^{(-)}\}$ from the solutions to the equations (34) (with a term $\xi R \Psi_\alpha$ added for gravitational coupling to the scalar field); (29) and (46). The upper $\pm$ indices signify the positive and negative frequency solutions as clarifed in [35, 36]. The orthonormal conditions are

$$(\Psi_\alpha^{(+)}, \Psi_\beta^{(+)}) = \mp \delta_{\alpha\beta}, \;\; (\Psi_\alpha^{(-)}, \Psi_\beta^{(-)}) = \delta_{\alpha\beta}, \;\; (\Psi_\alpha^{(+)}, \Psi_\beta^{(-)}) = 0 \tag{100}$$

where the integration is performed over a globally spacelike hypersurface. The field Ψ can then be expressed as

$$\Psi = \Sigma_\alpha [\Psi_\alpha^{(-)} a_\alpha^{(-)} + \Psi_\alpha^{(+)} a_\alpha^{(+)}] \tag{101}$$

where the expressions for antiparticle creation and particle annihilation operators follow from the orthonormality conditons

$$a_{\alpha}^{(+)} = \mp(\Psi_{\alpha}^{(+)}, \Psi), \;\; a_{\alpha}^{(-)} = (\Psi_{\alpha}^{(-)}, \Psi). \tag{102}$$

The $\mp$ signs refer to the boson and fermion cases, respectively. Quantization of the field requires the commutation (anticommutation) relations

$$[a_{\alpha}^{*(-)}, a_{\beta}^{(+)}]\mp = [a_{\alpha}^{(-)}, a_{\beta}^{*(+)}]\mp = \delta_{\alpha\beta}, \;\; [a_{\alpha}^{(\pm)}, a_{\beta}^{(\pm)}]\mp = [a_{\alpha}^{*(\pm)}, a_{\beta}^{*(\pm)}]\mp = 0. \tag{103}$$

These are equivalent to the equal time canonical commutation (anticommutation) relations

$$[\Psi(t,x), \Psi(t,x')]\mp = [\pi(t,x), \pi(t,x')]\mp = 0, \;\; [\Psi(t,x), \pi(t,x')]\mp = i\delta(x-x') \tag{104}$$

where π is the canonically conjugate momentum operator defined by

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi)} = \partial_0 \Psi. \tag{105}$$

In curved spacetime, there is no unique way of defining the positive and negative frequency modes and therefore no unique "vacuum state". The concept of old-new states is introduced and if mixing of positive and negative frequency solutions occurs, then "particles" are created by the gravitational field. The new particle density in the old vacuum state is

$$< 0|\Psi_{\beta}|0 >= \Sigma_{\mu} \mid \Psi_{\beta\mu} \mid^2 \tag{106}$$

where the Bogolubov coefficients $\Psi_{\beta\mu}$ connect the old-new modes.

We have proven that the modified Einstein equation describing GR and dark energy is hidden in the symmetric part of the KG equations for the spins noted in section III. The field operators and the commutation (anticommutation) operators of canonical QFT in curved spacetime are obtained from the free field solutions to the KG equation. Canonical quantization of QFT in curved spacetime is formulated from the infinite superposition of KG free field solutions with mixed frequencies. GR and dark energy are hidden in the canonical formalism of QFT.

8.2. Algebraic QFT in Curved Spacetime

As stated in Wald [37], Hollands and Wald [38], and Brunetti and Fredenhagen [39], the standard canonical formulation of QFT relies heavily on concepts like vacuum and particles. However, these concepts lose their meaning in curved spacetime. Although natural notions of "vacuum state" and "particles" can be defined for a free field in stationary spacetimes, no such notions exist in a general curved spacetime. It is not that "particles" cannot be defined at all in curved spacetime but rather that many definitions exist and none appears preferred. This difficulty is not present in the algebraic approach. Algebraic QFT can be formulated without requiring a preferred representation of the canonical commutation relations, and without requiring the definition of a preferred notion of "particles".

In [37], algebraic QFT begins with a real scalar field satisfying the KG equation in a formulation so that all the basic ideas carry over without any essential change to real, linear bosonic fields. Minor modifications are needed to treat complex, linear, bosonic fields. The modifications needed to formulate the theory of linear fermionic fields essentially consist of several key sign changes. Hollands and Wald [38] start with a real scalar free field solution of the KG equation. The KG equation with a source $j(x^{\alpha})$

$$(\nabla_{\mu}\nabla^{\mu} - k^2)\phi = j \tag{107}$$

on a globally hyperbolic spacetime has a well posed initial value formulation in the sense defined in the article. The field equation of the algebra of observables generated by the fundamental field ϕ and a distribution f as a test function, is defined in the algebra as

$$\phi((\nabla_{\mu}\nabla^{\mu} - k^2)f) = 0. \tag{108}$$

The algebraic theory in [39] is based on the locally covariant approach to quantum field theory which uses the language of categories to incorporate the principle of general covariance. The functor defining the quantum theory depends on the free field KG with an additional coupling term

$$(\nabla_\mu \nabla^\mu - k^2 - \xi R)\phi = 0 \tag{109}$$

where ξ is a coupling constant to the Riemann scalar R.

An algebraic approach to QFT in curved spacetime must involve the KG equation in some fundamental role because it is the basic equation of relativistic quantum mechanics in curved spacetime. Therefore, *GR and dark energy are hidden in a canonical or an algebraic QFT in curved spacetime.* Although each major theory of physics appears to act independently of the other, GR resides beautifully within QFT. The metric as a field variable does not appear in the spin-2 KG equation for a massless "particle". Massless spin-2 "particles" do not couple to a non-zero energy-momentum tensor as the force mediators for gravity. GR down to the detectable size of elementary particles, such as the quark radius [40] of $\sim 10^{-19}$m, is not quantized. In that sense, QFT in curved four dimensional spacetime *is* quantum gravity in the particle and long-range regimes. GR is unique among the four forces and geometrically describes the force of gravity without the exchange of massless particles.

However, a more general theory of quantum gravity is required to explain the high energy physics near the Planck length $\sim 10^{-35}$m. We see in subsection 3.4 that a massive graviton exists in the short-range high-energy regime described by the metric and the topological tensor $h_{\alpha\beta}$. Massless spin-2 "particles" are associated with the vacuum. If the KG equation is fundamental to the physics of the Planck regime, these results will be important.

9. Conclusion

The results in this article were obtained directly from the rudimentary KG equation of relativistic quantum mechanics in curved spacetime, with the wave function having spins $0, 1, 2, 1/2$ and $3/2$. Decomposition of the KG equation was focused on its symmetric part described by the tensor $\tilde{\Psi}_{\alpha\beta}$. An orthogonal decomposition of symmetric tensors on a time oriented 4-dimensional Lorentzian manifold was developed. The decomposition of $\tilde{\Psi}_{\alpha\beta}$ introduced a new tensor $\Phi_{\alpha\beta}$. By requiring the field equations contained in $\tilde{\Psi}_{\alpha\beta}$ to be determined from an action functional, and adhering to Einstein's postulate requiring the energy-momentum tensor to be the source of both the geometry of spacetime and the gravitational field, a modified equation of general relativity which includes the energy-momentum of the gravitational field and dark energy, was obtained from the trace of $\tilde{\Psi}_{\alpha\beta}$. The modified Einstein equation was then generated from an action functional. As the terms in $\tilde{\Psi}_{\alpha\beta}$ involving the modified Einstein equation add to zero, general relativity and dark energy are hidden in the symmetric part of the KG equation for each spin.

Thus, quantum theory intrinsically contains general relativity and dark energy in the Klein-Gordon equation in curved spacetime. This leads to the following conclusions:

1. Both the canonical and algebraic approaches to QFT fundamentally involve free field solutions to the KG equation. As the modified Einstein equation is hidden in the symmetric part of the KG equation for each spin, general relativity is hidden in the formalism of quantum field theory. Although each major theory of physics appears to act independently of the other, GR resides beautifully within QFT.

2. The metric as a field variable does not appear in the spin-2 KG equation for a massless particle in the low-energy long-range to particle regimes. Massless spin-2 particles do not couple to a non-zero energy-momentum tensor as the force mediators for gravity.

3. Massless gravitons do not exist which substantially or completely explains the hierarchy problem of particle physics. Unlike the other three known fundamental forces in nature, no particle exchange is required to explain the force of gravity; that is nicely done by the curvature of spacetime.

4. Massless spin-2 "particles" are associated with the vacuum. Massive gravitons exist in the short-range high-energy regime near the Planck length.

5. When the total matter tensor $\tilde{T}_{\alpha\beta}$ vanishes, $\Phi_{\alpha\beta} = -G_{\alpha\beta}$. The gravitational field has an intrinsic energy-momentum attributed to $\Phi_{\alpha\beta}$ which represents the energy-momentum of the gravitational field and dark energy.

6. The cosmological constant Λ is an integration constant and must vanish. It is dynamically replaced by Φ.

7. $\Phi > 0$ is gravitationally replulsive and represents dark energy.

8. $\Phi < 0$ is gravitationally attractive and negative dark energy may be associated with dark matter.

9. Φ is constrained by the condition $\int \Phi\sqrt{-g}d^4x = 0$. Φ can accelerate or decelerate the universe but has a net zero global affect on it.

10. The dark energy pressure explains the accelerating and decelerating phases of the cyclic universe developed after the Big Bang. The dark energy density explains the initial inflation of the universe and provides a natural explanation of why the vacuum energy density is so small.

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