A 3-Dimensional Excursion through 2-Dimensional "Flatland" as an Analogy for 4-Dimensional Light and Coulombic Force in Three Dimensions

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The 19th century novella "Flatland" has often inspired comparisons of inter-dimensional phenomena, particularly how phenomena in dimension $n + 1$ would be manifested to observers restricted to dimension n. With n = 2, a possible analogy for the phenomena of light and Coulombic repulsion and attraction is examined for its potential extension to $n = 3$, given the impossibility of our even physically imagining a fourth dimension, purely spatial (i.e., not-temporal, in contrast to Einstein's time or space-time) like the three we know (length, width and height).

1. Introduction

"*Flatland: A Romance of Many Dimensions* is a satirical novella by the English schoolmaster Edwin Abbott Abbott, first published in 1884 by Seeley & Co. of London. Written pseudonymously by 'A Square,' the book used the fictional two-dimensional world of Flatland to comment on the hierarchy of Victorian culture, but the novella's more enduring contribution is its examination of dimensions … The story describes a two-dimensional world occupied by geometric figures, whereof women are simple line-segments, while men are polygons with various numbers of sides. The narrator is a square named A Square, … who guides the readers through some of the implications of life in two dimensions. The first half of the story goes through the practicalities of existing in a two-dimensional universe … On New Year's Eve, the Square dreams about a visit to a one-dimensional world (Lineland) inhabited by 'lustrous points,' in which he attempts to convince the realm's monarch of a second dimension; but is unable to do so … Following this vision, he is himself visited by a three-dimensional sphere named A Sphere, which he cannot comprehend until he sees Spaceland (a tridimensional world) for himself. This Sphere visits Flatland at the turn of each millennium to introduce a new apostle to the idea of a third dimension in the hopes of eventually educating the population of Flatland … After the Square's mind is opened to new dimensions, he tries to convince the Sphere of the theoretical possibility of the existence of a fourth (and fifth, and sixth ...) spatial dimension …" [1]

The concept of "Flatland" as a 2-dimensional realm where 3-dimensional phenomena are projected as 2-dimensional manifestations offers an intriguing way to construct mathematically and geometrically an analogy for light and Coulombic repulsion and attraction in our 3-dimensional world. Assume there is a fourth spatial dimension (not time or space-time, but an equivalent unnamed, and physically non-representable, spatial dimension to our known three of length, width and height) whose phenomena are manifested in three dimensions by what we perceive for light and Coulombic forces. By exploring these manifestations in Flatland, might a window be opened to at least a mathematical and geometrical construct for a fourth spatial dimension explaining the phenomena of light and Coulombic forces in our three dimensions?

2. Light

As a resident of 2-dimensional Flatland (the plane), you see a "light" source (yellow point) spread out at constant speed uniformly and circularly in all directions, represented by the red circle in the plane of Flatland, concluding that light spreads out at constant speed in all directions with its intensity decreasing in proportion to the circumference of the circle, i.e., as $1/2\pi r$ (see Figure 1). What you do not realize is that this is the result of light in Flatland being a manifestation of the intersection of a 3-dimensional cone moving perpendicularly to Flatland at a constant speed in its third dimension. This is the analogy for a 4-dimensional "cone" moving perpendicularly to our 3-dimesnional space at a constant speed such that it manifests as "light" moving uniformly and spherically outward from its initial point (of intersection) with an intensity decreasing in proportion to the surface area of the sphere, i.e., as $1/4\pi r^2$.

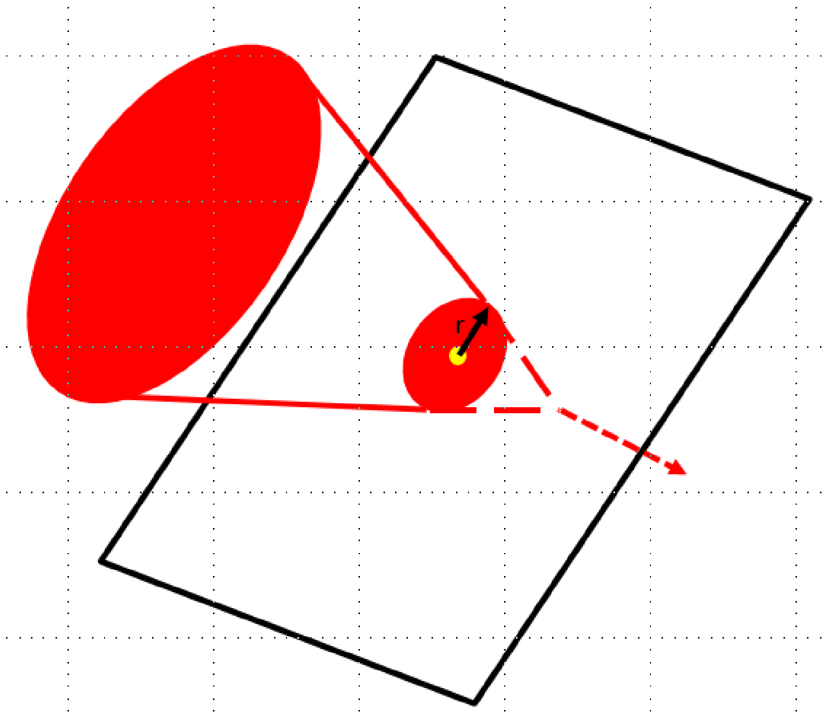


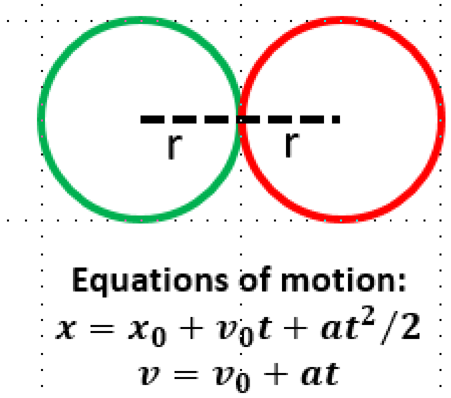
Figure 1. Representation of Light from a Point Source in Two Dimensions as the Result of Intersection between a Perpendicularly Incident 3-Dimensional Cone as it Passes "through" Flatland

3. Coulombic Repulsion (and Attraction)

The relatively simple representation for light spreading out spherically in three dimensions from a point source due to some 4-dimensional phenomenon (postulated as the "perpendicular" passing of a 4-diemsnional "cone" through our 3-dimensional world) offers an analogy for Coulombic repulsion (and attraction). We examine this from our realm of "Flatland," where an imperceptible 3-dimensional phenomena can only be "seen" in its 2-dimensional manifestation.

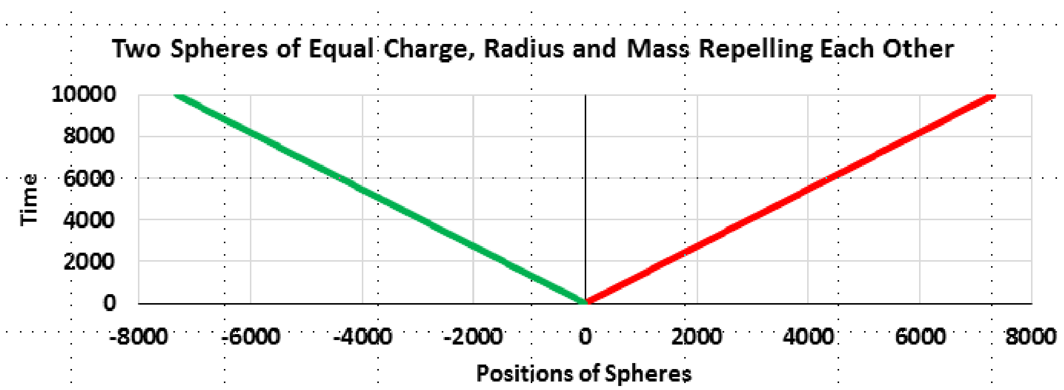
3.1. Case 1 – Equal Charge, Radius and Mass

As depicted in Figure 2, two spheres of equal charge (q), radius (r) and mass (m) start out touching, such that the distance between their centers is 2r. The Coulomb repulsion between them at time zero is $F = q^2/4\pi\varepsilon_0(2r)^2$. Setting the point where they touch as position x = 0, the center of each at time zero will be located r units away in opposite direction (i.e., $\pm r$). Each experiences a repulsive force F that imparts an initial acceleration a = F/m on each. Each has an initial speed of v = 0. After one time unit (t = 1), each will have moved outward to new position $x = r + (\frac{F}{m})\frac{(1)^2}{2}$ with speed $v = \left(\frac{F}{m}\right)(1)$ in the positive and negative directions.



<u>Figure 2</u>. Starting Position for Two Spheres (Manifested as Circles in Two Dimensions) of Equal Charge, Radius and Mass, with Equations of Motion, under Coulombic Repulsion

For simplicity, define q = 1 coulomb, r = 1 meter, $m = 1/4\pi\varepsilon_0$ kg, and $\varphi = 4\pi\varepsilon_0 F$ m/s^2, such that, at time zero, $\varphi = 1/4 = a$. After t = 1 s, $x = 1 + \frac{1}{4}(1)^2/2 = 1.125$ in the positive direction (-1.125 in the negative). The corresponding speed is $v = \frac{1}{4}(1) = 0.25$. We can repeat this recursion for subsequent time increments of 1 s, during which the acceleration will continuously decrease (since the spheres recede from each other, decreasing their repulsive force) and their speeds continuously increase (since there is no resistance and each time increment adds additional acceleration). A plot (Figure 3) of their respective positions over 10,000 s is essentially linear and symmetric with time in each direction (positive and negative).



<u>Figure 3</u>. Case 1 - Displacement vs. Time for Two Spheres of Equal Charge, Radius and Mass under Coulombic Repulsion

If we were to view this repulsion using a strobe light that flashes with a constant frequency over time starting when the two spheres (circles in Flatland), now shown as red and green, were touching at time zero, we would see them move linearly away from each other along a straight line at gradually increasing speed and distance (Figure 4). After a while, their speed would appear to be constant as would the increasing distance between them with each time increment (strobe flash). The reverse should be true for the two spheres if oppositely charged and starting at time zero from some great distance apart.

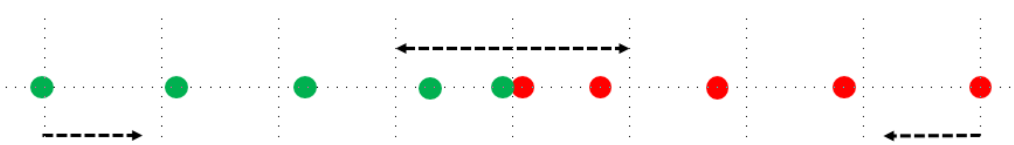


Figure 4. **Case 1 - Coulombic Repulsion (or Attraction) of Two Equal Spheres (Circles in Flatland) at Equal Time Increments**

3.2. Case 2 – Unequal Charge, but Equal Radius and Mass

Doubling the charge on one sphere while keeping their radii and masses the same does not change the symmetry, but shows both greater repulsion and attraction at each time increment, as manifested by the extension of the spheres' positions over the same time increments (Figure 5).

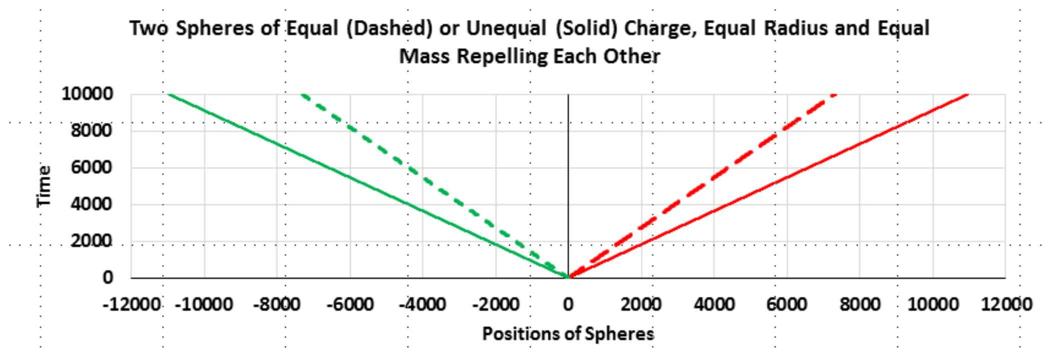


Figure 5. **Case 2 - Displacement vs. Time for Two Spheres of Unequal Charge but Equal Radius and Mass under Coulombic Repulsion**

3.3. Case 3 – Equal Charge and Radius, but Unequal Mass

Now we double the mass of one sphere while keeping their charge and radii equal. The sphere now with the doubled mass will experience only half the acceleration as its counterpart, and this will create an asymmetry between the positions of the spheres at each time increment (Figure 6). Additionally, the Coulomb repulsion between the spheres will increase relative to the case where the masses were equal since they will not separate as rapidly as previously. Assuming the green sphere has the doubled mass, we see that the position vs. time of the red sphere increases very slightly (since it is experiencing a greater force relative to previously at each time increment [previous shown as dashed black line, with difference more readily visible in Figure 7]), but that the displacement for the green sphere shows only half as much as previously, a consequence of its doubled mass reducing the acceleration.

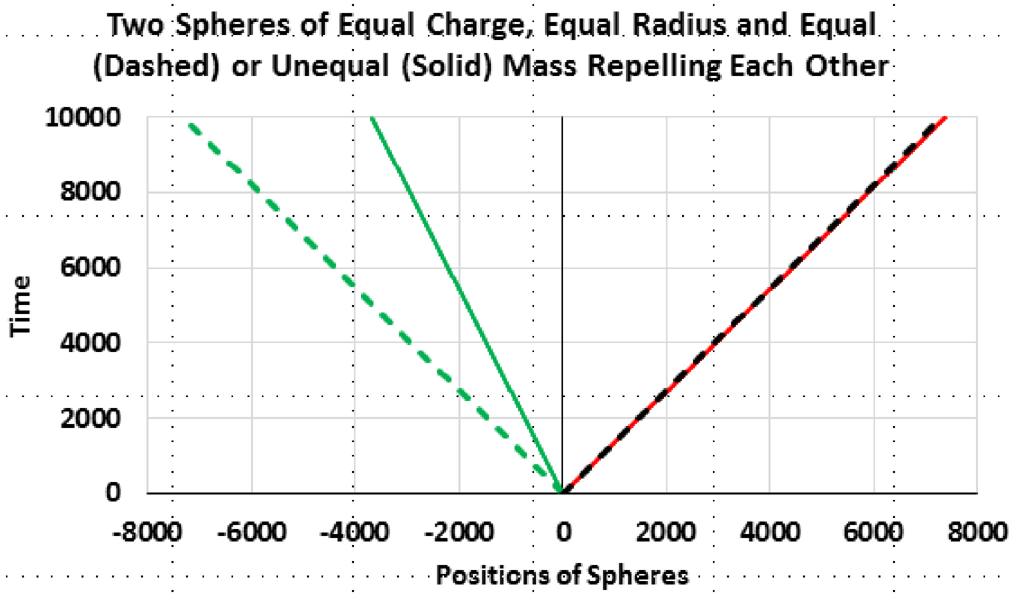


Figure 6. Case 3 - Displacement vs. Time for Two Spheres of Equal Charge and Radius, but Unequal Mass under Coulombic Repulsion

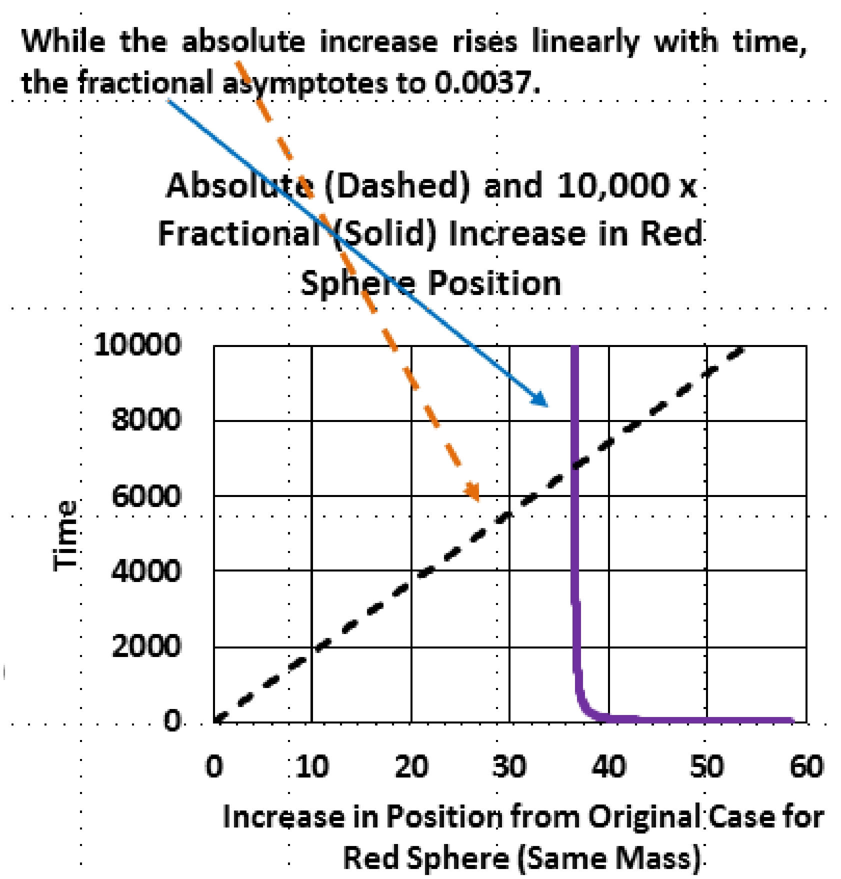


Figure 7. Case 3 - Absolute and 10,000 x Fractional (Solid) Increase in Red Sphere Position

If the “strobe” approach were applied now, one would see the green sphere moving as before, but at only half the rate as the red sphere (Figure 8).

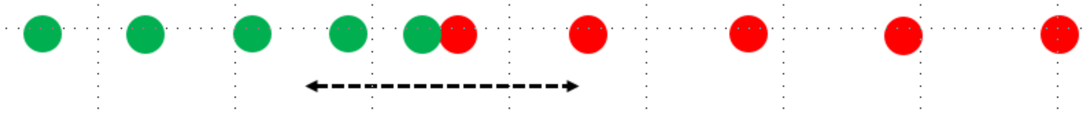


Figure 8. **Case 3 - Coulombic Repulsion of Two Unequal Mass Spheres (Circles in Flatland) at Equal Time Increments**

3.4. Case 4 – Equal Charge and Mass, but Unequal Radius

For the last perturbation, consider the relatively trivial case where the charges and masses are again equal, but one sphere has twice the radius of the other, such that the initial distance between their centers at time zero (when they are touching) is 3r rather than 2r. This will reduce the Coulomb repulsion between them at each time step relative to the original case, with this decrease becoming negligible as they spread farther apart and the initial extra distance of r is dwarfed by the increasing total distance. Figure 9 shows that the repulsion is essentially symmetric, but with reduced effect. Figure 10 shows this fractional difference between the two spheres’ positions becomes negligible with increasing time step.

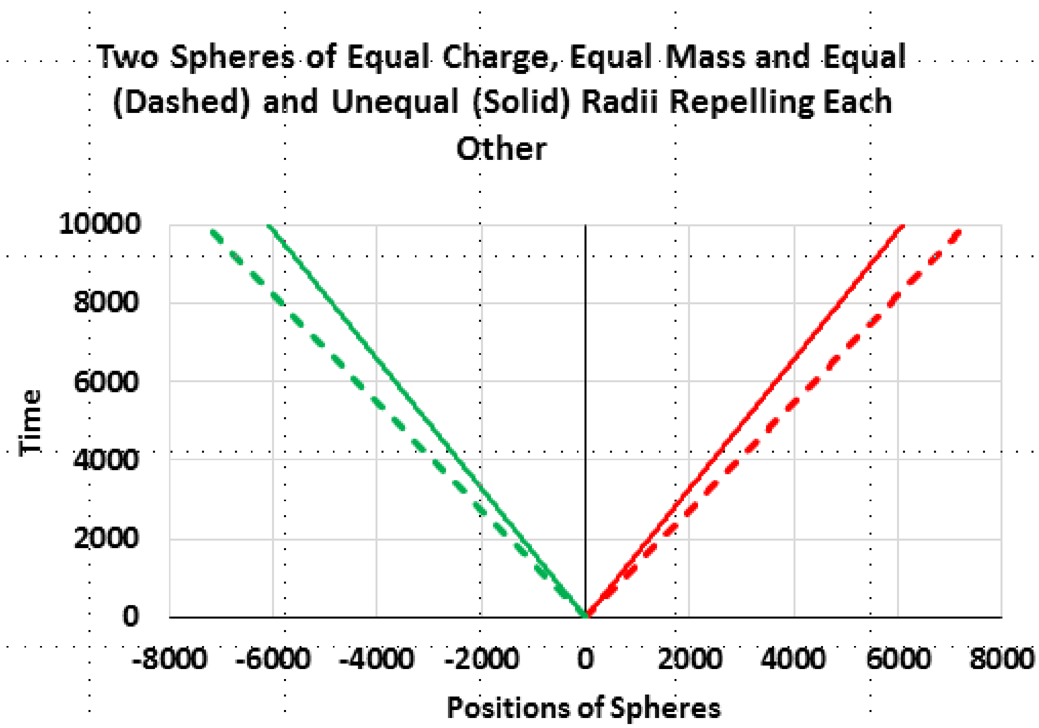


Figure 9. **Case 4 - Displacement vs. Time for Two Spheres of Equal Charge and Mass, but Unequal Radius under Coulombic Repulsion**

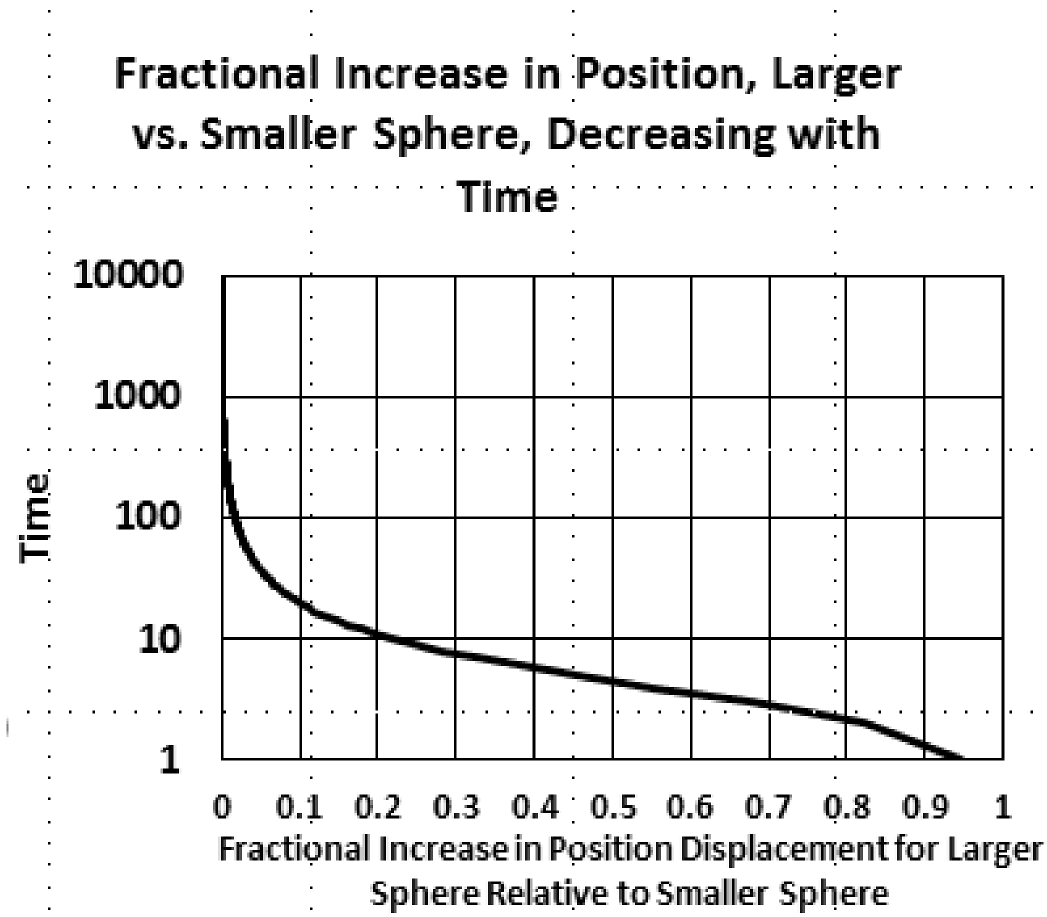


Figure 10. Case 4 – Fractional Increase in Displacement vs. Time for Two Spheres of Equal Charge and Mass, but Unequal Radius under Coulombic Repulsion

3.5. The Geometrical Construct for Flatland

For the simplest case (or any of the other symmetric ones), envision two slanted stacks of poker chips meeting at a right angle (Figure 11). These pass through "Flatland" along the perpendicular that bisects their right angle, such that, as each poker chip intersects Flatland, it appears as a circle, each of which recedes linearly away from the other at a speed proportional to that at which the "triangle" passes through Flatland. Initially, the speed is small, but increases with time until the two circles appear to be separating uniformly, i.e., at constant speed. This is analogous to the original case where the spreading circles, one red and one green, appear with each flash of the strobe. This can represent any of the symmetric spreading cases by varying the speed with which the triangle passes perpendicularly downward through Flatland, slower corresponding to lesser Coulombic repulsion, faster to greater Coulombic repulsion. And, if run in "reverse," we have a corresponding representation of Coulombic attraction.

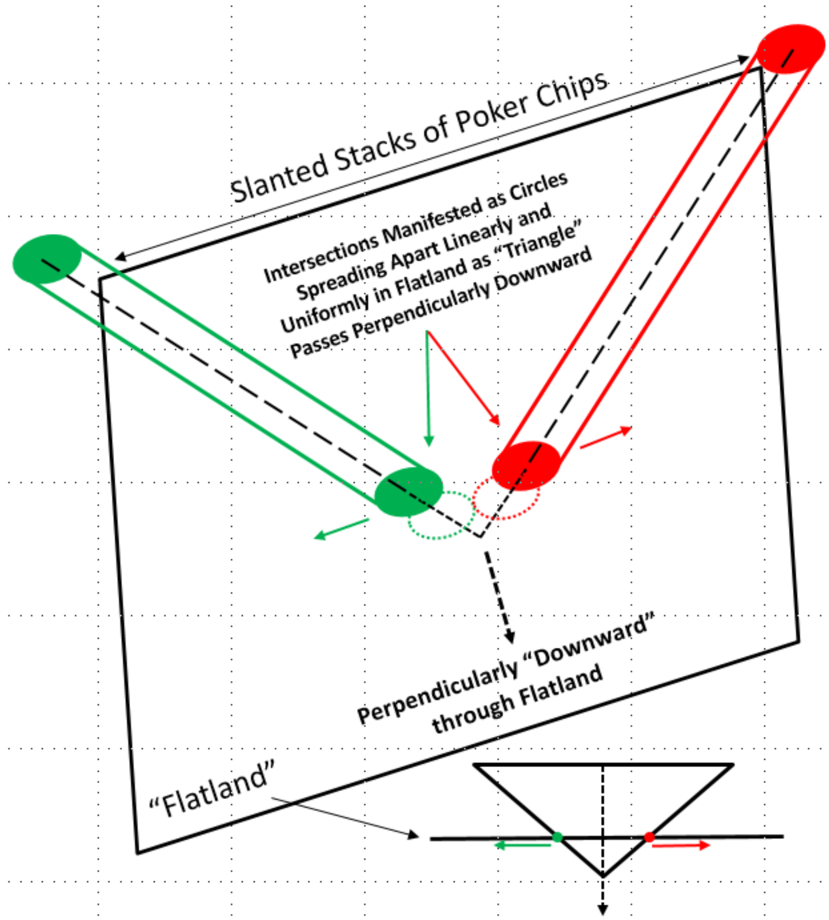


Figure 11. **Mathematical-Geometrical Construct for 3-Dimensional Coulombic Repulsion (or Attraction) as Manifested in 2-Dimensional Flatland**

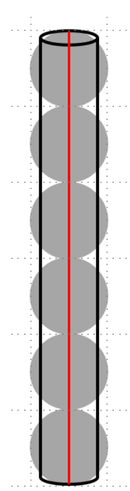


Figure 12. **Equivalent 3-Dimensional Cylinder for Stack of Spheres**

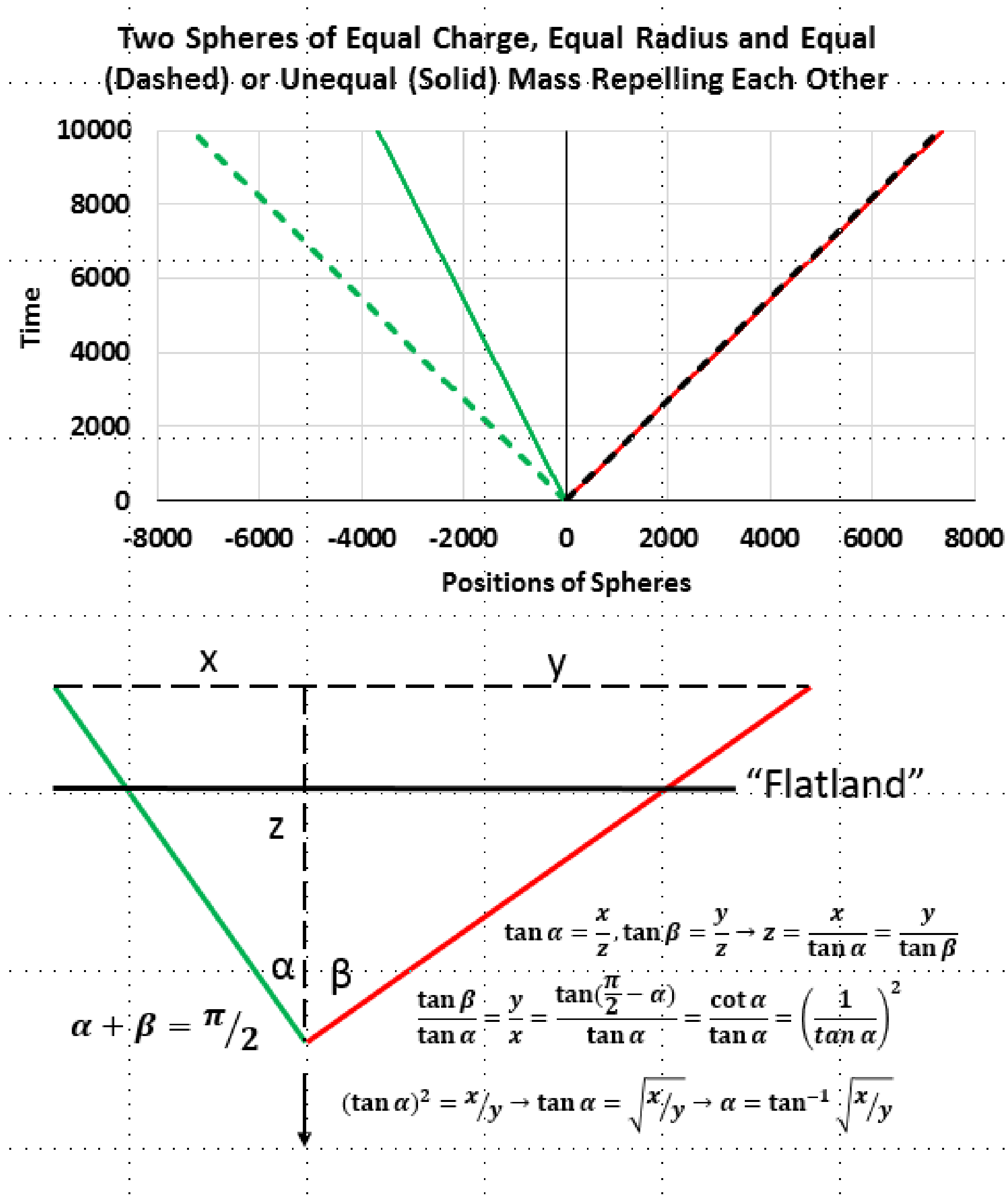
Strictly speaking, if a stack of spheres were passing perpendicularly through Flatland (ignoring the slant for now [Figure 12]), continuously growing and shrinking circles would manifest themselves rather than continuous circles of the same size. My construct here assumes that the stack of spheres can be represented by an "equivalent" stack of poker chips forming a cylinder of volume equivalent to the stack of spheres (again, ignoring the slant), such that what is manifest is continuously equal-sized circles. The justification for this is that I am conjuring a mathematical construct of the Coulombic force between two charged spheres, not a manifestation of the spheres themselves. And it is this Coulombic force that manifests as a series of circles (albeit slanted) as the projection of the cylinder passing through Flatland, such that their displacement corresponds to the effect of Coulombic repulsion (or attraction for the reverse case) between the circles.

An equivalent viewpoint would be to represent solely the Coulombic force depending solely upon the distance between the centers of the spheres or circles, with the cylinder now viewed as a line of infinitesimal width (shown in red), representing the increasing displacement of the positions of the centers of the spheres, which is what is actually measured by the Coulombic force equations.

Visually, both the symmetric and asymmetric cases can be seen in the plot in Figure 14 (Figure 6 from Case 3). The dashed lines represent the "triangle" passing perpendicularly along its bisector through Flatland. The asymmetric case can be similarly envisioned as a right triangle passing through Flatland, but now not perpendicularly to its bisector but perpendicularly to a line dividing the triangle into unequal parts based on the Coulombic force (triangle in Figure 14). Consider the case (#3) illustrated here, where the green sphere (now envisioned as a circle in Flatland), with double the mass of the red sphere, is displaced essentially only half as far from the two spheres' original point of contact as the red sphere. This can be shown by the triangle Figure 14 (again imagine two stacks of poker chips, slanted so as to always intersect Flatland in a circle). Again we have a right triangle, but no longer symmetric, such that: $\tan\alpha = \frac{x}{z}$ and $\tan\beta = \frac{y}{z},\ with\ \alpha + \beta = \frac{\pi}{2}$. The solution from the calculations shown with the triangle in Figure 14 is $\alpha = \tan^{-1}\sqrt{\frac{x}{y}}$. For the case where y = 2x, the solution is $\alpha = 0.615$ (35.3^{o}) and $\beta = 0.955$ (54.7^{o}).

4. Conclusion

The "Flatland" analogy represents a 3-dimensional phenomenon (the perpendicularly intersecting right triangle) manifesting as Columbic repulsion (or attraction) in two dimensions between two circles (slanted cylindrical projections). What appears as "action at a distance" in Flatland is really a limited 2-dimensional projection of a 3-dimensional phenomenon where the Coulombic force is not "disconnected," but rather is embedded in the right triangle composed of the fixed slanted stack of circular poker chips. Can this analogy be extended into the third and fourth dimensions, i.e., where, in three dimensions, we "see" two charged spheres repelling or attracting each other Coulombically via "action at a distance" when, in actuality, this is a projection in our limited world of 3-dimensional "Flatland" of a 4-dimensional, slanted stack of "poker spheres" forming a 4-dimensional right triangle intersecting perpendicularly (whatever that means in four dimensions) while passing through our 3-dimensional Flatland? Unlike the 2-dimensional Flatland analogy, this cannot be drawn or even visualized, other than just as the projection in three dimensions of two charged spheres repelling or attracting each other. Nonetheless, this exer4cise is offered as "food for thought" into the possibility of the existence of a fourth, purely spatial (i.e., non-temporal) dimension interacting with our 3-dimensional world in a way analogous to how an observer in 2-dimensional "Flatland" might perceive interaction with an imperceptible (to him/her)3-dimensional phenomenon.



<u>Figure 14</u>. Depiction of Asymmetric Case 3 as Right Triangle from Three Dimensions Intersecting Flatland Perpendicular Along Non-Bisecting Line Dividing Triangle into Two Unequal Parts

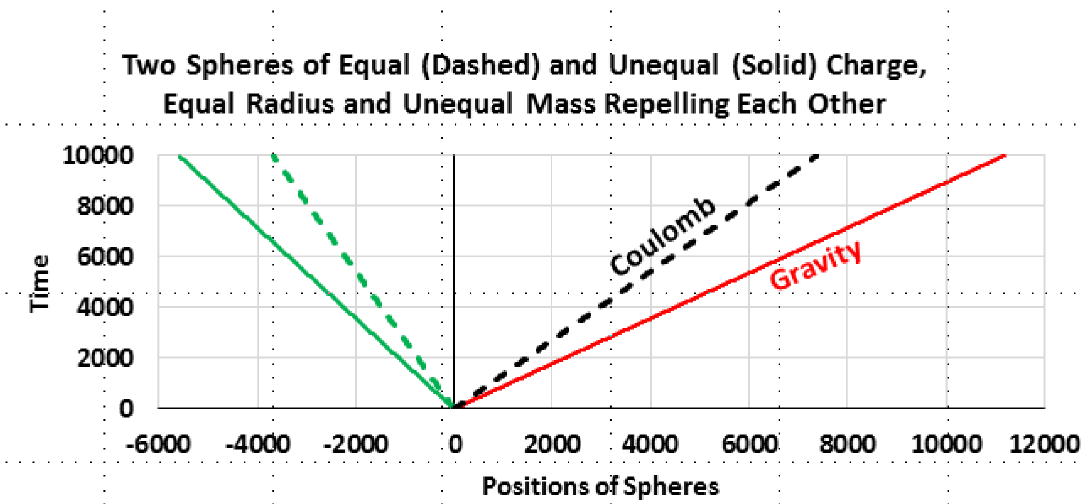
5. <u>Reference</u>

1. *https://en.wikipedia.org/wiki/Flatland*

Appendix: Gravity Too?

This admittedly far-fetched, purely mathematical-geometrical construct can be extended to gravity, albeit only in the attractive sense with respect to the Coulombic construct, and with a bit of a twist since mass plays the role of opposite charge in attraction. If the two masses are the same and have the same geometry, the symmetric "anti-repulsive," i.e., attractive, representation equally applies for gravity. However, if the masses are different, not only the case with greater attractive force manifests itself, but also the case with the asymmetry of different accelerations, and therefore displacements, for the reference point at which both spheres finally touch manifests. Below is the plot for the green sphere having double the charge and double the mass of the red sphere, the former simulating the effect of doubling the mass while keeping the geometry the same to simulate gravity.

If dealing only with Coulombic attraction, the effect of doubling the mass for equal charges (dashed) is less in terms of force and displacement (Figure 15). However, the gravitational equivalent is the same as if the charge on the green sphere were also doubled, showing greater force and displacement. Still, the asymmetry is preserved, and the construct of a slanted stack of poker chips forming a right triangle intersecting "Flatland" is preserved, albeit solely for repulsion (i.e., starting from a great distance).



<u>Figure 15</u>. Representing Attractive Effect of Gravity between Two Spheres of Equal Radius but Unequal Mass as Analogy with Coulombic Attraction between Two Unequally Charged Spheres of Same Radius but Unequal Mass