Confronting the Galilean Transformation with the Field Shapes of a Constant-Velocity Point Charge

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Abstract

The Galilean space-time transformation is predicated on a salient theme of Galilean/Newtonian physics: relative motion at constant velocity has no physical consequences beyond the minimum which is required by that motion's existence. Therefore, since the electric field produced by a point charge at rest is spherically symmetric around the charge's location, and since a point charge at rest produces zero magnetic field, Galilean physics implies that a point charge moving at constant velocity produces an electric field which is spherically symmetric around that charge's instantaneous location and that it produces zero magnetic field. But the Biot-Savart-Maxwell Law has it that a point charge moving at nonzero constant velocity produces nonzero magnetic field, and Faraday's Law has it that this time-varying magnetic field, which has zero component along the line of the charge's motion, produces an electric field which isn't spherically symmetric around the charge's instantaneous location. Thus the Galilean space-time transformation is violated by electromagnetic phenomena in a definite way, and must be modified. The needed modification produces the Lorentz space-time transformation, which can straightforwardly be shown to never change the speed of electromagnetic radiation. The fate of the Galilean/Newtonian constant-velocity relative-motion paradigm was actually already sealed when it was observed that the presence of direct current in a wire deflects an adjacent compass needle.

Introduction

The theoretical physics idea which underlies the Galilean space-time transformation of constant velocity $\mathbf{v}$, namely,

$$\mathbf{r}' = (\mathbf{r} - \mathbf{v}t), \;\; t' = t, \tag{1}$$

is that relative motion at constant velocity has no physical consequences beyond the minimum which is required by that motion's existence.

Therefore since the electric field of a point charge at rest, namely $e\mathbf{r}/|\mathbf{r}|^3$, manifests spherical symmetry around that charge's location, Galilean logic implies that the electric field of a point charge traveling at constant velocity $\mathbf{v}$ ought to manifest precisely the same spherical symmetry around that charge's instantaneous moving location $\mathbf{v}t$; indeed its electric field ought to be precisely $e(\mathbf{r} - \mathbf{v}t)/|(\mathbf{r} - \mathbf{v}t)|^3$. By the same token, the Galilean prediction for the constant-velocity point charge's electric potential would, in the appropriately chosen gauge, be $e/|(\mathbf{r} - \mathbf{v}t)|$. Since a point charge *at rest* produces *precisely zero magnetic field*, and also, in the appropriately-chosen gauge, *zero vector potential*, the Galilean prediction would be that a constant-velocity point charge produces *zero magnetic field* and also, in the appropriately chosen gauge, *zero vector potential*.

However an outright *zero* value for the magnetic field of a constant-velocity point charge flies utterly and completely in the face of the Biot-Savart-Maxwell Law of electromagnetism, which is,

$$\nabla \times \mathbf{B} = (4\pi\mathbf{j} + \dot{\mathbf{E}})/c. \tag{2a}$$

Direct current which is impelled by a battery in a length of copper wire can, to reasonable approximation, be thought of as consisting of well-shielded electric charges moving along at constant velocity, and there is no doubt whatsoever that they produce a magnetic field, one in accord with the Biot-Savart Law (the fact that the electric field of the charges in the wire is well-shielded *eliminates the effect of the Maxwell source term* $\dot{\mathbf{E}}/c$). There is thus no doubt that electromagnetic physics is at loggerheads with the theoretical physics idea which underlies the Galilean space-time transformation, and that that transformation *requires modification*.

The *way* in which the electric field of a constant-velocity point charge *differs* from $e(\mathbf{r} - \mathbf{v}t)/|(\mathbf{r} - \mathbf{v}t)|^3$, or the *way* in which its electric potential *differs* from $e/|(\mathbf{r} - \mathbf{v}t)|$ should *speak volumes* concerning *how* the Galilean space-time transformation of Eq. (1) *needs to be modified*.

The calculation of the electric and magnetic fields of a constant-velocity point charge was first carried out by Oliver Heaviside in 1888, and Heaviside's results were famously cited by George FitzGerald in 1889 and by Hendrik A. Lorentz in 1892 in the endeavors of those two physicists to attain theoretical understanding of the null reading for the "aether wind" of the 1887 Michelson-Morley experiment [1].

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While the fact that a constant-velocity point charge *produces* a nonzero *magnetic* field is *a first-order consequence of the Biot-Savart-Maxwell Law of electromagnetism*, the *alteration* of a constant-velocity point charge's *electric* field and *electric* potential *from what would be expected from the Galilean space-time transformation* is *a second-order consequence, via Faraday's Law*, namely via,

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}/c, \tag{2b}$$

of the *existence* of that first-order *magnetic* field. Furthermore, because that first-order magnetic field departs altogether from spherical symmetry—it has *zero* component along the line of the point charge's motion—its electric-field consequence that is produced by Faraday's Law *isn't spherically symmetric around the charge's instantaneous moving location*.

A sensible way to deal with such "fed-through" physical phenomena is to *combine* the *first-order* electromagnetic field equations, such as the one of Biot-Savart-Maxwell and the one of Faraday, into *second-order* "driven-wave" type equations. The treatment below of the electromagnetic potentials and fields of the constant-velocity point charge is carried out *entirely* using such *second-order* driven-wave type electromagnetic field equations.

We turn now to the *details* of calculating the electromagnetic potentials and fields of a constant-velocity point charge. A nonzero magnetic field is obtained, one which has zero component along the point charge's line of motion, and therefore *no semblance whatsoever of spherical symmetry*. Obtained along with that magnetic field is *the distortion induced by it*, via Faraday's Law, of the electric-field and electric-potential shapes $e(\mathbf{r}-\mathbf{v}t)/|\mathbf{r}-\mathbf{v}t|^3$ and $e/|\mathbf{r}-\mathbf{v}t|$ whose spherically-symmetric forms around the constant-velocity point charge's instantaneous location $\mathbf{v}t$ follow from the Galilean space-time transformation. Taking into account *the distortion which is obtained of these shapes that follow from the Galilean space-time transformation* enables *repair* of the Galilean space-time transformation to be effected.

The electromagnetic potentials and fields of a constant-velocity point charge

The two electromagnetic laws given by Eqs. (2a) and (2b) are *completed* by the *addition* of *both* Coulomb's Law,

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{2c}$$

and the Gauss Law,

$$\nabla \cdot \mathbf{B} = 0. \tag{2d}$$

The technical development of the second-order driven-wave type electromagnetic field equations begins by applying the curl operator to both sides of both the Biot-Savart-Maxwell and Faraday Laws. The resulting field divergence terms are then eliminated by insertions of the Coulomb and Gauss Laws. The electric and magnetic fields can then be fully decoupled from each other by eliminating the remaining field curl terms in the second-order equations by insertions of the original Biot-Savart-Maxwell and Faraday Laws. The two second-order driven-wave type electric and magnetic field equations which result are,

$$(1/c^2)\ddot{\mathbf{E}} - \nabla^2\mathbf{E} = -4\pi\left(\nabla\rho + (1/c^2)\partial\mathbf{j}/\partial t\right), \quad (1/c^2)\ddot{\mathbf{B}} - \nabla^2\mathbf{B} = 4\pi(\nabla \times \mathbf{j})/c. \tag{3}$$

Since these electric-field and magnetic-field driven-wave type equations *have complicated source terms*, it is convenient to introduce the electromagnetic scalar potential ϕ and the electromagnetic vector potential $\mathbf{A}$ *in Lorentz gauge*, which *also* satisfy driven-wave type equations, *albeit ones with much simpler source terms*,

$$(1/c^2)\ddot{\phi} - \nabla^2\phi = 4\pi\rho, \quad (1/c^2)\ddot{\mathbf{A}} - \nabla^2\mathbf{A} = 4\pi\mathbf{j}/c. \tag{4a}$$

In *addition* to satisfying these driven-wave type equations, ϕ and $\mathbf{A}$ *are related to each other by the Lorentz condition*,

$$(1/c)\dot{\phi} + \nabla \cdot \mathbf{A} = 0. \tag{4b}$$

The electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ *are obtained from* ϕ *and* $\mathbf{A}$ *via the two relations*,

$$\mathbf{E} = -\nabla\phi - (1/c)\dot{\mathbf{A}}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{4c}$$

Combining the two relations of Eq. (4c) with the simple driven-wave type equations for ϕ and $\mathbf{A}$ of Eq. (4a) is readily seen to yield the more complicated driven-wave type equations for $\mathbf{E}$ and $\mathbf{B}$ of Eq. (3). Also the second relation of Eq. (4c) implies the Gauss Law for $\mathbf{B}$ as given by Eq. (2d), while the first relation of

Eq. (4c) combined with *both* the Lorentz condition of Eq. (4b) *and* the simple driven-wave type equation for ϕ of Eq. (4a) yields Coulomb's Law for $\mathbf{E}$ as given by Eq. (2c).

Now a point charge of strength e traveling with constant vector velocity $\mathbf{v}$ has the charge density,

$$\rho(\mathbf{r}, t) = e\delta^{(3)}(\mathbf{r} - \mathbf{v}t), \tag{5a}$$

and it has the current density,

$$\mathbf{j}(\mathbf{r}, t) = e\mathbf{v}\delta^{(3)}(\mathbf{r} - \mathbf{v}t) = \mathbf{v}\rho(\mathbf{r}, t). \tag{5b}$$

It can be readily shown that the charge density of Eq. (5a) and the current density of Eq. (5b) together satisfy the required equation of continuity, namely $\dot{\rho}(\mathbf{r}, t) + \nabla\cdot\mathbf{j}(\mathbf{r}, t) = 0$. With these source terms, which are appropriate to a point charge of strength e traveling at constant vector velocity $\mathbf{v}$, the two simple driven-wave type equations of Eq. (4a) read,

$$(1/c^2)\ddot{\phi} - \nabla^2\phi = 4\pi e\delta^{(3)}(\mathbf{r} - \mathbf{v}t), \quad (1/c^2)\ddot{\mathbf{A}} - \nabla^2\mathbf{A} = (\mathbf{v}/c)4\pi e\delta^{(3)}(\mathbf{r} - \mathbf{v}t), \tag{5c}$$

which makes it apparent that,

$$\mathbf{A}(\mathbf{r}, t) = (\mathbf{v}/c)\phi(\mathbf{r}, t), \tag{5d}$$

so we only need to solve,

$$(1/c^2)\ddot{\phi}(\mathbf{r}, t) - \nabla^2\phi(\mathbf{r}, t) = 4\pi e\delta^{(3)}(\mathbf{r} - \mathbf{v}t), \tag{5e}$$

a task which we undertake *by applying Fourier transforms and their inverses*.

Insertion of the Fourier *ansatz*,

$$\phi(\mathbf{r}, t) = \int d^3\mathbf{k}\, d\omega\, e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)}\bar{\phi}(\mathbf{k}, \omega), \tag{6a}$$

into Eq. (5e) yields,

$$\begin{aligned}\left(|\mathbf{k}|^2 - (\omega/c)^2\right)\bar{\phi}(\mathbf{k}, \omega) &= 4\pi e(2\pi)^{-4}\int d^3\mathbf{r}\, dt\, e^{-i(\mathbf{k}\cdot\mathbf{r}+\omega t)}\delta^{(3)}(\mathbf{r} - \mathbf{v}t) = \\ 4\pi e(2\pi)^{-4}\int dt\, e^{-i((\mathbf{k}\cdot\mathbf{v})t+\omega t)} &= 4\pi e(2\pi)^{-3}\delta(\omega + (\mathbf{k}\cdot\mathbf{v})),\end{aligned} \tag{6b}$$

so the Fourier transform $\bar{\phi}(\mathbf{k}, \omega)$ of $\phi(\mathbf{r}, t)$ is evaluated to be,

$$\bar{\phi}(\mathbf{k}, \omega) = \frac{e}{2\pi^2}\,\frac{\delta(\omega + (\mathbf{k}\cdot\mathbf{v}))}{|\mathbf{k}|^2 - (\omega/c)^2}. \tag{6c}$$

Insertion of Eq. (6c) into Eq. (6a) produces,

$$\phi(\mathbf{r}, t) = \frac{e}{2\pi^2}\int d^3\mathbf{k}\, d\omega\, e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)}\frac{\delta(\omega + (\mathbf{k}\cdot\mathbf{v}))}{|\mathbf{k}|^2 - (\omega/c)^2}. \tag{7a}$$

Using the delta function to carry out the integration over ω then yields,

$$\phi(\mathbf{r}, t) = \frac{e}{2\pi^2}\int d^3\mathbf{k}\,\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)}}{|\mathbf{k}|^2 - (\mathbf{k}\cdot(\mathbf{v}/c))^2}. \tag{7b}$$

In order to evaluate the integral in Eq. 7b we choose a Cartesian coordinate system whose x-axis points in the direction of $\mathbf{v}$. In that Cartesian coordinate system, $\mathbf{v} = (v, 0, 0)$ and $\mathbf{k}\cdot\mathbf{v} = k^1 v$, so Eq. (7b) can be written,

$$\phi(x, y, z, t) = \frac{e}{2\pi^2}\int dk^1\, dk^2\, dk^3\,\frac{e^{i[k^1(x-vt)+k^2 y+k^3 z]}}{(k^1)^2(1 - (v/c)^2) + (k^2)^2 + (k^3)^2}. \tag{7c}$$

We now change the three integration variables from (k^1, k^2, k^3) to $(q^1, q^2, q^3) \stackrel{\text{def}}{=} (k^1(1 - (v/c)^2)^{\frac{1}{2}}, k^2, k^3)$, which implies that $(k^1, k^2, k^3) = (\gamma q^1, q^2, q^3)$, where $\gamma \stackrel{\text{def}}{=} (1-(v/c)^2)^{-\frac{1}{2}}$. With that change of the integration variables, Eq. (7c) becomes,

$$\phi(x, y, z, t) = \frac{e\gamma}{2\pi^2}\int dq^1\, dq^2\, dq^3\,\frac{e^{i[q^1\gamma(x-vt)+q^2 y+q^3 z]}}{(q^1)^2 + (q^2)^2 + (q^3)^2}, \tag{7d}$$

which can be written more compactly as,

$$\phi(x,y,z,t) = \frac{e\gamma}{2\pi^2}\int d^3\mathbf{q}\,\frac{e^{i\mathbf{q}\cdot\mathbf{R}(x,y,z,t;v,\gamma)}}{|\mathbf{q}|^2}, \tag{7e}$$

where the time-dependent radius-like vector field $\mathbf{R}(x,y,z,t;v,\gamma)$ is of course seen from Eq. (7d) to be,

$$\mathbf{R}(x,y,z,t;v,\gamma) \stackrel{\text{def}}{=} (\gamma(x-vt),y,z), \tag{8a}$$

in which γ is defined as,

$$\gamma \stackrel{\text{def}}{=} (1-(v/c)^2)^{-\frac{1}{2}}. \tag{8b}$$

The integral over $d^3\mathbf{q}$ which appears in Eq. (7e) is a standard one which is familiar from the case of the Coulomb potential for a point charge at rest; as a matter of fact, in the *special case* that the point charge e has its *speed* $|v|$ put equal to *zero*, $\phi(x,y,z,t)$ *must* reduce to the straightforward static Coulomb result $e/|\mathbf{r}|$, where $\mathbf{r} = (x,y,z)$. That fact implies that the *integral* which occurs in Eq. (7e) has the value $2\pi^2/|\mathbf{R}(x,y,z,t;v,\gamma)|$, namely that,

$$\int d^3\mathbf{q}\,\frac{e^{i\mathbf{q}\cdot\mathbf{R}(x,y,z,t;v,\gamma)}}{|\mathbf{q}|^2} = 2\pi^2/|\mathbf{R}(x,y,z,t;v,\gamma)|,$$

a result which alternatively can be obtained by simply carrying out the relatively straightforward integration over $d^3\mathbf{q}$. From this result and Eq. (7e) it follows that,

$$\phi(x,y,z,t) = \frac{e\gamma}{|\mathbf{R}(x,y,z,t;v,\gamma)|} = \frac{e\gamma}{(\gamma^2(x-vt)^2+y^2+z^2)^{\frac{1}{2}}}, \tag{9a}$$

where we have used Eq. (8a) to obtain the second equality.

Since $\gamma = (1-(v/c)^2)^{-\frac{1}{2}} > 1$, the electric potential ϕ of Eq. (9a) *deviates from the form* $e/((x-vt)^2+y^2+z^2)^{\frac{1}{2}}$ *predicted by the Galilean space-time transformation*. Eq. (9a) shows that the details of electromagnetic theory *have distorted the spherical symmetry around the constant-velocity point charge's instantaneous location which is predicted by the Galilean transformation*, with that distortion occurring along the charge's line of motion.

We next obtain the constant-velocity point charge's vector potential and magnetic field. Since from Eq. (5d), $\mathbf{A}(x,y,z,t) = (\mathbf{v}/c)\phi(x,y,z,t)$, and from the discussion below Eq. (7b), $\mathbf{v} = (v,0,0)$, we obtain from Eq. (9a) that,

$$\mathbf{A}(x,y,z,t) = \frac{e\gamma((v/c),0,0)}{(\gamma^2(x-vt)^2+y^2+z^2)^{\frac{1}{2}}}. \tag{9b}$$

From Eq. (9b) it is apparent that the three components of $\mathbf{A}(x,y,z,t)$ are,

$$A^1(x,y,z,t) = \frac{e\gamma(v/c)}{(\gamma^2(x-vt)^2+y^2+z^2)^{\frac{1}{2}}},\ \ A^2(x,y,z,t)=0,\ \ A^3(x,y,z,t)=0,$$

and therefore the magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$ produced by the moving point charge is given by,

$$\mathbf{B} = \nabla\times\mathbf{A} = (0,\partial A^1/\partial z,-\partial A^1/\partial y) = \frac{e\gamma(v/c)(0,-z,y)}{(\gamma^2(x-vt)^2+y^2+z^2)^{\frac{3}{2}}}. \tag{9c}$$

This magnetic field clearly *vanishes entirely* when the point charge is *at rest*, namely when $v=0$, and *it has no semblance whatsoever of spherical symmetry around the point charge's instantaneous location since it has vanishing component along the point charge's line of motion*. But as we have strongly emphasized in the Introduction, the fact that this magnetic field *is nonzero at all* contradicts the theoretical physics idea which underlies the Galilean space-time transformation, namely that constant relative velocity has no physical consequences beyond the minimum required by its existence. As we have *also* emphasized, this nonzero *time-varying* magnetic field $\mathbf{B}$, *which departs so strongly from spherical symmetry around the point charge's instantaneous location* $\mathbf{v}t = (vt,0,0)$, modifies, via Faraday's Law $\nabla\times\mathbf{E} = -\dot{\mathbf{B}}/c$, the constant-velocity point charge's electric field $\mathbf{E}$ and electric potential ϕ (shown in Eq. (9a)) in a way *that distorts the electric-field shape* $e(\mathbf{r}-\mathbf{v}t)/|\mathbf{r}-\mathbf{v}t|^3$ *and the electric-potential shape* $e/|\mathbf{r}-\mathbf{v}t|$ *which follow from the Galilean space-time transformation* (and are consequently spherically-symmetric around the constant-velocity point charge's instantaneous location $\mathbf{v}t$). That distortion is clearly apparent in the Eq. (9a) expression for the electric potential ϕ of the constant-velocity point charge; the distortion occurs because of the presence of the factor $\gamma \stackrel{\text{def}}{=} (1-(v/c)^2)^{-\frac{1}{2}} > 1$. The result for the electric *field* $\mathbf{E}$ of the constant-velocity point charge will be similarly distorted by the presence of that factor $\gamma > 1$.

We next work out $\mathbf{E}$, this constant-velocity point charge's electric field. Since $\mathbf{E} = -\nabla\phi - (1/c)\dot{\mathbf{A}}$, we need to use Eq. (9a) to calculate,

$$-\nabla\phi(x,y,z,t) = \frac{e\gamma(\gamma^2(x-vt),y,z)}{(\gamma^2(x-vt)^2+y^2+z^2)^{\frac{3}{2}}},$$

and *in addition* we need to use Eq. (9b) to calculate,

$$-(1/c)\dot{\mathbf{A}}(x,y,z,t) = \frac{e\gamma(-(v/c)^2\gamma^2(x-vt),0,0)}{(\gamma^2(x-vt)^2+y^2+z^2)^{\frac{3}{2}}}.$$

We can now assemble the above two intermediate results to obtain $\mathbf{E}$, the constant-velocity point charge's electric field,

$$\mathbf{E} = -\nabla\phi - (1/c)\dot{\mathbf{A}} = \frac{e\gamma((x-vt),y,z)}{(\gamma^2(x-vt)^2+y^2+z^2)^{\frac{3}{2}}} = \frac{e\gamma((x-vt),y,z)}{|\mathbf{R}(x,y,z,t;v,\gamma)|^3}. \tag{9d}$$

Comparison of Eq. (9c) with Eq. (9d) shows that $(\mathbf{B}\cdot\mathbf{E})$ *always vanishes identically*. This is an interesting result which has to do with the exact *non-Galilean* space-time transformation properties of the electromagnetic fields; we will have more to say further on about those exact *non-Galilean* space-time field transformations.

We note that $\gamma \stackrel{\text{def}}{=} (1-(v/c)^2)^{-\frac{1}{2}}$ is very close to unity when $(v/c)^2 \ll 1$, so under those circumstances Eq. (9d) shows that the electric field $\mathbf{E}$ is nearly spherically symmetric around the instantaneous location $(vt,0,0)$ of the constant-velocity point charge. However, as $(v/c)^2 \to 1$, γ can become *arbitrarily large*, which can *markedly distort the electric field* $\mathbf{E}$ *away from spherical symmetry*, weakening it along the line of motion of the point charge relative to its strength *perpendicular* to that line of motion.

We note that *the key entity which underlies the character of both the electric field* $\mathbf{E}$ *of* Eq. (9d) *and the electric potential* ϕ *of* Eq. (9a) *is the constant-rate translating, radius-like vector field* $\mathbf{R}(x,y,z,t;v,\gamma) = (\gamma(x-vt),y,z)$ *of* Eq. (8a), *which is distorted away from spherical symmetry along its line of motion*. A crucial property of $(\gamma(x-vt),y,z)$ is that when $(v/c)^2 \ll 1$, so that γ is very close to unity, *it reduces to the three spatial components* $(x',y',z') = ((x-vt),y,z)$ *of the Galilean space-time transformation which is given by* Eq. (1) (bearing in mind that we have meantime *specialized* to a Cartesian coordinate system where $\mathbf{v} = (v,0,0)$, so that $(\mathbf{r}-\mathbf{v}t) = (x,y,z)-(vt,0,0) = ((x-vt),y,z)$—see the discussion below Eq. (7b)).

However, although $(\gamma(x-vt),y,z)$ *reduces* to the three spatial components $(x',y',z') = ((x-vt),y,z)$ of the Galilean space-time transformation when $(v/c)^2 \ll 1$, $(\gamma(x-vt),y,z)$ becomes *vastly different* from $((x-vt),y,z)$ as $(v/c)^2 \to 1$. Thus the Galilean transformation becomes *less and less consistent* with electromagnetic theory as the transformation speed $|v|$ approaches the universal constant c of electromagnetic theory.

It is therefore clear that the Galilean transformation must be *replaced* by a transformation which has the property that,

$$(x',y',z') = (\gamma(x-vt),y,z), \tag{10a}$$

where, of course,

$$\gamma \stackrel{\text{def}}{=} (1-(v/c)^2)^{-\frac{1}{2}}. \tag{10b}$$

The *three* requirements of Eq. (10a) for the replacement transformation for the Galilean transformation *aren't sufficient to determine* t' *in terms of* t *and* x. However, the Galilean transformation, as given by Eq. (1) with $\mathbf{v} = (v,0,0)$, namely,

$$(x',y',z',t') = ((x-vt),y,z,t),$$

is readily verified to have the *inverse*,

$$(x,y,z,t) = ((x'+vt'),y',z',t').$$

Thus the *inverse* of the Galilean transformation is the *same* as the Galilean transformation itself *except that* v *is replaced by* $-v$. That fact is completely sensible from a physical standpoint; indeed, from a physical standpoint it is well-nigh *inconceivable* that replacing v by $-v$ *wouldn't* invert the transformation. There is no reason which is apparent to doubt *that the correct replacement transformation for the Galilean* *transformation is as well inverted by replacing* v *by* $-v$. Therefore in *addition* to Eqs. (10a) and (10b), we *also* expect the correct replacement transformation to satisfy,

$$(x, y, z) = (\gamma(x' + vt'), y', z'), \tag{10c}$$

which takes into account the fact that γ *doesn't change* when v is replaced by $-v$, as is seen from Eq. (10b).

In consequence of Eq. (10c) we obtain that $x' = -vt' + x/\gamma$, while, of course, Eq. (10a) states that $x' = \gamma(x - vt)$. We can therefore *deduce* that $-vt' + x/\gamma = \gamma(-vt + x)$, and from *that* we uniquely work out t' in terms of t and x, which is exactly what we *need* to fully specify the definition of the *replacement transformation* for the *electromagnetically invalid* Galilean space-time transformation. The *result* of thus working out t' is,

$$t' = \gamma\left(t - (x/v)\left(1 - \left(1/\gamma^2\right)\right)\right) = \gamma\left(t - \left(vx/c^2\right)\right),$$

where the last equality follows from Eq. (10b).

We combine the above result for t' in terms of t and x with Eq. (10a) to obtain the full expression for the replacement transformation,

$$(x', y', z', t') = \left(\gamma(x - vt), y, z, \gamma\left(t - \left(vx/c^2\right)\right)\right), \tag{11a}$$

where,

$$\gamma \stackrel{\text{def}}{=} (1 - (v/c)^2)^{-\frac{1}{2}}. \tag{11b}$$

The replacement transformation given by Eqs. (11a) and (11b) above is, of course, *the Lorentz space-time transformation* [2]. It is readily seen that the Lorentz space-time transformation of Eqs. (11a) and (11b) goes over into the Galilean space-time transformation of Eq. (1) when $|v/c| \ll 1$.

With a modicum of algebraic effort, it can be verified from Eqs. (11a) and (11b) that the quadratic form,

$$x^2 + y^2 + z^2 - (ct)^2,$$

is *preserved* by the Lorentz transformations, namely that,

$$(x')^2 + (y')^2 + (z')^2 - (ct')^2 = x^2 + y^2 + z^2 - (ct)^2.$$

The physical relevance of the preservation of this quadratic form lies with the fact that that $x^2+y^2+z^2 = (ct)^2$ is the three-dimensional analytic-geometry equation of a spherical shell of electromagnetic radiation whose radius is expanding at speed c. Thus a Lorentz transformation of velocity $(v, 0, 0)$ never alters the speed c of electromagnetic radiation by a single iota, *regardless of what value* v *has*! Of course the speed of electromagnetic radiation is determined by the two Eq. (3) driven-wave type electric and magnetic field equations at locations far away from their driving sources, so it is absolutely patent from Eq. (3) that that speed can never deviate from the universal constant c of electromagnetic theory. Einstein treated the universal speed of electromagnetic radiation as a postulate to be used to *obtain* the Lorentz transformation [3]; here we noted that both the universal speed of electromagnetic radiation and the Lorentz transformation are in fact built into the laws of electromagnetism, but of course they aren't built into the Galilean laws of Newton. The universal speed of electromagnetic radiation is clearly in complete harmony with the *inability* of Michelson-Morley type of experiments to obtain positive evidence for the "aether wind" [3].

The *entire set* of four-dimensional homogeneous linear transformations which preserve the quadratic form $x^2 + y^2 + z^2 - (ct)^2$ comprise *the Lorentz-transformation group*.

The locus of *four-dimensional* space-time points described by the equation $x^2+y^2+z^2 = (ct)^2$ is sometimes referred to as "the light cone", so the Lorentz-transformation group is sometimes said to "preserve the light cone".

The equations of electromagnetic theory turn out to be covariant under Lorentz transformations, and can be reexpressed in a Lorentz tensor notation which reflects that covariance—the $\mathbf{E}$ and the $\mathbf{B}$ fields are incorporated into a single 4×4 second-rank antisymmetric tensor $F^{\alpha\beta}$. Likewise, the charge density ρ and the current density $\mathbf{j}$ are incorporated into a single current-density four-vector j^μ, and the scalar potential ϕ and the vector potential $\mathbf{A}$ are both incorporated into a single electromagnetic four-vector potential A^μ. Lorentz-transformation *invariants* which can be formed from the electromagnetic field tensor can be of special physical interest; one of these Lorentz-transformation *invariants* of the electromagnetic field tensor turns out to be equal to $(\mathbf{B} \cdot \mathbf{E})$, and another turns out to be equal to $(|\mathbf{E}|^2 - |\mathbf{B}|^2)$. The Lorentz-transformation *invariant* property of $(\mathbf{B} \cdot \mathbf{E})$ enables us to understand *why* this entity vanishes identically in the case of a point charge moving at constant velocity, a fact we have previously noted from Eqs. (9c) and (9d). This entity of course vanishes in *the rest frame* of the point charge, where $\mathbf{B} = \mathbf{0}$, and it can be verified to be *equal* to a constant times $\varepsilon_{\alpha\beta\kappa\lambda}F^{\alpha\beta}F^{\kappa\lambda}$, whose *completely tensor-index contracted form* is what *makes* it a Lorentz-transformation invariant entity. Being a Lorentz-transformation *invariant* entity, if it vanishes in one inertial frame of reference, it must vanish in all inertial frames of reference.

Finally we would like to understand exactly *which* aspect of electromagnetic physics is incompatible with Galilean principles. Taking the Galilean limit $c \to \infty$ of the four electromagnetic laws, which are,

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}/c, \qquad \nabla \cdot \mathbf{B} = \mathbf{0}, \qquad \nabla \times \mathbf{B} = (4\pi\mathbf{j} + \dot{\mathbf{E}})/c,$$

produces,

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \nabla \times \mathbf{E} = \mathbf{0}, \qquad \nabla \cdot \mathbf{B} = \mathbf{0}, \qquad \nabla \times \mathbf{B} = \mathbf{0},$$

so we see that in the $c \to \infty$ Galilean limit *all of the sources and all of the effects of the magnetic field* $\mathbf{B}$ *fall away*! Thus it definitely is *magnetism* which is incompatible with Galilean principles. We can understand the *details* of this incontrovertible fact by scrutinizing the right-hand side of the Biot-Savart-Maxwell Law,

$$\nabla \times \mathbf{B} = (4\pi\mathbf{j} + \dot{\mathbf{E}})/c,$$

which tells us that it is *motion* which produces magnetism, specifically the *motion* of electric charge and the *motion* of the electric field; *without motion* there is patently *no source* for the magnetic field. At the *same time*, the magnetic field *isn't picky in the slightest about the character of the motion of its sources*; the Biot-Savart-Maxwell Law attests that *a previously static* assortment of electric charge and electric field is, *once set into uniform motion at nonzero constant velocity*, a perfectly *effective source of magnetic field*—our magnetic-field result $\mathbf{B}$ of Eq. (9c) *is an exact example of that fact*. But as we have emphasized in the discussion below Eq. (9c), it is a fact which flies utterly in the face of Galilean principles and the Galilean space-time transformation, which have it that relative motion at constant velocity has no physical consequences beyond the minimum which is required by the existence of that motion: the *production* of completely detectable and physically consequential magnetic field *merely* by dint of *the magnitude of constant velocity of relative motion* is *just as unequivocal* and *startling* a contradiction of Galilean/Newtonian principles as are the much more *celebrated* slowing of clocks and contractions of lengths which *also* occur *merely* by dint of *the magnitude of constant velocity of relative motion*. Indeed the production of magnetic field by dint of the magnitude of constant velocity of relative motion is a *first-order* effect of that velocity magnitude, while the distortion of electric fields, the slowing of clocks and the contraction of lengths are vastly more subtle *second-order* effects of that velocity magnitude. To bluntly put this matter in perspective, the discovery that the presence of direct current in a wire deflects an adjacent compass needle *already sealed the fate of the Galilean/Newtonian paradigm*, notwithstanding that scientists understandably didn't have the wit to *realize* that for a very, very long time.

Précis

By studying the behavior of the electric potential and field produced by a point charge traveling at constant velocity $(v, 0, 0)$, we learn that their forms *aren't* functions of the time-dependent spherically-symmetric radius vector field $((x - vt), y, z)$ which translates with velocity $(v, 0, 0)$, *as one would expect them to be if the Galilean transformation were valid*, unless $|v| \ll c$, where c is the universal constant of electromagnetic theory which has the dimension of speed. *Instead* we find that, more generally, due to the electric-field *knock-on effect* of the time-varying cylindrical *magnetic field* which a constant-velocity point charge *also produces*, they are functions of the constant-rate translating, radius-like vector field $((x-vt)/(1-(v/c)^2)^{\frac{1}{2}}, y, z)$, *which is distorted away from spherical symmetry along its line of motion*.

Due to that *distortion*, we see that the Galilean transformation, which is,

$$(x', y', z', t') = ((x - vt), y, z, t),$$

must be *replaced* by,

$$(x', y', z', t') = \left(\gamma(x - vt), y, z, \kappa(v)\left(t - \left(v_1(v)x/c^2\right)\right)\right),$$

where $\gamma \stackrel{\text{def}}{=} (1-(v/c)^2)^{-\frac{1}{2}}$, and the coefficient functions $\kappa(v)$ and $v_1(v)$ are *determined* by the physically unassailable requirement that *inverting* the transformation is achieved by replacing *the transformation velocity* v *by* $-v$, namely that,

$$(x, y, z, t) = \left(\gamma(x' + vt'), y', z', \kappa(-v)\left(t' - \left(v_1(-v)x'/c^2\right)\right)\right).$$

We obtain from *the replacement transformation ansatz* above that $x' = \gamma(x - vt)$, and from *the inverse replacement transformation ansatz* above that $x' = -vt' + x/\gamma$, which two equations allow us to *eliminate* x', and then *to determine* t' *in terms of* t *and* x, which is exactly what we *need* to do to remove the ambiguity from the above replacement transformation ansatz. The *result* of thus determining t' in terms of t and x is,

$$t' = \gamma\left(t - (x/v)\left(1 - \left(1/\gamma^2\right)\right)\right) = \gamma\left(t - \left(vx/c^2\right)\right),$$

where the last equality follows from $\gamma \stackrel{\text{def}}{=} (1 - (v/c)^2)^{-\frac{1}{2}}$. Therefore, $\kappa(v) = \gamma$, $\kappa(-v) = \gamma$, $v_1(v) = v$ and $v_1(-v) = -v$, which when inserted into the above replacement transformation and inverse replacement transformation *ansätze* produce the Lorentz transformation,

$$(x', y', z', t') = \left(\gamma(x - vt), y, z, \gamma\left(t - \left(vx/c^2\right)\right)\right),$$

and its inverse,

$$(x, y, z, t) = \left(\gamma(x' + vt'), y', z', \gamma\left(t' + \left(vx'/c^2\right)\right)\right).$$

We then note the readily verified algebraic fact that the Lorentz transformation takes the quadratic form $x^2 + y^2 + z^2 - (ct)^2$ into itself,

$$(x')^2 + (y')^2 + (z')^2 - (ct')^2 = x^2 + y^2 + z^2 - (ct)^2.$$

The equation $x^2 + y^2 + z^2 = (ct)^2$ in three-dimensional analytic geometry is that of a spherical shell of electromagnetic radiation whose radius is expanding at speed c, so a Lorentz transformation never alters the speed c of electromagnetic radiation by a single iota *irrespective of the velocity* v *of the transformation*. The universal speed of electromagnetic radiation is clearly in complete harmony with the *inability* of Michelson-Morley type experiments to obtain positive evidence for the "aether wind". Einstein treated the universal speed of electromagnetic radiation as a *postulate* to be used to *obtain* the Lorentz transformations; here we have wrested the Lorentz transformations from the laws of electromagnetism, and have noted that the Lorentz transformations imply the universal speed of electromagnetic radiation.

The *Lorentz-transformation group* is the *entire set* of four-dimensional homogeneous linear transformations which preserve the quadratic form $x^2 + y^2 + z^2 - (ct)^2$.

The *four-dimensional* space-time locus of the equation $x^2 + y^2 + z^2 = (ct)^2$ is sometimes referred to as "the light cone", so the Lorentz transformation group is sometimes said to "preserve the light cone".

The equations of electromagnetism are *covariant* under the Lorentz transformations, which is best shown and taken advantage of by restating them in terms of Lorentz-tensor formalism—the $\mathbf{E}$ and $\mathbf{B}$ fields are both incorporated into a single 4×4 second-rank antisymmetric electromagnetic-field tensor $F^{\alpha\beta}$. Likewise, the charge density ρ and the current density $\mathbf{j}$ are incorporated into a single current-density four-vector j^μ, and the scalar potential ϕ and the vector potential $\mathbf{A}$ are both incorporated into a single electromagnetic four-vector potential A^μ. Lorentz-transformation *invariants* which can be formed from the electromagnetic field tensor can be of special physical interest; one of these Lorentz-transformation *invariants* of the electromagnetic field tensor turns out to be equal to $(\mathbf{B} \cdot \mathbf{E})$, and another turns out to be equal to $(|\mathbf{E}|^2 - |\mathbf{B}|^2)$.

To pinpoint *which* aspect of electromagnetic physics is incompatible with Galilean principles, we take the Galilean limit $c \to \infty$ of the four electromagnetic laws, which are,

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thus producing,

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \nabla \times \mathbf{E} = \mathbf{0}, \qquad \nabla \cdot \mathbf{B} = \mathbf{0}, \qquad \nabla \times \mathbf{B} = \mathbf{0}.$$

We see that in the $c \to \infty$ Galilean limit *all of the sources and all of the effects of the magnetic field* $\mathbf{B}$ *fall away*, so it definitely is *magnetism* which is incompatible with Galilean principles. To understand the *details* of this fact, we ponder the right-hand side of the Biot-Savart-Maxwell Law,

$$\nabla \times \mathbf{B} = (4\pi\mathbf{j} + \dot{\mathbf{E}})/c,$$

which tells us that *motion* is *key* to magnetism, specifically the *motion* of electric charge and of the electric field; *without motion* there is patently *no source* for the magnetic field. However, the right-hand side of the Biot-Savart-Maxwell Law attests that *a previously static* assortment of electric charge and electric field is, *once set into motion at nonzero constant velocity*, a perfectly *effective source of magnetic field*—the magnetic-field result of Eq. (9c) *is an exact example of that fact*. But it is a fact which flies utterly in the face of Galilean principles and the Galilean space-time transformation, which have it that relative motion at constant velocity has no physical consequences beyond the minimum which is required by the existence of that motion: the *production* of completely detectable and physically consequential magnetic field *merely* by dint of *the magnitude of constant velocity of relative motion* is *just as unequivocal* and *startling* a contradiction of Galilean/Newtonian principles as are the much more *celebrated* slowing of clocks and contractions of lengths which *also* occur *merely* by dint of *the magnitude of constant velocity of relative motion*. Indeed the production of magnetic field by dint of the magnitude of constant velocity of relative motion is a *first-order* effect of that velocity magnitude, while the distortion of electric fields, the slowing of clocks and the contraction of lengths are vastly more subtle *second-order* effects of that velocity magnitude. To bluntly put this matter in perspective, the discovery that the presence of direct current in a wire deflects an adjacent compass needle *already sealed the fate of the Galilean/Newtonian paradigm*, notwithstanding that scientists understandably didn't have the wit to *realize* that for a very, very long time.

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