The difference of any real transcendental number and complex number $e^i$ is always a complex transcendental number.

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Section 1. From Euler’s formula,

$$e^{ix} = \cos x + \sin x$$

we can derive the following equation

$$ae^{ix} = 2c - a + 2i\sqrt{ca - c^2} \quad (1)$$

Where $a$, $c$ and $x$ are real numbers and can be defined as

$$x = \cos^{-1} \frac{2c-a}{a}$$

$$c = \frac{a(\cos x + 1)}{2}$$ and $a$ is the diameter of the circle on complex plane such that $a \leq c \leq 0 \leq c \leq a$ and $0 \leq x \leq \pi$ for positive values of $a$ and $c$ and $\pi \leq x \leq 2\pi$ for negative values of $a$ and $c$.

The equation (1) can be obtained as follows. Consider the following circle on a complex plane with center O touching the imaginary axis at zero.
Let length LM is $c$ and MN is $b$ then diameter LN should be $c+b$ and radius OP or ON is $\frac{c+b}{2}$. In this way, the length OM is

$$OM = ON - MN = \frac{c+b}{2} - b = \frac{c-b}{2}$$

Using Pythagoras theorem, the length PM can be obtained as follows:

$$PM = \sqrt{OP^2 - OM^2}$$

$$PM = \sqrt{\left(\frac{c+b}{2}\right)^2 - \left(\frac{c-b}{2}\right)^2} = \sqrt{cb}$$

Hence

$$\sin x = \frac{PM}{OP} = \frac{2\sqrt{cb}}{c+b}$$

and

$$\cos x = \frac{OM}{OP} = \frac{c-b}{c+b}$$

Insert (2) and (3) in Euler’s formula to get

$$e^{ix} = \frac{c-b}{c+b} + l\frac{2\sqrt{cb}}{c+b}$$

$$(c + b)e^{ix} = c - b + 2i\sqrt{cb}$$

Let $c + b = a$ then $b = a - c$; so we have,

$$ae^{ix} = c - (a - c) + 2i\sqrt{c(a - c)}$$

Or

$$ae^{ix} = 2c - a + 2i\sqrt{ca - c^2}$$

**Section 2. Checking the validity of the equation:**

(A) Let $a = 1$ then we have

$$e^{ix} = 2c - 1 + 2i\sqrt{c - c^2}$$

Now we check the equality of the equation for different values of $c$ by solving R.H.S first then L.H.S

(i) $c = 0$

$$e^{ix} = 0 - 1 + 0$$
\[ x = \cos^{-1} \frac{2c-a}{a} \]

\[ x = \cos^{-1} \frac{-1}{a_1} = -1 \text{ therefore } x = \pi \]

\[ e^{i\pi} = -1, \text{ hence L.H.S = R.H.S} \]

(ii) \( c = 0.25 \)

\[ e^{ix} = 0.5 - 1 + 2i\sqrt{0.25 - 0.0625} \]
\[ = -0.5 + 2i\sqrt{0.1875} \]
\[ = -0.5 + 0.866i \]

\[ x = \cos^{-1} \frac{0.5-1}{1} \]

\[ x = \cos^{-1} -0.5, \text{ therefore } x = \frac{2\pi}{3} \]

\[ e^{i2\pi/3} = -0.5 + 0.866i \text{ hence L.H.S = R.H.S} \]

(iii) \( c = 0.5 \)

\[ e^{ix} = 1 - 1 + 2i\sqrt{0.5 - 0.25} \]
\[ = i \]

\[ x = \cos^{-1} \frac{1-1}{1} \text{, therefore } x = \frac{\pi}{2} \]

\[ e^{i\pi/2} = i \]

(iv) \( c = 0.75 \)

\[ e^{ix} = 1.5 - 1 + 2i\sqrt{0.75 - 0.5625} \]
\[ = 0.5 + 2i\sqrt{0.1875} \]
\[ = 0.5 + 0.866i \]

\[ x = \cos^{-1} \frac{1.5-1}{1} \]

\[ x = \cos^{-1} 0.5, \text{ therefore } x = \frac{\pi}{3} \]

\[ e^{i\pi/3} = 0.5 + 0.866i \text{ hence L.H.S = R.H.S} \]

(v) \( c = 1 \)

\[ e^{ix} = 2 - 1 + 2i\sqrt{1-1} \]
= 1

\[ x = \cos^{-1} \frac{2 - 1}{1} \], therefore \( x = 0 \)

\[ e^0 = 1 \], hence L.H.S = R.H.S

(B) Let \( a = -1 \) then we have

\[ -e^{ix} = 2c + 1 + 2i\sqrt{-c - c^2} \] or

\[ e^{ix} = -2c - 1 - 2i\sqrt{-c - c^2} \]

(i) \( c = 0 \)

\[ e^{ix} = -1 \]

\[ x = \cos^{-1} \frac{0 + 1}{-1} \], therefore \( x = \pi \)

\[ e^{i\pi} = -1 \], hence L.H.S = R.H.S

(ii) \( c = -0.25 \)

\[ e^{ix} = 0.5 - 1 - 2i\sqrt{0.25 - 0.0625} \]

\[ = -0.5 - 2i\sqrt{0.1875} \]

\[ e^{ix} = -0.5 - 0.866i \]

\[ x = \cos^{-1} \frac{-0.5 + 1}{-1} \], therefore \( x = \frac{4\pi}{3} \)

\[ e^{i\pi/3} = -0.5 - 0.866i \], hence L.H.S = R.H.S

(iii) \( c = -0.5 \)

\[ e^{ix} = 1 - 1 - 2i\sqrt{0.5 - 0.25} \]

\[ e^{ix} = 1 - 1 - 2i\sqrt{0.25} \]

\[ e^{ix} = -i \]

\[ x = \cos^{-1} \frac{-1 + 1}{-1} \], \( x = \frac{3\pi}{2} \)

\[ e^{i\pi/2} = -i \], hence L.H.S = R.H.S

(iv) \( c = -0.75 \)
\[ e^{ix} = 1.5 - 1 - 2i\sqrt{0.75 - 0.5625} \]
\[ e^{ix} = 1.5 - 1 - 2i\sqrt{0.1875} \]
\[ e^{ix} = 0.5 - 0.866i \]
\[ x = \cos^{-1} \left( \frac{-1.5 + 1}{-1} \right) = -0.5, \therefore x = \frac{5\pi}{3} \]
\[ e^{\frac{5\pi}{3}} = 0.5 - 0.866i, \text{ hence L.H.S} = \text{R.H.S} \]

(v) \[ c = -1 \]
\[ e^{ix} = 2 - 1 - 2i\sqrt{1 - 1} \]
\[ e^{ix} = 1 \]
\[ x = \cos^{-1} \left( \frac{-2 + 1}{-1} \right) = 1, \therefore x = 2\pi \]
\[ e^{i2\pi} = 1 \]

Therefore we can conclude that
\[ ae^{ix} = 2c - a + 2i\sqrt{ca - c^2} \] such that \( a \leq c \leq 0 \leq c \leq a \) and \( x \leq \pi \) for positive values of \( a \) and \( c \) and
\[ \pi \leq x \leq 2\pi \] for negative values of \( a \) and \( c \)

Based on above values, the following diagram can be presented.
From the diagram 2, we can see that equation 1 completes half cycle for positive values of $a$ and $c$ and completes other half for negative values of $a$ and $c$. The other half cycle is the mirror image of dotted cosine wave. That means when $c$ falls from 1 to 0 the circle flips to the negative side of the real number line and the value of $c$ starts falling from 0 to -1. This behavior can be seen in the following diagram.

![Diagram 3]

The left circle is flipped circle or the mirror image of dotted circle and therefore rotating clockwise, the angle $x$ is increasing from $\pi$ to $2\pi$ and the value of $c$ is falling from 0 to -1. The dotted circle depicts the dotted cosine wave in diagram 2 and is rotating anticlockwise.

**Section 3 : If $c$ is transcendental then $c-e^i$ or $-c+e^i$ is also transcendental**

**Lemma 1.** If $x$ is algebraic and $x \in \left\{ \cos^{-1} \frac{2c-a}{a} \right\}$ then $a$ is algebraic and $c$ is transcendental.

**Proof:** Consider equation 3

$$\cos x = \frac{c-b}{c+b}$$

Or $$\cos x = \frac{c-b}{a} = \frac{2c-a}{a}$$ because $b = a - c$.

Therefore $$x = \cos^{-1} \frac{2c-a}{a}$$
According to Lindemann’s theorem, for all algebraic values of $x$ the trigonometric function $\cos x$ is transcendental. But $\frac{2c-a}{a}$ can always be transcendental only if $a$ is algebraic and $c$ is transcendental. This implies that set of algebraic values of $x$ is subset of $\left\{ \cos^{-1} \frac{2c-a}{a} \right\}$ when $a$ is algebraic and $c$ is transcendental.

It is not impossible to make number $a$ algebraic of any desired value by adding some unknown transcendental number $b$ in $c$. Similarly we can obtain any desired algebraic value of $x$ by adjusting the value of $a$.

**Lemma 2.** $2i\sqrt{ca - c^2}$ is always transcendental if $a$ is algebraic and $c$ is transcendental.

**Proof.** Let $ca - c^2 = y$

Where we assume $y$ is algebraic. We get the following quadratic equation:

$$c^2 - ac + y = 0$$

Therefore

$$c = \frac{-(-a) \pm \sqrt{(-a)^2 - 4y}}{2} \quad (4)$$

Since $c$ is transcendental and $a$ is algebraic then equation (4) can only be transcendental if $y$ is transcendental. Therefore our assumption that $y$ or $ca - c^2$ is algebraic is wrong. Hence term $2i\sqrt{ca - c^2}$ is transcendental.

**Proposition:** $c - e^i$ or $-c + e^i$ is a complex transcendental number where $c$ is any real transcendental number.

**Proof:** We can re-write equation 1 as follows:

$$a = 2i\sqrt{ca - c^2} + 2c - ae^{ix} \quad (5)$$

Let $a$ is algebraic and $c$ is transcendental.

We have $x = \cos^{-1} \frac{2c-a}{a}$

According to lemma 1 we can make $x$ algebraic of value of one radian by adjusting the value of $a$. Hence equation 5 becomes—

$$a = 2i\sqrt{ca - c^2} + 2c - ae^i$$
Since $a$ is algebraic therefore both the terms $2i\sqrt{ca - c^2}$ and $2c - ae^i$ should be algebraic or transcendental.

But from lemma 2 we know $2i\sqrt{ca - c^2}$ is transcendental therefore $2c - ae^i$ is also transcendental.

We can put some algebraic number $n$ in place of $2$ and $a$ without affecting the value of the term $2c - ae^i$. Hence we can write --

$$n(c - e^i) = 2c - ae^i$$

If $c$ is negative then $a$ is also negative therfore above equation becomes as follows

$$n(-c + e^i) = -2c + ae^i$$

In this way we conclude that the difference $c - e^i$ or $-c + e^i$ is a complex transcendental number where $c$ is any real transcendental number.