The difference of any real transcendental number and complex number $e^i$ is always a complex transcendental number.

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From Euler’s formula,

$$e^{ix} = \cos x + isinx$$

we can derive the following equation

$$ae^{ix} = 2c - a + 2i\sqrt{ca - c^2}$$

Where, $a$, $c$, and $x$ are real numbers such that

$$x = \cos^{-1} \frac{2c-a}{a} \quad \text{and} \quad c = \frac{a(cosx+1)}{2}$$

The above equation can be obtained as follows. Consider the following circle on a complex plane with center O touching the imaginary axis at zero.

Let length LM is $c$ and MN is $b$ then diameter LN should be $c+b$ and radius OP or ON is $\frac{c+b}{2}$. In this way, the length OM is
OM = ON – MN

\[ OM = \frac{c+b}{2} - b = \frac{c-b}{2} \]

Using Pythagoras theorem, the length PM can be obtained as follows:

\[ PM = \sqrt{OP^2 - OM^2} \]

\[ PM = \sqrt{\left(\frac{c+b}{2}\right)^2 - \left(\frac{c-b}{2}\right)^2} = \sqrt{cb} \]

Hence

\[ \sin x = \frac{PM}{OP} = \frac{2\sqrt{cb}}{c+b} \]

(1)

and

\[ \cos x = \frac{OM}{OP} = \frac{c-b}{c+b} \]

(2)

Insert (1) and (2) in Euler’s formula to get

\[ e^{ix} = \frac{c-b}{c+b} + i \frac{2\sqrt{cb}}{c+b} \]

\[ (c+b)e^{ix} = c - b + 2i\sqrt{cb} \]

Let \( c + b = a \) then \( b = a - c \); so we have,

\[ ae^{ix} = c - (a - c) + 2i\sqrt{c(a-c)} \]

Or

\[ ae^{ix} = 2c - a + 2i\sqrt{ca - c^2} \]

(3)

**Lemma 1.** If \( x \) is algebraic and \( x \in \left\{ \cos^{-1} \frac{2c-a}{a} \right\} \) then \( a \) is algebraic and \( c \) is transcendental.

**Proof:** Consider equation 2

\[ \cos x = \frac{c-b}{c+b} \]

Or \( \cos x = \frac{c-b}{a} = \frac{2c-a}{a} \) because \( b = a - c \).

Therefore \( x = \cos^{-1} \frac{2c-a}{a} \)
According to Lindemann’s theorem, for all algebraic values of $x$ the trigonometric function $\cos x$ is transcendental. But $\frac{2c-a}{a}$ can always be transcendental only if $a$ is algebraic and $c$ is transcendental. This implies that set of algebraic values of $x$ is subset of $\left\{ \cos^{-1} \frac{2c-a}{a} \right\}$ when $a$ is algebraic and $c$ is transcendental.

It is not impossible to make number $a$ algebraic of any desired value by adding some unknown transcendental number $b$ in $c$. Similarly we can obtain any desired algebraic value of $x$ by adjusting the value of the number $a$.

**Lemma 2.** $2i\sqrt{ca - c^2}$ is always transcendental if $a$ is algebraic and $c$ is transcendental.

**Proof.** Let $ca - c^2 = y$

Where we assume $y$ is algebraic. We get the following quadratic equation:

$$c^2 - ac + y = 0$$

Therefore

$$c = \frac{-(a) \pm \sqrt{(-a)^2 - 4y}}{2}$$ \hspace{1cm} (4)

Since $c$ is transcendental and $a$ is algebraic then equation (4) can only be transcendental if $y$ is transcendental. Therefore our assumption that $y$ or $ca - c^2$ is algebraic is wrong. Hence term $2i\sqrt{ca - c^2}$ is transcendental.

**Proposition:** $c - e^{i}$ is a complex transcendental number where $c$ is any real transcendental number.

**Proof:** We can re-write equation 3 as follows:

$$a = 2i\sqrt{ca - c^2} + 2c - ae^{ix}$$ \hspace{1cm} (5)

Let $a$ is algebraic and $c$ is transcendental.

We have $x = \cos^{-1} \frac{2c-a}{a}$

According to lemma 1 we can make $x$ algebraic of value of one radian by adjusting the value of $a$. Hence equation 5 becomes—

$$a = 2i\sqrt{ca - c^2} + 2c - ae^{i}$$
Since $a$ is algebraic therefore both the terms $2i\sqrt{ca - c^2}$ and $2c - ae^i$ should be algebraic or transcendental.

But from lemma 2 we know $2i\sqrt{ca - c^2}$ is transcendental therefore $2c - ae^i$ is also transcendental.

We can adjust the values of coefficients $2$ and $a$ to make them equal to some algebraic number $n$ without affecting the value of the term $2c - ae^i$. Hence we can write --

$$n(c - e^i) = 2c - ae^i$$

In this way we conclude that the difference $c - e^i$ is a complex transcendental number where $c$ is any real transcendental number.