A special hexagon with two-fold symmetry must have an inscribed square satisfying the Toeplitz conjecture

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Abstract: In this paper, we generate a special hexagon with two-fold symmetry by diagonally juxtaposing two squares of different dimensions so that they share exactly one common vertex and their adjacent sides are perpendicular to one another. We connect in specific pairs, the vertices adjacent to common vertex of both squares to generate a hexagon that is symmetrical about a line connecting the unconnected vertices. We show that this special hexagon must have one square whose points lie on its sides. With suitable modifications, it may be possible to use this technique to prove the Toeplitz conjecture for a simple closed curve generated by connecting the same six vertices of this special hexagon.

Results: Consider two squares of different lengths for its sides AFGB (length of each side is a) and ACDE (length of each side is b). Juxtapose the two squares at the common vertex A such that the points G, A and D are collinear and the angles BAC and FAE are right angles. Connect the points B with C and F with C to create a hexagon GBCDEF that has a two-fold symmetry around the diagonal GAD (Figure 1).

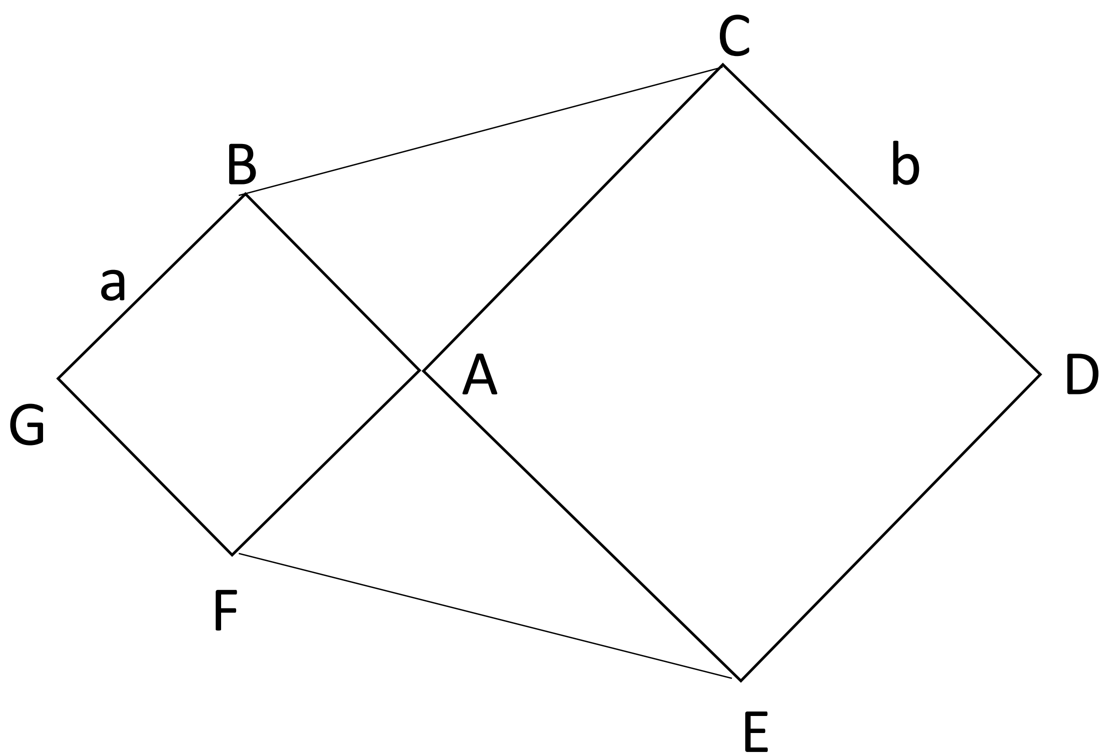

Figure 1

Does this special hexagon GBCDEF with two-fold symmetry contain an inscribed square and thereby is special simple closed curve for which the Toeplitz conjecture is true?

We prove that the larger four sides of the regular hexagon must each contain one point on itself that together define a square consistent with the Toeplitz conjecture for a simple closed curve that states the presence of atleast a single inscribed square PQRS within such curves.

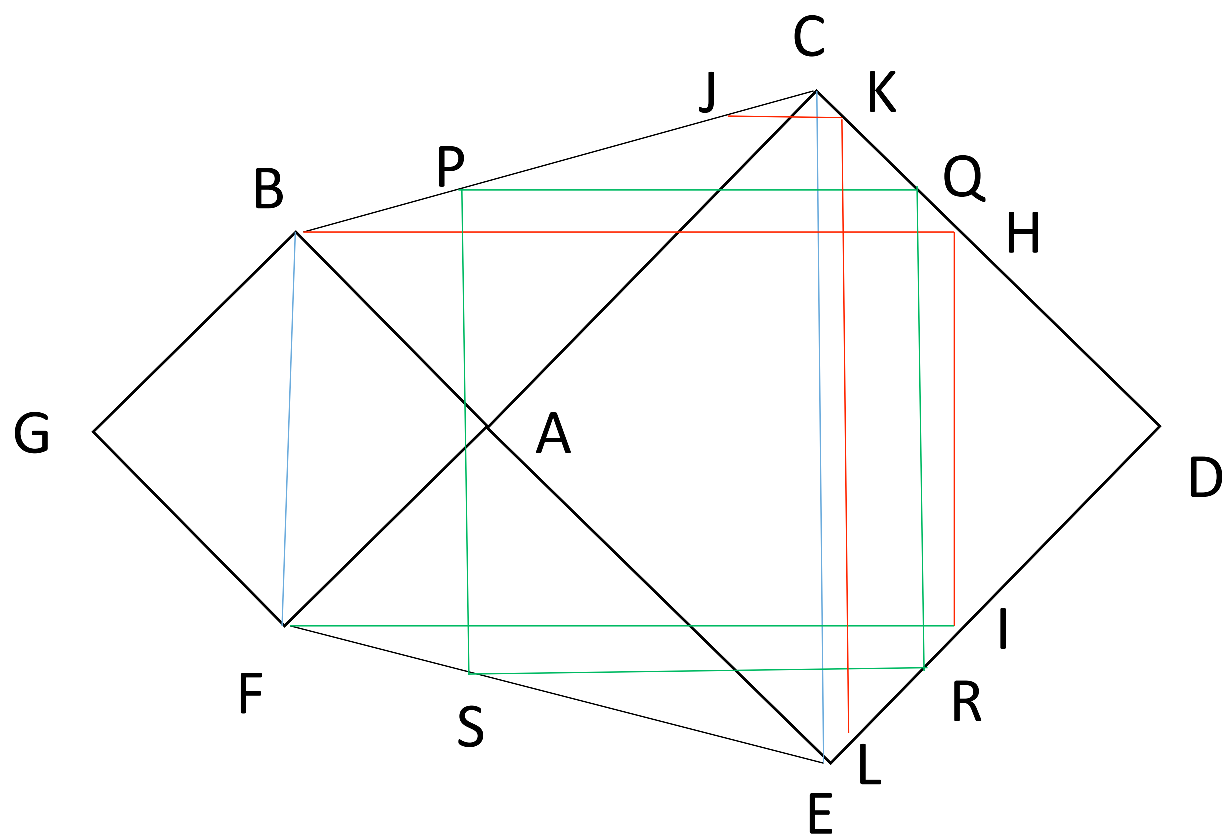

Figure 2

The proof is as follows:

The small square has a side of length “a”. Therefore the segment BF, its diagonal must have length $a\sqrt{2}$. Consider segments BH and FI which are parallel to line segment GAD, the axis of symmetry. Segment HI is parallel to segment BF and has same length of $a\sqrt{2}$.

Similarly the large square has length “b”. Therefore the segment CE, its diagonal has length $b\sqrt{2}$.

Consider all horizontal segments formed by connecting a point on CB with a point on CH. One such horizontal segment is segment JK and it is associated with a vertical segment KL perpendicular to it where L is a point on the side DE. Consider all such pairs of horizontal and vertical segments from top to the bottom. So for the interval that we are interested in, the first horizontal segment at the top is literally the point C and the last horizontal segment is segment BH. As the length of the horizontal segment increases smoothly from one segment to the next from top to bottom, the length of the corresponding vertical segment decreases continuously and it shifts rightward. So for the top horizontal segment approximated by point C (and therefore length = 0), the corresponding vertical distance is the segment CE of length b$\sqrt{2}$. So for the bottom most horizontal segment in the interval discussed, that is segment BH has a length b$\sqrt{2}$ and the corresponding vertical distance is segment HI with length a$\sqrt{2}$.

So the horizontal segment increases smoothly in length from 0 to b$\sqrt{2}$ while the vertical distance decreases in length smoothly from b$\sqrt{2}$ to a$\sqrt{2}$. So there must be a horizontal segment PQ of length "s" between a$\sqrt{2}$ and b$\sqrt{2}$ where there is a corresponding vertical segment QR that has the same length "s". By symmetry, the segment SR has the same length as segment PQ and therefore the segment PS must also have the same length "s". Therefore the square PQRS is inscribed in this special hexagon and satisfies the Toeplitz conjecture for this special hexagon.

It is possible that a similar approach may be useful to prove the Toeplitz conjecture for a special simple closed curve running through the same points as the vertices of the hexagon decribed above or other cases.