Question 383: Nonlinear Equation , Euler Numbers , Number Pi

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abstract

This note presents some formulas for π .

1. Introduction

- The nonlinear equation:

$$\sinh x - a\cos x = 0 \tag{1}$$

- Unknown: $x := x(a)$.
- Assumptions: $|x| < \pi / 2\sqrt{2}$, $a > 0$.
- Particular cases:

$$a = \left\{1, \frac{1}{\sqrt{3}}, \sqrt{2}-1, 2-\sqrt{3}\right\} \tag{2}$$

$$a = 1 \Rightarrow \sinh x - \cos x = 0 \Rightarrow x = \alpha = 0.70329065... \tag{3}$$

$$a = \frac{1}{\sqrt{3}} \Rightarrow \sinh x - \frac{1}{\sqrt{3}}\cos x = 0 \Rightarrow x = \beta = 0.48969168... \tag{4}$$

$$a = \sqrt{2}-1 \Rightarrow \sinh x - \left(\sqrt{2}-1\right)\cos x = 0 \Rightarrow x = \gamma = 0.37628974... \tag{5}$$

$$a = 2-\sqrt{3} \Rightarrow \sinh x - \left(2-\sqrt{3}\right)\cos x = 0 \Rightarrow x = \delta = 0.25637378... \tag{6}$$

- Iterative method:

$$f(a,x) = \sinh^{-1}(a\cos x) \tag{7}$$

$$x_{n+1} = f(1, x_n) \, , \, x_1 = 0 \Rightarrow x_n \to \alpha \tag{8}$$

$$x_{n+1} = f\left(\frac{1}{\sqrt{3}}, x_n\right), x_1 = 0 \Rightarrow x_n \to \beta \tag{9}$$

$$x_{n+1} = f\left(\sqrt{2}-1, x_n\right), x_1 = 0 \Rightarrow x_n \to \gamma \tag{10}$$

$$x_{n+1} = f\left(2-\sqrt{3}, x_n\right), x_1 = 0 \Rightarrow x_n \to \delta \tag{11}$$

❖ Remark: $\sinh^{-1} y = \ln\left(y+\sqrt{1+y^2}\right), y \in \mathbb{R}$.

2. Euler Numbers

❖ Definition Euler numbers $E_n, n \in \mathbb{N}$:

$$\frac{2}{e^x + e^{-x}} = 1 - \frac{E_1}{2!}x^2 + \frac{E_2}{4!}x^4 - \frac{E_3}{6!}x^6 + \ldots \quad , |x| < \frac{\pi}{2} \tag{12}$$

$$E_n = \{1, 5, 61, 1385, 50521, \ldots\} \tag{13}$$

$$E_n = (-1)^n \sum_{k=1}^{n} 2^{-k+1} \sum_{m=1}^{k} (-1)^m \binom{2k}{k-m} m^{2n} \quad , n \in \mathbb{N} \tag{14}$$

3. Pi Formulas

$$\pi = 4\alpha + 4\sum_{n=1}^{\infty} \frac{E_n 2^n}{(2n+1)!} \alpha^{2n+1} \operatorname{Im}\left(i^n (1+i)\right) \tag{15}$$

$$\pi = 6\beta + 6\sum_{n=1}^{\infty} \frac{E_n 2^n}{(2n+1)!} \beta^{2n+1} \operatorname{Im}\left(i^n (1+i)\right) \tag{16}$$

$$\pi = 8\alpha + 8\sum_{n=1}^{\infty} \frac{E_n 2^n}{(2n+1)!} \gamma^{2n+1} \operatorname{Im}\left(i^n (1+i)\right) \tag{17}$$

$$\pi = 12\delta + 12\sum_{n=1}^{\infty} \frac{E_n 2^n}{(2n+1)!} \delta^{2n+1} \operatorname{Im}\left(i^n (1+i)\right) \tag{18}$$

❖ Remark: $i=\sqrt{-1}$, $z=x+iy\in\mathbb{C},\operatorname{Im}(z)=y$.

$$u_n=\operatorname{Im}\left(i^n\left(1+i\right)\right)\quad ,n\in\mathbb{N} \tag{19}$$

$$u_n=\left\{1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,...\right\} \tag{20}$$

$$u_1=1,u_{4n-2}=u_{4n-1}=-1,u_{4n}=u_{4n+1}=1 \tag{21}$$

$$u_{n+2}=-u_n\ ,u_1=1,u_2=-1 \tag{22}$$

4. Other Formulas

For $\ln\left(1+\sqrt{2}\right)<x<\pi/2$:

$$\pi=2x+2\sum_{n=1}^{\infty}\frac{\left(-1\right)^n E_n}{\left(2n+1\right)!}x^{2n+1}+4\sum_{n=0}^{\infty}e^{-(2n+1)x}\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{\left(-4\right)^k}{2k+1} \tag{23}$$

Example $x=1$:

$$\pi=2+2\sum_{n=1}^{\infty}\frac{\left(-1\right)^n E_n}{\left(2n+1\right)!}+4\sum_{n=0}^{\infty}e^{-2n-1}\sum_{k=0}^{n}\binom{n+k}{n-k}\frac{\left(-4\right)^k}{2k+1} \tag{24}$$

$$\pi=4\ln\left(1+\sqrt{2}\right)+4\sum_{n=1}^{\infty}\frac{\left(-1\right)^n E_n}{\left(2n+1\right)!}\left(\ln\left(1+\sqrt{2}\right)\right)^{2n+1} \tag{25}$$

$$\frac{\ln 2}{2}+i\frac{\pi}{4}=z+\sum_{n=1}^{\infty}\frac{E_n}{\left(2n+1\right)!}z^{2n+1}\quad ,z=\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+i\frac{\ln 5}{2} \tag{26}$$

If $\sinh\left(\frac{r\sqrt{3}}{2}\right)=\cos\left(\frac{r}{2}\right),r=0.9289002...$ then:

$$\pi=4r\sin\left(\frac{\pi}{3}\right)+4\sum_{n=1}^{\infty}\frac{E_n}{\left(2n+1\right)!}r^{2n+1}\sin\left(\frac{\left(2n+1\right)\pi}{3}\right) \tag{27}$$

If $\sinh\left(\frac{r}{2}\right)=\cos\left(\frac{r\sqrt{3}}{2}\right),r=1.1010843...$ then:

$$\pi=4r\sin\left(\frac{\pi}{6}\right)+4\sum_{n=1}^{\infty}\frac{E_n}{\left(2n+1\right)!}r^{2n+1}\sin\left(\frac{\left(2n+1\right)\pi}{6}\right) \tag{28}$$

If $\sinh\left(r\sin 1\right)=\cos\left(r\cos 1\right),r=0.9383295...$ then:

$$\pi = 4r\sin 1 + 4\sum_{n=1}^{\infty} \frac{E_n}{(2n+1)!} r^{2n+1} \sin(2n+1) \tag{29}$$

If $\sinh(\sin\theta) = \cos(\cos\theta), \theta = 0.7687389...$ then:

$$\pi = 4\sin\theta + 4\sum_{n=1}^{\infty} \frac{E_n}{(2n+1)!} \sin((2n+1)\theta) \tag{30}$$

If $z = 1 + i\ln\left(\cos 1 + \sqrt{1+(\cos 1)^2}\right)$ then:

$$\pi = 4\operatorname{Im}(z) + 4\sum_{n=1}^{\infty} \frac{E_n}{(2n+1)!} \operatorname{Im}\left(z^{2n+1}\right) \tag{31}$$

$$\pi = 6\ln\left(\sec\left(\frac{\ln 3}{2}\right) + \tan\left(\frac{\ln 3}{2}\right)\right) - 12\sum_{n=1}^{\infty} \frac{E_{2n-1}}{(4n-1)!} \left(\frac{\ln 3}{2}\right)^{4n-1} \tag{32}$$

$$\pi = 6\ln 3 - 6\ln\left(\sec\left(\frac{\ln 3}{2}\right) + \tan\left(\frac{\ln 3}{2}\right)\right) + 12\sum_{n=1}^{\infty} \frac{E_{2n}}{(4n+1)!} \left(\frac{\ln 3}{2}\right)^{4n+1} \tag{33}$$

References

1. Abramowitz, M., and Stegun, I. A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathemetical Tables. Applied Mathematical Series 55, National Bureau of Standards, Washington, DC; Repr. Dover, New York, 1965.
2. Spanier, J., and Oldham, K.B.: An Atlas of Functions , Hemisphere Publishing,1987.