Analyzing the monotonicity of belief interval based uncertainty measures in belief function theory

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Abstract

Measuring the uncertainty of pieces of evidence is an open issue in belief function theory. A rational uncertainty measure for belief functions should meet some desirable properties, where monotonicity is an very important property. Recently, measuring the total uncertainty of a belief function based on its associated belief intervals becomes a new research idea and have attracted increasing interest. Several belief interval based uncertainty measures have been proposed for belief functions. In this paper, we summarize the properties of these uncertainty measures and especially investigate whether the monotonicity is satisfied by the measures. This study provide a comprehensive comparison to these belief interval based uncertainty measures and is very useful for choosing the appropriate uncertainty measure in the practical applications.

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1. Introduction

Extracting information from uncertain data is a challenging problem which cannot be avoided in practice. Various theories have been developed to deal with different types of uncertainties. In particular, the Dempster–Shafer theory of evidence [1, 2], which is also called belief function theory, is a popular mathematical tool for representing and handling uncertain information [3–10] because it has the advantage of directly expressing "uncertainty" by assigning a probability to the subsets of a set comprising multiple solutions, rather than to each of the individual solution. Primarily, belief function theory allows a representation with both imprecision and ignorance by using a belief measure and a plausibility measure which respectively express the lower bound and upper bound of the support degree to each proposition in the given information. Secondly, this theory has the ability to combine information from multiple sources without the requirement for a priori knowledge.

In belief function theory, measuring the uncertainty pertaining to a given belief function is an open issue [11, 12]. Thus, the desirable properties for a rational uncertainty measure of belief functions have attracted much interest. Representative properties include monotonicity, additivity, maximum entropy, and so forth. There are several typical systems of properties for uncertainty measures based on different semantics consistency requirements. A well-known one is proposed by Klir and Wierman [13], which is consistent with semantics of probability theory and includes five basic properties, i.e., probabilistic consistency, set consistency, range, subadditivity, and additivity. An uncertainty measure meeting the five properties is definitely compatible with the Shannon's entropy for probability function. Subsequently, based on credal set semantics of belief functions, Abellán and Masegosa [11, 12] further extended Klir and Wierman's system to a new axiomatic system additionally containing a new property called monotonicity and a set of behaviour properties to overcome the imperfections of Klir and Wierman's system. Another representative system of properties for the uncertainty measures of belief functions was given by Jiroušek and Shenoy [14, 15] based on semantics consistent with Dempster-Shafer theory of belief functions where Dempster's product-intersection rule is used as the combination rule to aggregate knowledge. A plausibility transform-based [16] entropy definition for belief functions is proved to meet Jiroušek and Shenoy's system of properties.

Based on the properties mentioned above, different uncertainty measures have been developed. According to previous studies, a belief function can have two types of uncertainties, i.e., conflict and nonspecificity. As explained in [17], the conflict mainly stands for the disagreement in choosing among several alternatives, and the nonspecificity refers to that two or more alternatives are left unspecified. Previous studies have proposed many uncertainty measures for quantifying these types of uncertainties for belief functions. With respect to conflict, representative uncertainty measures include "dissonance" (Diss) proposed by Yager [18] and "Confusion" (Conf) presented by Höhle [19]. For nonspecificity, Dubois and Prade's generalized Hartley measure [20] is a standard solution for measuring this type of uncertainty. Regarding the total uncertainty, the aggregated uncertainty (AU) [21] and ambiguity measure (AM) [17] are typical widely used measure functions. In addition, some other uncertainty measures were proposed by [14, 22, 23] for the total uncertainty. However, most of these uncertainty measures have been criticized because of their low sensitivity, high computational complexity, the concealment of conflict and nonspecificity, and other issues. For example, AU is known to be highly insensitive to changes in evidence. Moreover, a basic difficulty with an uncertainty measure is that it is usually required to be consistent between the frameworks of belief function theory and probability theory.

Recently, belief intervals constituted by the belief measure and plausibility measure have attracted people's attention for calculating the total uncertainty of belief functions. Several belief interval based uncertainty measures have been proposed in recent studies. Relying on the belief intervals of a belief function, Yang and Han [24] proposed a distance-based total uncertainty measure TU^I which is based solely on the framework of belief function theory, but without considering the switch between the frameworks of belief function theory and probability theory. Whereafter, Deng et al. [25] found some deficiencies of TU^I and given an improved uncertainty measure TU_E^I which is also based on the belief intervals. In [26], an uncertainty measure SU has been developed by Wang and Song on the basis of the belief intervals associated with a belief function, and SU could reduce to the Shannon'e entropy under a certain conditions. These studies provide a new research idea on measuring the total uncertainty of belief functions.

In this paper, these belief interval based uncertainty measures are concerned. We investigate them from the property of monotonicity. The monotonicity is one of the most important properties for a rational uncertainty measures, which requires that an uncertainty measure for belief functions must not decrease the quantity of the total uncertainty when a clear decrease in information (increment of uncertainty) occurs [11]. In theory and application, the monotonicity is a crucial basis for information ordering. In the study, the monotonicity of uncertainty measures TU^I, SU and TU_E^I is analyzed respectively, and their other properties are summarized for the sake of choosing an appropriate uncertainty measure in the practical applications.

The rest of the paper is organized as follows. Section 2 gives a brief introduction about Dempster-Shafer evidence theory and some existing uncertainty measures for belief functions. Then, the definition of monotonicity is given in Section 3. In Section 4, the monotonicity of recent proposed belief interval based uncertainty measures is analyzed, and other properties of these uncertainty measures are summarized as well. At last, Section 5 concludes this paper.

2. Basics of belief function theory

In belief function theory, a FOD is a set of mutually exclusive and collectively exhaustive events denoted by $\Theta = \{\theta_1, \theta_2, \cdots, \theta_n\}$. The power set of Θ is denoted as 2^Θ. Given a FOD, a mapping $m : 2^\Theta \to [0, 1]$ is a mass function, which is also called a basic belief assignment (BBA), defined on Θ

if it satisfies

$$m(\emptyset) = 0 \quad \text{and} \quad \sum_{A \subseteq \Theta} m(A) = 1. \tag{1}$$

If $m(A) > 0$, then A is called a focal element, and the union of all focal elements is called the core of the BBA. We note that the constraint of $m(\emptyset) = 0$ is only required in a closed world environment where all possible results are included in the FOD. Thus, the constraint is not required if we set an open world assumption, which allows events to occur that do not belong to the FOD. In belief function theory, $m(A)$ measures the belief assigned exactly to A and it represents how strongly the evidence supports A.

The belief measure Bel and plausibility measure Pl associated with a BBA express the lower bound and upper bound of the support degree for each proposition in a BBA, respectively. They are defined as

$$Bel(A) = \sum_{B \subseteq A} m(B), \tag{2}$$

$$Pl(A) = 1 - Bel(\bar{A}) = \sum_{B \cap A \neq \emptyset} m(B), \tag{3}$$

where $\bar{A} = \Omega - A$. Obviously, $Pl(A) \geq Bel(A)$ for each $A \subseteq \Theta$. The belief $Bel(A)$ and plausibility $Pl(A)$ constitute a belief interval $[Bel(A), Pl(A)]$. The length of $[Bel(A), Pl(A)]$ represents the degree of imprecision for the proposition or focal element A.

Measuring the uncertainty of evidence is an open issue in belief function theory. In the following, the mathematical definitions of some representative uncertainty measures are given.

In order to measure the nonspecificity included in a BBA, Dubois and Prade [20] gave a generalization of Hartley measure as follows:

$$\mathrm{NS}(m) = \sum_{A \subseteq \Theta} m(A) \log_2(|A|). \tag{4}$$

"Dissonance" (Diss) [18] and "Confusion" (Conf) [19] are two typical uncertainty measures for the conflict, which extend the Shannon's entropy to belief function theory:

$$\mathrm{Diss}(m) = -\sum_{A \subseteq \Theta} m(A) \log_2(Pl(A)), \tag{5}$$

$$\mathrm{Conf}(m) = -\sum_{A \subseteq \Theta} m(A) \log_2(Bel(A)). \tag{6}$$

Regarding the total uncertainty associated with a BBA, the aggregated uncertainty (AU) [21] and ambiguity measure (AM) [17] are the representatives. The mathematical definition of AU is

$$\begin{aligned} &\mathrm{AU}(m) = \max\left[-\sum_{\theta \in \Theta} p_\theta \log_2 p_\theta\right] \\ &s.t. \begin{cases} p_\theta \in [0,1],\ \forall \theta \in \Theta \\ \sum_{\theta \in \Theta} p_\theta = 1 \\ Bel(A) \leq \sum_{\theta \in A} p_\theta \leq Pl(A),\ \forall A \subseteq \Theta \end{cases} \end{aligned} \tag{7}$$

and AM is defined as

$$\mathrm{AM}(m) = -\sum_{\theta \in \Theta} BetP_m(\theta) \log_2(BetP_m(\theta)), \tag{8}$$

where

$$BetP_m(\theta) = \sum_{\theta \in A \subseteq \Theta} \frac{m(A)}{|A|}. \tag{9}$$

3. Monotonicity for uncertainty measures

As mentioned above, in belief function theory a rational uncertainty measure should meet some properties, such as non-negativity, boundedness, sensitivity, an so on. Among them, the monotonicity is a basic property for uncertainty measures from a practical application point of view and it is defined as follows [11].

Definition 1 Given an uncertainty measure UM and two arbitrary BBAs m_1 and m_2 over FOD Θ, UM is a monotonic uncertainty measure for belief functions if

$$\forall A \subseteq \Theta : \quad [Bel_{m_1}(A), Pl_{m_1}(A)] \subseteq [Bel_{m_2}(A), Pl_{m_2}(A)], \tag{10}$$

$UM(m_1) \leq UM(m_2)$ exists.

Note that in the closed world assumption (i.e., $m(\emptyset) = 0$), given two BBAs m_1 and m_2 over a FOD Θ, $Bel_{m_1}(A) \geq Bel_{m_2}(A)$ is equivalent to $Pl_{m_1}(A) \leq Pl_{m_2}(A)$ for any $A \subseteq \Theta$, based on the partial ordering relations between belief functions introduced in literatures [27–29]. Therefore, Eq. (10) can be simply rewritten as

$$\forall A \subseteq \Theta : \ Bel_{m_1}(A) \geq Bel_{m_2}(A) \quad \text{or} \quad Pl_{m_1}(A) \leq Pl_{m_2}(A). \tag{11}$$

Essentially, the property of monotonicity means that the total uncertainty of a BBA cannot decrease if the information content (or determinacy) of the BBA decreases. In previous study [11], it has shown that the monotonicity is satisfied in AU but violated by AM.

4. Recent proposed belief interval based uncertainty measures and their monotonicity analysis

Measuring the total uncertainty of a BBA based on its associated belief intervals is a new research idea in belief function theory. So far, three belief interval based uncertainty measures have been proposed by Yang and Han [24], Wang and Song [26], and Deng et al. [25], respectively. In this section, we will investigate the three uncertainty measures and verify whether the monotonicity is satisfied in these measures one by one.

4.1. Yang and Han's uncertainty measure

In [24], Yang and Han proposed a belief interval based total uncertainty measure denoted as TU^I directly for BBAs and based wholly on the framework of belief function theory. TU^I is formally defined as follows.

Definition 2 (TU^I). *If we let m be a BBA defined on the FOD $\Theta = \{\theta_1, \theta_2, \cdots, \theta_n\}$, then the total uncertainty of m is*

$$TU^I(m) = 1 - \frac{1}{n} \cdot \sqrt{3} \cdot \sum_{i=1}^{n} d^I\left([Bel(\{\theta_i\}), Pl(\{\theta_i\})], [0,1]\right) \tag{12}$$

with

$$d^I\left([a_1, b_1], [a_2, b_2]\right) = \sqrt{\left[\frac{a_1+b_1}{2} - \frac{a_2+b_2}{2}\right]^2 + \frac{1}{3}\left[\frac{b_1-a_1}{2} - \frac{b_2-a_2}{2}\right]^2}, \tag{13}$$

where $\sqrt{3}$ is a normalization factor derived from $1/d^I\left([0,0],[0,1]\right)$.

Theoretically, the measure TU^I utilizes information from the belief intervals of all elements in the FOD to measure the total uncertainty in a BBA.

In [24], TU^I is considered to be monotonic. However, via an in-depth study we find that TU^I actually fails to satisfy the monotonicity. Let us consider the following example.

Example 1 Assume that we have three BBAs defined on FOD $\Theta = \{\theta_1, \theta_2\}$:

$$m_1(\{\theta_1\}) = 0.4, \quad m_1(\{\theta_2\}) = 0.3, \quad m_1(\{\theta_1, \theta_2\}) = 0.3;$$

$$m_2(\{\theta_1\}) = 0.4, \quad m_2(\{\theta_2\}) = 0.2, \quad m_2(\{\theta_1, \theta_2\}) = 0.4;$$

$$m_3(\{\theta_1\}) = 0.4, \quad m_3(\{\theta_2\}) = 0.0, \quad m_3(\{\theta_1, \theta_2\}) = 0.6.$$

Then, according to the belief measure Bel and plausibility measure Pl, we can obtain the belief intervals of all the propositions in m_1, m_2, and m_3. For m_1, we have

$$[Bel_{m_1}(\{\theta_1\}), Pl_{m_1}(\{\theta_1\})] = [0.4, 0.7],$$
$$[Bel_{m_1}(\{\theta_2\}), Pl_{m_1}(\{\theta_2\})] = [0.3, 0.6],$$
$$[Bel_{m_1}(\{\theta_1, \theta_2\}), Pl_{m_1}(\{\theta_1, \theta_2\})] = [1, 1].$$

For m_2, we have

$$[Bel_{m_2}(\{\theta_1\}), Pl_{m_2}(\{\theta_1\})] = [0.4, 0.8],$$
$$[Bel_{m_2}(\{\theta_2\}), Pl_{m_2}(\{\theta_2\})] = [0.2, 0.6],$$
$$[Bel_{m_2}(\{\theta_1, \theta_2\}), Pl_{m_2}(\{\theta_1, \theta_2\})] = [1, 1].$$

For m_3, we have

$$[Bel_{m_3}(\{\theta_1\}), Pl_{m_3}(\{\theta_1\})] = [0.4, 1],$$
$$[Bel_{m_3}(\{\theta_2\}), Pl_{m_3}(\{\theta_2\})] = [0, 0.6],$$
$$[Bel_{m_3}(\{\theta_1, \theta_2\}), Pl_{m_3}(\{\theta_1, \theta_2\})] = [1, 1].$$

Therefore,

$$\forall A \subseteq \Theta: \ [Bel_{m_1}(A), Pl_{m_1}(A)] \subseteq [Bel_{m_2}(A), Pl_{m_2}(A)] \subseteq [Bel_{m_3}(A), Pl_{m_3}(A)].$$

Based on the definition of monotonicity given above, we should have $\text{TotalUncertainty}(m_1) \leq \text{TotalUncertainty}(m_2) \leq \text{TotalUncertainty}(m_3)$. However, in terms of TU^I, we obtain that $TU^I(m_1) = 0.6394$, $TU^I(m_2) = 0.6536$, $TU^I(m_3) = 0.6000$. This result shows that the monotonicity between m_1 and m_2 is satisfied in TU^I, but the monotonicity between m_3 and m_1, m_2 is violated. Therefore, we can conclude that the monotonicity is not satisfied in TU^I.

4.2. Wang and Song's uncertainty measure

Recently, Wang and Song [26] developed a new uncertainty measure SU which is also based on the belief interval. The definition of this new uncertainty measure is shown as follows.

Definition 3 (SU). *If we let m be a BBA defined on the FOD $\Theta = \{\theta_1, \theta_2, \cdots, \theta_n\}$, then the total uncertainty of m is*

Given a BBA m defined on the FOD $\Theta = \{\theta_1, \theta_2, \cdots, \theta_n\}$, the total uncertainty degree of m can be expressed by

$$SU(m) = \sum_{i=1}^{n} \left[-\frac{Bel(\{\theta_i\}) + Pl(\{\theta_i\})}{2} \log_2 \frac{Bel(\{\theta_i\}) + Pl(\{\theta_i\})}{2} + \frac{Pl(\{\theta_i\}) - Bel(\{\theta_i\})}{2} \right] \tag{14}$$

Different from TU^I, the uncertainty measure SU satisfies the probability consistency which means that SU is identical to Shannon's entropy when m reduces to a Bayesian BBA. Since in [26] the authors do not give a strict proof on the monotonicity of SU, it impels us to verify if SU is monotonic or not. However, a counter example is found directly which shows that the monotonicity is also violated by SU.

Example 2 Let FOD Θ be $\{\theta_1, \theta_2\}$, assume there are two BBAs over Θ:

$$m_1(\{\theta_2\}) = 0.9, \quad m_1(\{\theta_1, \theta_2\}) = 0.1;$$

$$m_2(\{\theta_1\}) = 0.1, \quad m_2(\{\theta_2\}) = 0.9.$$

Then, according to the belief measure Bel and plausibility measure Pl, for m_1 we have

$$\begin{aligned}
&[Bel_{m_1}(\{\theta_1\}), Pl_{m_1}(\{\theta_1\})] = [0, 0.1], \\
&[Bel_{m_1}(\{\theta_2\}), Pl_{m_1}(\{\theta_2\})] = [0.9, 1], \\
&[Bel_{m_1}(\{\theta_1, \theta_2\}), Pl_{m_1}(\{\theta_1, \theta_2\})] = [1, 1].
\end{aligned}$$

For m_2, we have

$$\begin{aligned}
&[Bel_{m_2}(\{\theta_1\}), Pl_{m_2}(\{\theta_1\})] = [0.1, 0.1], \\
&[Bel_{m_2}(\{\theta_2\}), Pl_{m_2}(\{\theta_2\})] = [0.9, 0.9], \\
&[Bel_{m_2}(\{\theta_1, \theta_2\}), Pl_{m_2}(\{\theta_1, \theta_2\})] = [1, 1].
\end{aligned}$$

Hence,

$$\forall A \subseteq \Theta : \; [Bel_{m_1}(A), Pl_{m_1}(A)] \supseteq [Bel_{m_2}(A), Pl_{m_2}(A)].$$

In terms of the monotonicity given in Definition 1, there should be $\text{TotalUncertainty}(m_1) \geq \text{TotalUncertainty}(m_2)$. By using SU, however we have $SU(m_1) = 0.3864$, $SU(m_2) = 0.4690$, namely $SU(m_1) < SU(m_2)$. So, the monotonicity between m_1 and m_2 is violated by the uncertainty measure SU.

4.3. Deng et al.'s uncertainty measure

Apart from the uncertainty measures given by Yang and Han, Wang and Song, in our previous study [25] we have also proposed a belief interval based uncertainty measure for belief function theory on the basis of TU^I. The uncertainty measure proposed by us is denoted as TU_E^I, it is defined as below.

Definition 4 (TU_E^I). *Let m be a BPA defined on FOD $\Theta = \{\theta_1, \theta_2, \cdots, \theta_n\}$, then the total uncertainty of m is*

$$TU_E^I(m) = \sum_{i=1}^{n} \left[\frac{d_E^I([0,0],[0,1]) - d_E^I([Bel(\{\theta_i\}), Pl(\{\theta_i\})],[0,1])}{d_E^I([0,0],[0,1])} \right] \tag{15}$$

with

$$d_E^I\left([a_1, b_1], [a_2, b_2]\right) = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}. \tag{16}$$

Since $d_E^I\left([0,0],[0,1]\right) = 1$, then Eq. (15) can be simply rewritten as

$$TU_E^I(m) = \sum_{i=1}^{n} \left[1 - d_E^I\left([Bel(\theta_i), Pl(\theta_i)], [0,1]\right)\right]. \tag{17}$$

It has been shown that TU_E^I processes many desirable properties and has a good performance in measuring the uncertainty of mass functions compared with other uncertainty measures. Please refer to [25] for more details. Different from TU^I and SU, we find that TU_E^I meets the monotonicity. Here, a detailed proof on the monotonicity of TU_E^I is given as follows.

Property 1 (Monotonicity of TU_E^I). *Let m_1 and m_2 be two BBAs defined on $\Theta = \{\theta_1, \theta_2, \cdots, \theta_n\}$, and if $\forall A \subseteq \Theta : [Bel_{m_1}(A), Pl_{m_1}(A)] \subseteq [Bel_{m_2}(A), Pl_{m_2}(A)]$, then $TU_E^I(m_1) \leq TU_E^I(m_2)$.*

PROOF. Since $\forall A \subseteq \Theta : [Bel_{m_1}(A), Pl_{m_1}(A)] \subseteq [Bel_{m_2}(A), Pl_{m_2}(A)]$, then we have $\forall \theta_i \in \Theta : [Bel_{m_1}(\{\theta_i\}), Pl_{m_1}(\{\theta_i\})] \subseteq [Bel_{m_2}(\{\theta_i\}), Pl_{m_2}(\{\theta_i\})]$.

Because $0 \leq Bel_{m_2}(\{\theta_i\}) \leq Bel_{m_1}(\{\theta_i\}) \leq Pl_{m_1}(\{\theta_i\}) \leq Pl_{m_2}(\{\theta_i\}) \leq 1$, $\forall\theta_i \in \Theta$ we have

$$(Bel_{m_1}(\{\theta_i\}))^2 \geq (Bel_{m_2}(\{\theta_i\}))^2$$
$$(1 - Pl_{m_1}(\{\theta_i\}))^2 \geq (1 - Pl_{m_2}(\{\theta_i\}))^2$$

then

$$(Bel_{m_1}(\{\theta_i\}))^2 + (1 - Pl_{m_1}(\{\theta_i\}))^2 \geq (Bel_{m_2}(\{\theta_i\}))^2 + (1 - Pl_{m_2}(\{\theta_i\}))^2,$$

so

$$\sqrt{(Bel_{m_1}(\{\theta_i\}))^2 + (1 - Pl_{m_1}(\{\theta_i\}))^2} \geq \sqrt{(Bel_{m_2}(\{\theta_i\}))^2 + (1 - Pl_{m_2}(\{\theta_i\}))^2},$$

i.e.,

$$d_E^I\left([Bel_{m_1}(\theta_i), Pl_{m_1}(\theta_i)], [0,1]\right) \geq d_E^I\left([Bel_{m_2}(\theta_i), Pl_{m_2}(\theta_i)], [0,1]\right), \forall\theta_i \in \Theta.$$

Therefore,

$$1 - d_E^I\left([Bel_{m_1}(\theta_i), Pl_{m_1}(\theta_i)], [0,1]\right) \leq 1 - d_E^I\left([Bel_{m_2}(\theta_i), Pl_{m_2}(\theta_i)], [0,1]\right), \forall\theta_i \in \Theta.$$

and thus

$$\sum_{i=1}^{n}\left[1 - d_E^I([Bel_{m_1}(\{\theta_i\}), Pl_{m_1}(\{\theta_i\})], [0,1])\right] \leq \sum_{i=1}^{n}\left[1 - d_E^I([Bel_{m_2}(\{\theta_i\}), Pl_{m_2}(\{\theta_i\})], [0,1])\right],$$

i.e.,

$$TU_E^I(m_1) \leq TU_E^I(m_2).$$

Hence, the monotonicity of TU_E^I has been proved.

Now, let us numerically illustrate the monotonicity of TU_E^I in terms of Examples 1 and 2. For the three BBAs given in Example 1,

$$m_1(\{\theta_1\}) = 0.4, \quad m_1(\{\theta_2\}) = 0.3, \quad m_1(\{\theta_1, \theta_2\}) = 0.3;$$

$$m_2(\{\theta_1\}) = 0.4, \quad m_2(\{\theta_2\}) = 0.2, \quad m_2(\{\theta_1, \theta_2\}) = 0.4;$$

$$m_3(\{\theta_1\}) = 0.4, \quad m_3(\{\theta_2\}) = 0.0, \quad m_3(\{\theta_1, \theta_2\}) = 0.6.$$

According to the definition of TU_E^I, we have $TU_E^I(m_1) = 1.0$, $TU_E^I(m_2) = 1.1056$, $TU_E^I(m_3) = 1.2$, namely $TU_E^I(m_1) < TU_E^I(m_2) < TU_E^I(m_3)$. Therefore, the monotonicity of the total uncertainty among m_1, m_2, and m_3 is satisfied. For the two BBAs in Example 2,

$$m_1(\{\theta_2\}) = 0.9, \quad m_1(\{\theta_1, \theta_2\}) = 0.1;$$

$$m_2(\{\theta_1\}) = 0.1, \quad m_2(\{\theta_2\}) = 0.9.$$

It is calculated that $TU_E^I(m_1) = 0.2$, $TU_E^I(m_2) = 0.1889$, i.e. $TU_E^I(m_1) > TU_E^I(m_2)$. Therefore, the monotonicity is successfully satisfied in TU_E^I.

4.4. summary of the properties of belief interval based uncertainty measures

Now, let us summarize the properties of belief interval based uncertainty measures mentioned above. The results are shown in Table 1, which is very useful for selecting appropriate uncertainty measures in the practical applications.

5. Conclusion

In this study, we investigated several belief interval based uncertainty measures of belief functions including TU^I, SU and TU_E^I. Mainly, the monotonicity of these uncertainty measures has been analyzed. We have found counter examples which demonstrate that the monotonicity is isolated by TU^I and SU. In contrast, TU_E^I successfully satisfies the monotonicity through strict proof and numerical examples. In addition, other properties of these uncertainty measures have been summarized, which gives a useful

Table 1: Properties of belief interval based uncertainty measures

Properties	TU^I	SU	TU_E^I
Range	$[0, 1]$	$[0, \|\Theta\|]$	$[0, \|\Theta\|]$
Probability consistency	Unsatisfied	Satisfied	Unsatisfied
Monotonicity	Unsatisfied	Unsatisfied	Satisfied
BBA having largest uncertainty	$m(\Theta) = 1$	$m(\Theta) = 1$	$m(\Theta) = 1$
Generalized set consistency [26]	Satisfied	Satisfied	Satisfied
Invariance [25]	Unsatisfied	Satisfied	Satisfied
Completeness and non-concealment [25]	Complete, concealed	Complete, concealed	Complete, concealed
Computing complexity	Low	Low	Low

reference to choose an appropriate uncertainty measure in practical applications.

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