ON THE COSMOLOGICAL EVOLUTION

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Abstract: In this work, we discuss the possibility to investigate the cosmological evolution using a covariant Ricci flow. Even though we have focused on the evolution of large-scale structures, the results obtained can be applied equally to microscopic structures of quantum particles.

Based on observations from the observable universe, the cosmological evolution can be investigated by assuming the cosmological principle, the Weyl's postulate and Einstein's field equations of general relativity [1]. The cosmological principle states that at large scale the spatial component of the observable universe at any given cosmic time is homogeneous and isotropic. The Weyl's postulate requires that the geodesics of the fluid substance are orthogonal to a family of spacelike hypersurfaces and there is only one geodesic passing through each point of spacetime with a unique velocity, therefore the fluid substance is a perfect fluid. Einstein's field equations of general relativity are given as [2]

$$R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R-\Lambda g_{\alpha\beta}=\frac{8\pi G}{c^2}T_{\alpha\beta} \qquad (1)$$

With these assumptions, it is shown that the cosmological evolution can be described by the Robertson-Walker metric [3,4,5]

$$ds^2=(cdt)^2-a^2(t)\left(\frac{1}{1-\frac{r^2}{K^2}}(dr)^2+r^2(d\theta)^2+r^2\sin^2\theta(d\phi)^2\right) \qquad (2)$$

with the energy-momentum tensor of the form

$$T_{\alpha\beta}=(\rho+p)u_\alpha u_\beta-pg_{\alpha\beta} \qquad (3)$$

While Newton's theory of gravity uses only the gravitational potential $\phi(r)$ to describe the gravitational field via Poisson's equation $\nabla^2\phi(r)=4\pi G\rho$, the gravitational field in general relativity is described by the ten components of the metric tensor $g_{\alpha\beta}$. However, as in the case of the metric given by the line element in Equation (2), there is only one unknown function, which is $a(t)$, it seems more reasonable that the evolution should be formulated in terms of one equation. Furthermore, as shown in our previous works [6], a classical potential can be identified with the Ricci scalar, therefore the most appropriate equation would be an equation that involves only the Ricci scalar. By contracting Einstein's field equations with the inverse metric tensor $g^{\alpha\beta}$, we obtain

$$-R - 4\Lambda = \frac{8\pi G}{c^2}T \quad (4)$$

If we apply Einstein's field equations given by Equation (1) with the metric given by Equation (2) and the energy-momentum tensor given by Equation (3) then, from the results in the appendix 1, we obtain the following evolution equations

$$\frac{1}{a^2c^2}\left(\frac{da}{dt}\right)^2 + \frac{1}{K^2a^2} - \frac{\Lambda}{3} = \frac{8\pi G\rho}{3} \quad (5)$$

$$\frac{1}{ac^2}\frac{d^2a}{dt^2} + \frac{1}{2a^2c^2}\left(\frac{da}{dt}\right)^2 + \frac{1}{2K^2a^2} - \frac{\Lambda}{2} = -4\pi Gp \quad (6)$$

However, if we apply Equation (4) for the Ricci scalar with the metric given by Equation (2) and the energy-momentum tensor given by Equation (3) then we obtain the following single evolution equation

$$\frac{1}{ac^2}\frac{d^2a}{dt^2} + \frac{1}{a^2c^2}\left(\frac{da}{dt}\right)^2 + \frac{1}{K^2a^2} - \frac{2\Lambda}{3} = \frac{4\pi G}{3c^2}(\rho - 3p) \quad (7)$$

Now, as shown in our works on the Ricci flow [7,8], the cosmological evolution can be described by the evolution equation of the Ricci flow [9]. It was shown that by applying the Lie differentiation with respect to a vector field X^μ, we may propose the following tensor equation a covariant Ricci flow

$$L_X g_{\alpha\beta} = -\kappa R_{\alpha\beta} \quad (8)$$

where κ is a dimensional constant. With the Lie derivative of the metric tensor given by the relation $L_X g_{\alpha\beta} = X^\mu \partial_\mu g_{\alpha\beta} + g_{\mu\alpha}\partial_\beta X^\mu + g_{\mu\beta}\partial_\alpha X^\mu$, Equation (8) can also be written as

$$X^\mu \partial_\mu g_{\alpha\beta} + g_{\mu\alpha}\partial_\beta X^\mu + g_{\mu\beta}\partial_\alpha X^\mu = -\kappa R_{\alpha\beta} \quad (9)$$

Applying Weyl's postulate by introducing comoving synchronous coordinate systems, the covariant Ricci flow given by Equation (9) is reduced to the evolution equation

$$\frac{\partial g_{\alpha\beta}}{c\partial t} = kR_{\alpha\beta} \quad (10)$$

As discussed above, in the case when the metric depends only on one function, the cosmological evolution should be described by an evolution equation that involves the Ricci scalar. By contracting Equation (10), we obtain

$$g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{c\partial t} = kR \quad (11)$$

Using the results in the appendix 1 for the line element given in Equation (2), we arrive at the following evolution equation

$$\frac{d^2a}{dt^2} + \frac{1}{k}\frac{da}{dt} + \frac{1}{a}\left(\frac{da}{dt}\right)^2 + \frac{1}{K^2 a} = 0 \tag{12}$$

It is possible to speculate that the evolution of the universe may be more subtle than those that can be observed and restricted by the cosmological principle. As a generalised line element of the Robertson-Walker metric, we may assume a cosmological evolution with a line element of the form

$$ds^2 = D(cdt)^2 - A(x, y, z, t)((dx)^2 + (dy)^2 + (dx)^2) \tag{13}$$

where D is constant. Using the equation given by Equation (11) above and Equation (12) in the appendix 2, we obtain the following evolution equation

$$-\frac{3}{c^2 D}\frac{\partial^2 A}{\partial t^2} + \frac{2}{A}\nabla^2 A + \frac{3}{ck}\frac{\partial A}{\partial t} + \frac{3}{2A^2}(\boldsymbol{\nabla} A)^2 = 0 \tag{14}$$

As a further remark, we would like to mention here that even though our work has been focused on the evolution of large-scale structures, the results obtained can be applied equally to microscopic structures of quantum particles.

Appendix 1

With the line element given in Equation (2) we have

$$g_{00} = 1, \quad g_{11} = -\frac{a^2}{1 - \frac{r^2}{K^2}}, \quad g_{22} = -a^2 r^2, \quad g_{33} = -a^2 r^2 \sin^2\theta$$

$$g^{00} = 1, \quad g^{11} = -\frac{1 - \frac{r^2}{K^2}}{a^2}, \quad g^{22} = -\frac{1}{a^2 r^2}, \quad g^{33} = -\frac{1}{a^2 r^2 \sin^2\theta}$$

$$g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{c\partial t} = \frac{6}{ca}\frac{da}{dt} \tag{1}$$

Using the affine connection defined in terms of the metric tensor

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right) \tag{2}$$

the non-zero components of the affine connection are found as [4,5]

$$\Gamma^0_{11} = \frac{a}{1-\frac{r^2}{K^2}}\frac{da}{cdt}, \qquad \Gamma^0_{22} = r^2 a\frac{da}{cdt}, \qquad \Gamma^0_{33} = r^2\sin^2\theta a\frac{da}{cdt}$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{a}\frac{da}{cdt}$$

$$\Gamma^1_{11} = \frac{r}{K^2\left(1-\frac{r^2}{K^2}\right)}, \quad \Gamma^1_{22} = -r\left(1-\frac{r^2}{K^2}\right), \quad \Gamma^1_{33} = -r\left(1-\frac{r^2}{K^2}\right)\sin^2\theta$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^0_{11} = -\sin\theta\cos\theta, \qquad \Gamma^0_{11} = \Gamma^0_{11} = \frac{\cos\theta}{\sin\theta} \tag{3}$$

With the components of the affine connection given in Equation (3), using the Ricci curvature tensor defined in terms of the affine connection

$$R_{\mu\nu} = \frac{\partial\Gamma^\sigma_{\mu\nu}}{\partial x^\sigma} - \frac{\partial\Gamma^\sigma_{\mu\sigma}}{\partial x^\nu} + \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\lambda\nu} \tag{4}$$

we obtain the following non-zero components

$$R_{11} = \left(\frac{1}{1-\frac{r^2}{K^2}}\right)\left(\frac{a}{c^2}\frac{d^2a}{dt^2} + \frac{2}{c^2}\left(\frac{da}{dt}\right)^2 + \frac{2}{K^2}\right)$$

$$R_{22} = \frac{r^2 a}{c^2}\frac{d^2a}{dt^2} + \frac{2r^2}{c^2}\left(\frac{da}{dt}\right)^2 + \frac{2r^2}{K^2}$$

$$R_{33} = \frac{r^2\sin^2\theta a}{c^2}\frac{d^2a}{dt^2} + \frac{2r^2\sin^2\theta}{c^2}\left(\frac{da}{dt}\right)^2 + \frac{2r^2\sin^2\theta}{K^2}$$

$$R_{00} = -\frac{3}{ac^2}\frac{d^2a}{dt^2} \tag{5}$$

The Ricci scalar is obtained as

$$R = g^{\alpha\beta}R_{\alpha\beta} = -\frac{6}{ac^2}\frac{d^2a}{dt^2} - \frac{6}{a^2c^2}\left(\frac{da}{dt}\right)^2 - \frac{6}{K^2a^2} \tag{6}$$

Appendix 2

With the line element given in Equations (13), we have

$$g_{00} = D, \quad g_{ii} = -A \quad \text{where} \quad i = 1,2,3 \tag{7}$$

$$g^{00} = \frac{1}{D}, \quad g^{ii} = -\frac{1}{A} \quad \text{where} \quad i = 1,2,3 \tag{8}$$

$$g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{c\partial t}=\frac{3}{A}\frac{\partial A}{c\partial t} \tag{9}$$

The non-zero components of the affine connection are

$$\Gamma^1_{01}=\Gamma^1_{10}=\frac{1}{2cA}\frac{\partial A}{\partial t},\qquad \Gamma^2_{02}=\Gamma^2_{20}=\frac{1}{2cA}\frac{\partial A}{\partial t},\qquad \Gamma^3_{03}=\Gamma^3_{30}=\frac{1}{2cA}\frac{\partial A}{\partial t}$$

$$\Gamma^0_{11}=\frac{1}{2cD}\frac{\partial A}{\partial t},\qquad \Gamma^1_{11}=\frac{1}{2A}\frac{\partial A}{\partial x},\qquad \Gamma^2_{11}=-\frac{1}{2A}\frac{\partial A}{\partial y},\qquad \Gamma^3_{11}=-\frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^1_{12}=\Gamma^1_{21}=\frac{1}{2A}\frac{\partial A}{\partial y},\quad \Gamma^2_{12}=\Gamma^2_{21}=\frac{1}{2A}\frac{\partial A}{\partial x},\quad \Gamma^1_{13}=\Gamma^1_{31}=\frac{1}{2A}\frac{\partial A}{\partial z},\quad \Gamma^3_{13}=\Gamma^3_{31}=\frac{1}{2A}\frac{\partial A}{\partial x}$$

$$\Gamma^0_{22}=\frac{1}{2cD}\frac{\partial A}{\partial t},\qquad \Gamma^1_{22}=\frac{1}{2A}\frac{\partial A}{\partial x},\qquad \Gamma^2_{22}=\frac{1}{2A}\frac{\partial A}{\partial y},\qquad \Gamma^3_{22}=-\frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^0_{33}=\frac{1}{2cD}\frac{\partial A}{\partial t},\qquad \Gamma^1_{33}=-\frac{1}{2A}\frac{\partial A}{\partial x},\qquad \Gamma^2_{33}=-\frac{1}{2A}\frac{\partial A}{\partial y},\qquad \Gamma^3_{33}=\frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^2_{23}=\Gamma^2_{32}=\frac{1}{2A}\frac{\partial A}{\partial z},\qquad \Gamma^3_{23}=\Gamma^3_{32}=\frac{1}{2A}\frac{\partial A}{\partial y} \tag{10}$$

The non-zero components of the Ricci curvature tensor are

$$R_{11}=\frac{1}{2c^2D}\frac{\partial^2 A}{\partial t^2}-\frac{1}{A}\frac{\partial^2 A}{\partial x^2}-\frac{1}{2A}\frac{\partial^2 A}{\partial y^2}-\frac{1}{2A}\frac{\partial^2 A}{\partial z^2}+\frac{3}{4c^2AD}\left(\frac{\partial A}{\partial t}\right)^2+\frac{1}{A^2}\left(\frac{\partial A}{\partial x}\right)^2+\frac{1}{4A^2}\left(\frac{\partial A}{\partial y}\right)^2+\frac{1}{4A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{22}=\frac{1}{2c^2D}\frac{\partial^2 A}{\partial t^2}-\frac{1}{2A}\frac{\partial^2 A}{\partial x^2}-\frac{1}{A}\frac{\partial^2 A}{\partial y^2}-\frac{1}{2A}\frac{\partial^2 A}{\partial z^2}+\frac{3}{4c^2AD}\left(\frac{\partial A}{\partial t}\right)^2+\frac{1}{4A^2}\left(\frac{\partial A}{\partial x}\right)^2+\frac{1}{A^2}\left(\frac{\partial A}{\partial y}\right)^2+\frac{1}{4A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{33}=\frac{1}{2c^2D}\frac{\partial^2 A}{\partial t^2}-\frac{1}{2A}\frac{\partial^2 A}{\partial x^2}-\frac{1}{2A}\frac{\partial^2 A}{\partial y^2}-\frac{1}{A}\frac{\partial^2 A}{\partial z^2}+\frac{3}{4c^2AD}\left(\frac{\partial A}{\partial t}\right)^2+\frac{1}{4A^2}\left(\frac{\partial A}{\partial x}\right)^2+\frac{1}{4A^2}\left(\frac{\partial A}{\partial y}\right)^2+\frac{1}{A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{00}=-\frac{3}{2c^2A}\frac{\partial^2 A}{\partial t^2}+\frac{3}{4c^2A^2}\left(\frac{\partial A}{\partial t}\right)^2 \tag{11}$$

Using the relation $R=g^{00}R_{00}+g^{11}R_{11}+g^{22}R_{22}+g^{33}R_{33}$ we obtain

$$R=-\frac{3}{c^2DA}\frac{\partial^2 A}{\partial t^2}+\frac{2}{A^2}\nabla^2 A+\frac{3}{2A^3}(\boldsymbol{\nabla} A)^2 \tag{12}$$

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