Was Polchinski wrong? Colombeau disributional Rindler spacetime with disributional Levi-Cività connection induced a strong vacuum dominance. Unruh effect revisited.

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Abstract:The vacuum energy density of free scalar quantum field Φ in a Rindler distributional spacetime with distributional Levi-Cività connection is considered.It has been widely believed that, except in very extreme situations, the influence of acceleration on quantum fields should amount to just small, sub-dominant contributions. Here we argue that this belief is wrong by showing that in a Rindler distributional background spacetime with distributional Levi-Cività connection the vacuum energy of free quantum fields is forced, by the very same background distributional spacetime such a Rindler distributional background spacetime, to become dominant over any classical energy density component. This semiclassical gravity effect finds its roots in the singular behavior of quantum fields on a Rindler distributional spacetimes with distributional Levi-Cività connection. In particular we obtain that the vacuum fluctuations $\langle\Phi^2\rangle$ has a singular behavior at a Rindler horizon $\delta_+ = 0 : \langle\Phi^2(\delta)\rangle \sim \delta^{-4}, \delta \approx c^2/a, a \to \infty$.Therefore sufficiently strongly accelerated observer burns up near the Rindler horizon. Thus Polchinski's account doesn't violation of the Einstein equivalence principle.

Key words.vacuum energy density; Rindler distributional spacetime; Levi-Cività connection; semiclassical gravity effect; Einstein equivalence principle spacetime; Levi-Cività connection; semiclassical gravity effect; Einstein equivalence principle.

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1.Introduction

In March 2012, Joseph Polchinski claimed that the following three statements cannot all be true : (i) Hawking radiation is in a pure state, (ii) the information carried by the radiation is emitted from the region near the horizon, with low energy effective field theory valid beyond some microscopic distance from the horizon, (iii) the infalling observer encounters nothing unusual at the horizon. Joseph Polchinski argue that the most conservative resolution is that: the infalling observer burns up at the horizon. In Polchinski's account, quantum effects would turn the event horizon into a seething maelstrom of particles. Anyone who fell into it would hit a wall of fire and be burned to a crisp in an instant. As pointed out by physics community such firewalls would violate a foundational tenet of contemporary physics known as the equivalence principle, it states in part that an observer falling in a gravitational field — even the powerful one inside a black hole — will see exactly the same phenomena as an accelerated observer floating in empty space.

In this paper we argue that Polchinski not was wrong, but Unruh effect revision is needed.

1.1.What is Colombeau distributional semi-Riemannian geometry?

Recall that the classical Cartan's structural equations show in a compact way the relation between a connection and its curvature, and reveals their geometric interpretation in terms of moving frames. In order to study the mathematical properties of singularities, we need to study the geometry of manifolds endowed on the tangent bundle with a symmetric bilinear form which is allowed to become degenerate (singular).

Remark 1.**1**.**1**.But if the fundamental tensor is allowed to be degenerate (singular), there are some obstructions in constructing the geometric objects normally associated to the fundamental tensor. Also,local orthonormal frames and coframes no longer exist, as well as the metric connection and its curvature operator.

Remark 1.**1**.**2**."Singular Semi-Riemannian Geometry"- the main brunch of contemporary semi-Riemannian geometry in which have been studied a smooth manifolds M furnished with a degenerate (singular) on a smooth submanifold $M' \subsetneqq M$ metric tensor of arbitrary signature [1].

Remark 1.1.3. In order to solve problems of the gravitational singularity in classical general relativity the singular semi-Riemannian geometry based on Colombeau calculas and Colombeau generalized functions was many developed,see [2]-[23].

Remark 1.1.4. Let $\mathbf{G}(M')$ be algebra of Colombeau generalized functions on $M' \subset M$,let $\widetilde{\mathbb{R}}$ be the ring of Colombeau generalized numbers [2]-[5]. Let $(g_\varepsilon)_\varepsilon$ be Colombeau generalized metric tensor on M and let $\mathbf{Ric}_{M'}(p)$ be generalized Ricci tensor of the metric $(g_\varepsilon(p))_\varepsilon|_{M'}$ [21]-[22].The main properties of such nonclassical manifolds with a degenerate (singular) metric tensor that is $\mathbf{Ric}_{M'}(p) \in \mathbf{G}(M')\backslash C^\infty(M')$, i.e. for all $p \in M' : \mathbf{Ric}_{M'}(p) \in \widetilde{\mathbb{R}}\backslash\mathbb{R}$.

Definition 1.1.1. . Let $\mathbf{G}(M')$ be algebra of Colombeau generalized functions on $M' \subset M,$ and let $(g_\varepsilon)_\varepsilon$ be Colombeau generalized metric tensor on M such that $(g_\varepsilon)_\varepsilon$ is the Colombeau solution of the Einstein field equations,(see Remark 1.3.5). We define now the Colombeau distributional scalar curvature $\mathbf{R}_M(p)$ (or distributional Ricci [21]-[22] scalar) as the trace of $\mathbf{Ric}_M(p) : \mathbf{R}_M(p) = \mathbf{tr}(\mathbf{Ric}_M(p))$. Assume that $\mathbf{R}_{M'}(p) \in \mathbf{G}(M')\backslash C^\infty(M')$.

Then we say that: (i) gravitational field $(g_\varepsilon)_\varepsilon$ (or corresponding distributional spacetime) has a gravitational singularity on a smooth compact submanifold $M_c \subset M$ iff $\mathbf{R}_{M_c}(p) \in \mathbf{G}(M_c)\backslash C^\infty(M_c)$; (ii) gravitational field $(g_\varepsilon)_\varepsilon$ has a gravitational singularity with compact support iff $\mathbf{R}_{M_c}(p) \in D'(\mathbb{R}^3)$.

Remark 1.1.4. It turns out that the distributional Schwarzschild spacetime has a gravitational singularity with compact support at origin $\{r = 0\}$ [6]-[11] and at Schwarzschild horizon $\mathbf{S}^2 \times \{r = 2m\}$ [18]-[19].

1.2.Distributional Møller's geometry as Colombeau extension of the classical Møller's spacetime.

As importent example of Colombeau extension of the singular semi-Riemannian geometry mentioned above, we consider now Møller's uniformly accelerated frame given by Møller's line element [24]:

$$ds^2 = -(a+gx)^2dt^2 + dx^2 + dy^2 + dz^2. \tag{1.2.1}$$

Of couse Møller's metric (1.2.1) degenerate at Møller horizon $x_{hor}^{M\phi l} = -(a/g)^{-1}$. Note that formally corresponding to the metric (1.2.1) classical Levi-Civit'a connection is [1]

$$\Gamma^1_{44}(x) = (a+gx), \Gamma^4_{14}(x) = \Gamma^4_{41}(x) = \frac{1}{a+gx} \tag{1.2.2}$$

and therefore classical Levi-Civit'a connection (1.2.2) of couse is not available at

Møller horizon $x_{hor}^{Møl} = -a \cdot g^{-1}$. Recall that fundamental tensor corresponding to the metric (1.2.1) were obtained in Møller's paper [24] as a vacuum solution of the classical Einstein's field equations

$$G_i^k = R_i^k - \frac{1}{2}\delta_i^k R = 0, \tag{1.2.3}$$

where R_i^k is the contracted Riemann-Christoffel tensor formally calculated by canonical way by using classical Levi-Civit'a connection (1.2.2) and $R = R_i^i$. Using Dingle's formula [24] in case of the metric (1.2.1) we get

$$\begin{gathered} G_2^2(x) = G_3^3(x) = -\frac{1}{2\Delta(x)}\left\{\Delta''(x) - \frac{[\Delta'(x)]^2}{2\Delta(x)}\right\}, \\ \Delta(x) = (a+gx)^2, \end{gathered} \tag{1.2.4}$$

where $\Delta'(x) = \partial\Delta(x)/\partial x$ and all ather components of G_i^k vanishes identically. Note that

$$\Delta'(x) = 2g(a+gx), \Delta''(x) = 2g^2. \tag{1.2.5}$$

Thus for any $x \neq -a \cdot g^{-1}$ we get a classical result

$$G_2^2(x) = G_3^3(x) = -\frac{1}{2\Delta(x)}\left\{2g^2 - \frac{4g^2(a+gx)^2}{2\Delta(x)}\right\} \equiv 0. \tag{1.2.6}$$

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} x_n = -a \cdot g^{-1}, x_n \neq -a \cdot g^{-1}, n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ we get

$$\Im(x_n) = G_2^2(x_n) = G_3^3(x_n) = -\frac{1}{2\Delta(x_n)}\left\{2g^2 - \frac{4g^2(a+gx_n)^2}{2\Delta(x_n)}\right\} \equiv 0, \tag{1.2.7}$$

and therefore $\lim_{n\to\infty} \Im(x_n) \equiv 0$. However

$$\lim_{n\to\infty} \Gamma_{14}^4(x_n) = \lim_{n\to\infty} \Gamma_{41}^4(x_n) = \lim_{n\to\infty} \frac{1}{a+gx_n} = \infty, \tag{1.2.8}$$

i.e. classical Levi-Civit'a connection given by (1.2.2) unavaluble at Møller horizon.

Remark 1.2.1. In order to avoid difficultness which mentioned above, we consider now

the regularized Møller's metric

$$\begin{gathered} d_\varepsilon s^2 = -\Delta_\varepsilon(x)dt^2 + dx^2 + dy^2 + dz^2, \\ \Delta_\varepsilon(x) = \left[(a+gx)^2 + \varepsilon^2\right], \varepsilon \in (0,1]. \end{gathered} \tag{1.2.9}$$

Using now Dingle's formula [24] in case of (1.2.9) we get

$$\begin{gathered} \Im(x;\varepsilon) = G_2^2(x;\varepsilon) = G_3^3(x;\varepsilon) = -\frac{1}{2\Delta_\varepsilon(x)}\left\{\Delta_\varepsilon''(x) - \frac{[\Delta_\varepsilon'(x)]^2}{2\Delta_\varepsilon(x)}\right\}, \\ \Delta_\varepsilon(x) = \left[(a+gx)^2 + \varepsilon^2\right]. \end{gathered} \tag{1.2.10}$$

Note that

$$\Delta_\varepsilon' = 2g(1+gx), \Delta_\varepsilon'' = 2g^2 \tag{1.2.11}$$

and therefore

$$\Im(x;\varepsilon) = -\frac{1}{2\Delta_\varepsilon(x)}\left\{2g^2 - \frac{2g^2(a+gx)^2}{\Delta_\varepsilon(x)}\right\} =$$
$$-\frac{1}{2\Delta_\varepsilon(x)}\left\{2g^2 - \frac{2g^2\left[(a+gx)^2+\varepsilon^2\right]-2g^2\varepsilon^2}{\Delta_\varepsilon(x)}\right\} = \tag{1.2.12}$$
$$= -\frac{g^2\varepsilon^2}{\Delta_\varepsilon^2(x)}.$$

Remark 1.2.2. Note that $(\Im(x;\varepsilon))_\varepsilon, \varepsilon \in (0,1]$ is Colombeau generalized function such that

$\mathbf{cl}[(\Im(x;\varepsilon))_\varepsilon] \in \mathbf{G}(\mathbb{R})$ and $\mathbf{cl}[(\Im(-g^{-1};\varepsilon))_\varepsilon] = \mathbf{cl}[(\varepsilon^{-2})_\varepsilon] \in \widetilde{\mathbb{R}}$.

Remark 1.2.3. Note that $\mathbf{cl}[(\Im(x;\varepsilon))_\varepsilon] \sim \dfrac{\delta(a+gx)}{a+gx} \notin D'(\mathbb{R})$.

Remark 1.2.4. Thus Colombeau generalized fundamental tensor $(g_{ik}(\varepsilon))_\varepsilon$ corresponding to Colombeau metric

$$(d_\varepsilon s^2) = -(\Delta_\varepsilon(x)dt^2)_\varepsilon + dx^2 + dy^2 + dz^2,$$
$$(\Delta_\varepsilon(x))_\varepsilon = \left(\left[(a+gx)^2+\varepsilon^2\right]\right)_\varepsilon, \varepsilon \in (0,1] \tag{1.2.13}$$

that is non vacuum Colombeau solution (see [18] section 6 and [19] subsection 2.3 Distributional general relativity) of the Einstein's field equations

$$(G_i^k(\varepsilon))_\varepsilon = (R_i^k(\varepsilon))_\varepsilon - \frac{1}{2}\delta_i^k(R(\varepsilon))_\varepsilon = -g^2\left(\frac{\varepsilon^2}{\Delta_\varepsilon^2(x)}\right)_\varepsilon. \tag{1.2.14}$$

For Rindler metric $a = 0, g = 1$ and we get

$$(G_i^k(\varepsilon))_\varepsilon = (R_i^k(\varepsilon))_\varepsilon - \frac{1}{2}\delta_i^k(R(\varepsilon))_\varepsilon = -\left(\frac{\varepsilon^2}{(x^2+\varepsilon^2)^2}\right)_\varepsilon \in \mathbf{G}(\mathbb{R}). \tag{1.2.15}$$

Definition 1.2.1. Distributional Møller's geometry that is Colombeau extension of the classical Møller's spacetime given by Colombeau generalized fundamental tensor (1.2.13).

1.3. Distributional Schwarzschild geometry as Colombeau extension of the classical singular Schwarzschild spacetime.

As another importent example of Colombeau extension of the singular semi-Riemannian geometry we consider now classical singular Schwarzschild spacetime given by degenerate and singular Schwarzschild metric

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2d\Omega^2 \tag{1.3.1}$$

Remark 1.3.1.Note that formally corresponding to the metric (1.3.1) classical Levi-Civit‘a connection given by canonical Christoffel symbols are [25]:

$$\begin{gathered}\Gamma^1_{00}(r)|_{r=2m} = \lim_{r\to 2m} \frac{m(r-2m)}{r^3} = 0, \Gamma^1_{11}(r)|_{r=2m} = \lim_{r\to 2m} \frac{-m}{r(r-2m)} = \infty,\\ \Gamma^0_{01}(r)|_{r=2m} = \lim_{r\to 2m} \frac{m}{r(r-2m)} = \infty,\\ \Gamma^2_{12}(r)|_{r=2m} = \lim_{r\to 2m} \frac{1}{r} = 2^{-1}m^{-1}, \Gamma^1_{22}|_{r=2m} = -\lim_{r\to 2m}(r-2m) = 0,\\ \Gamma^3_{13}|_{r=2m} = \lim_{r\to 2m} \frac{1}{r} = 2^{-1}m^{-1}, \Gamma^1_{33}|_{r=2m} = -\lim_{r\to 2m}(r-2m)\sin^2\theta = 0,\\ \Gamma^1_{00}(r)|_{r=0} = \lim_{r\to 0} \frac{m(r-2m)}{r^3} = \infty \Gamma^1_{11}(r)|_{r=0} = \lim_{r\to 0} \frac{-m}{r(r-2m)} = \infty,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \Gamma^2_{33} = -\sin\theta\cos\theta, \Gamma^3_{23} = \frac{\cos\theta}{\sin\theta}.\end{gathered} \tag{1.3.2}$$

i.e. classical Levi-Civita connection given by Eq.(1.3.2) unavaluble at Schwarzschild horizon.

Remark 1.3.2.Newertheles in classical handbooks ([25-30]) mistakenly assumed that classical semi-Riemannian geometry holds on whole Schwarzschild manifold and therefore canonical formal calculation gives

$$R_{abcd}(r)R^{abcd}(r) = \frac{16m^2}{r^6}. \tag{1.3.3}$$

By Eq.(1.3.2) mistakenly pointed out that the Schwarzschild metric has only a coordinate singularity at $r = 2m$ and there is no gravitational singularity at Schwarzschild horizon.

Remark 1.3.2.Note that canonical formal calculation gives

$$\begin{gathered}R_{abcd}(r)R^{abcd}(r) = \frac{16m^2}{r^6} + 4\left[-\frac{m}{r\left(1-\frac{2m}{r}\right)}\right]\left[-\frac{m}{r^5}\left(1-\frac{2m}{r}\right)\right]\\ +4\left[-\frac{m}{r\left(1-\frac{2m}{r}\right)}\sin^2\theta\right]\left[-\frac{m}{r^5\sin^2\theta}\left(1-\frac{2m}{r}\right)\right]+\\ \frac{4m}{r}\left(1-\frac{2m}{r}\right)\frac{m}{r^5\left(1-\frac{2m}{r}\right)} + 8mr\sin^2\theta\frac{2m}{r^7\sin^2\theta}+\\ \frac{4m}{r}\left(1-\frac{2m}{r}\right)\sin^2\theta\frac{m}{r^5\sin^2\theta\left(1-\frac{2m}{r}\right)}\end{gathered} \tag{1.3.4}$$

Assume that $r \neq 0$ and $r \neq 2m$, i.e. $1 - \frac{2m}{r} \neq 0, \sin^2\theta \neq 0,$ then from Eq.(1.3.4) one obtains
directly

$$R_{abcd}(r)R^{abcd}(r) = \frac{16m^2}{r^6} + 4\left[-\frac{m}{r}\right]\left[-\frac{m}{r^5}\right]$$
$$+4\left[-\frac{m}{r}\right]\left[-\frac{m}{r^5}\right] + \frac{4m^2}{r^6} + \frac{16m^2}{r^6} + \frac{4m^2}{r^6} = \frac{48m^2}{r^6} = \frac{12r_s^2}{r^6}. \quad (1.3.5)$$

Remark 1.3.3. Notice that: if $r = 2m$ then RHS of the Eq.(1.3.4) become uncertainty

$$R_{abcd}(r)R^{abcd}(r) = \frac{16m^2}{r^6} + 4\left[-\frac{m}{r0}\right]\left[-\frac{m}{r^5}0\right] + 4\left[-\frac{m}{r0}\sin^2\theta\right]\left[-\frac{m}{r^5\sin^2\theta}0\right] +$$
$$\frac{4m}{r}0\frac{m}{r^5 0} + 8mr\sin^2\theta\frac{2m}{r^7\sin^2\theta} + \frac{4m}{r}0\sin^2\theta\frac{m}{(r^5\sin^2\theta)0} = \frac{16m^2}{r^6} + \frac{0}{0}. \quad (1.3.6)$$

In order to avoid this difficultness mentioned above one defines $R_{abcd}(r)R^{abcd}(r)$ at $r = 2m$ by the limit

$$\lim_{r\to 2m} R_{abcd}(r)R^{abcd}(r) = \lim_{r\to 2m}\left\{\frac{16m^2}{r^6} + 4\left[-\frac{m}{r\left(1-\frac{2m}{r}\right)}\right]\left[-\frac{m}{r^5}\left(1-\frac{2m}{r}\right)\right]\right.$$
$$+4\left[-\frac{m}{r\left(1-\frac{2m}{r}\right)}\sin^2\theta\right]\left[-\frac{m}{r^5\sin^2\theta}\left(1-\frac{2m}{r}\right)\right] +$$
$$\frac{4m}{r}\left(1-\frac{2m}{r}\right)\frac{m}{r^5\left(1-\frac{2m}{r}\right)} + 8mr\sin^2\theta\frac{2m}{r^7\sin^2\theta} + \quad (1.3.7)$$
$$\left.\frac{4m}{r}\left(1-\frac{2m}{r}\right)\sin^2\theta\frac{m}{r^5\sin^2\theta\left(1-\frac{2m}{r}\right)}\right\} = \frac{48m^2}{r^6}$$

However Eq.(1.3.7) doesn’t holds becouse classical Levi-Civit‘a connection (1.3.2) of couse is not available at Schwarzschild horizon, see Remark 1.3.1.

Remark 1.3.4. Thus from Eq.(1.3.4) for $r \neq 0$ and $r \neq 2m$ we get

$$R_{abcd}(r)R^{abcd}(r) = \frac{16m^2}{r^6} \Leftrightarrow (r \neq 0) \wedge (r \neq 2m), \quad (1.3.8)$$

and we get nothing at Schwarzschild horizon. Therefore semi-Riemannian geometry break
down at Schwarzschild horizon [18]-[19].

Remark 1.3.5. Recall that canonical derivation of the canonical singular Schwarzschild

metric in classical handbooks alwais based on assumption that:
Assumption 1.3.1.Classical semi-Riemannian geometry holds on whole semi-Riemannian manifold, see for example [27].
Let ds^2 be the metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2, \tag{1.3.9}$$

where $A,B \to 1$ as $r \to \infty$. Then under Assumption 1.3.1 one obtains [27]:

(i) all $\Gamma^1_{\mu\nu}$ are zero except

$$\Gamma^1_{00} = A'/2B, \Gamma^1_{11} = B'/2B, \Gamma^1_{22} = -r/B, \Gamma^1_{33} = -(r/B)\sin^2\theta, \tag{1.3.10}$$

The equations $R_{\mu\nu} = 0, \mu,\nu = 0,1,2,3$ are

$$\begin{gathered} R_{00} = \Gamma^1_{00,1} - 2\Gamma^1_{00}\Gamma^0_{01} + \Gamma^1_{00}\left(\log\sqrt{-g}\right)_{,1} = \\ \left(\frac{A'}{2B}\right)' - \frac{A'^2}{2AB} + \frac{A'}{2B}\left(\frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r}\right) = \\ \frac{1}{2B}\left(A'' - \frac{A'B'}{2B} - \frac{A'^2}{2A} + \frac{2A'}{r}\right) = 0, \end{gathered} \tag{1.3.11}$$

and

$$\begin{gathered} R_{11} = -\left(\log\sqrt{-g}\right)_{,1,1} + \Gamma^1_{11,1} - (\Gamma^0_{10})^2 - (\Gamma^1_{11})^2 - (\Gamma^2_{21})^2 - \\ -(\Gamma^3_{31})^2 + \Gamma^1_{11}\left(\log\sqrt{-g}\right)_{,1} = \\ \frac{1}{2A}\left(-A'' + \frac{A'B'}{2B} + \frac{A'^2}{2A} + \frac{2AB'}{rB}\right) = 0, \end{gathered} \tag{1.3.12}$$

and

$$\begin{gathered} R_{22} = -\left(\log\sqrt{-g}\right)_{,2,2} + \Gamma^1_{22,1} - 2\Gamma^1_{22}\Gamma^2_{21} - (\Gamma^3_{23})^2 + \Gamma^1_{22}\left(\log\sqrt{-g}\right)_{,1} = \\ -\frac{d\cot\theta}{d\theta} - \left(\frac{r}{B}\right)' + \frac{2}{B} - \cot^2\theta - \frac{r}{B}\left(\frac{2}{r} + \frac{(AB)'}{2AB}\right) = 0. \end{gathered} \tag{1.3.13}$$

From Eq.(1.3.11)-Eq.(1.3.12) one obtains

$$\frac{2(AB)'}{rB} = 0. \tag{1.3.14}$$

Therefore $AB =$ constant. Since at $r \to \infty$ we have A and $B \to 1$ one obtains $B = A^{-1}$.
From Eq.(1.3.14) one obtains

$$\left(\frac{r}{B}\right)' = 1, \tag{1.3.15}$$

and by integration one obtains $r/B = r - 2m$, where $2m$ is an integration constant.
Finally one obtains well known classical result

$$A(r) = 1 - \frac{2m}{r}, B(r) = \left(1 - \frac{2m}{r}\right)^{-1}. \quad (1.3.16)$$

From Eq.(1.3.16) and consideration above (see Remark1.3.4) Assumption 1.3.1 wrong,

otherwise one obtains the contradiction.

Remark 1.3.5. In order avoid this difficultnes:

(i) we have introduced instead a classical Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi GT_{\mu\nu}, \quad (1.3.17)$$

[where the sign of the energy-momentum tensor is defined by (ρ is the energy density)]

$$T_{44} = -T_{00} = T_0^0 = \rho, \quad (1.3.18)$$

apropriate Colombeau generalization of the Eq.(1.3.17)-Eq.(1.3.18) such that

$$(R_{\mu\nu}(\varepsilon))_\varepsilon - \frac{1}{2}(R(\varepsilon)g_{\mu\nu}(\varepsilon))_\varepsilon = -8\pi G(T_{\mu\nu}(\varepsilon))_\varepsilon, \quad (1.3.19)$$

where the sign of the distributional energy-momentum tensor is defined by

$$T_{44}(\varepsilon) = -T_{00}(\varepsilon) = T_0^0(\varepsilon) = \rho(\varepsilon) \in \mathbf{G}(M), \quad (1.3.20)$$

see [18]-[19].

(ii) we have introduced instead Assumption 1.3.1 the following assumption.

Assumption 1.3.2. Distributional semi-Riemannian geometry holds on whole distributional semi-Riemannian manifold.

Definition 1.3.1. Let $A_\varepsilon^\pm(r), \varepsilon \in [0,1]$ and $B_\varepsilon^\pm(r), \varepsilon \in [0,1]$ the regularization of the functions $A^\pm(r)$ and $B^\pm(r)$ [defined above by Eq.(1.3.16)] such that the followin conditions

are satisfied:

(i) $(A_\varepsilon^\pm(r))_\varepsilon \in \mathbf{G}(\mathbb{R}_+)$ and $(B_\varepsilon^\pm(r))_\varepsilon \in \mathbf{G}(\mathbb{R}_+), \varepsilon \in (0,1]$ are Colombeau generalizad functions;

(ii)

$$A_0^\pm(r) = 1 - \frac{2m}{r}, B_0^\pm(r) = \left(1 - \frac{2m}{r}\right)^{-1}; \quad (1.3.21)$$

(iii) $(A_\varepsilon^\pm(2m))_\varepsilon = \mp(\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}, (B_\varepsilon^\pm(2m))_\varepsilon = \pm(\varepsilon^{-1})_\varepsilon \in \widetilde{\mathbb{R}};$

(iv) $(A_\varepsilon^\pm(0))_\varepsilon = 1 - \dfrac{2m}{(\varepsilon)_\varepsilon} \in \widetilde{\mathbb{R}}, (B_\varepsilon^\pm(0))_\varepsilon = \left(1 - \dfrac{2m}{(\varepsilon)_\varepsilon}\right)^{-1} \in \widetilde{\mathbb{R}}.$

Let ds_ε^2 be the Colombeau metric

$$(ds_\varepsilon^2)_\varepsilon = -(A_\varepsilon^\pm(r)dt^2)_\varepsilon + (B_\varepsilon^\pm(r)dr^2)_\varepsilon + r^2 d\Omega^2, \quad (1.3.22)$$

and let $(\Gamma_{\mu\nu}^\eta(\varepsilon))_\varepsilon$ be the distributional Levi-Civita connection [18]-[19] corresponding to

Colombeau metric (1.3.22). Then under Assumption 1.3.2 one obtains:

(i) all $(\Gamma_{\mu\nu}^1(\varepsilon))_\varepsilon$ are zero except

$$(\Gamma^1_{00}(\varepsilon))_\varepsilon = (A'_\varepsilon)_\varepsilon/2(B_\varepsilon)_\varepsilon, (\Gamma^1_{11}(\varepsilon))_\varepsilon = (B'_\varepsilon)_\varepsilon/2(B_\varepsilon)_\varepsilon,$$
$$(\Gamma^1_{22}(\varepsilon))_\varepsilon = -r/(B_\varepsilon), (\Gamma^1_{33}(\varepsilon))_\varepsilon = -(r/B_\varepsilon)_\varepsilon \sin^2\theta, \quad (1.3.23)$$

$$\begin{gathered}(R_{00}(\varepsilon))_\varepsilon = (\Gamma^1_{00,1}(\varepsilon))_\varepsilon - 2(\Gamma^1_{00}(\varepsilon))_\varepsilon(\Gamma^0_{01}(\varepsilon))_\varepsilon + \\ (\Gamma^1_{00}(\varepsilon))_\varepsilon\left(\log\sqrt{-(g^{\pm}(\varepsilon))_\varepsilon}\right)_{,1} = \\ \left(\frac{(A'_\varepsilon)_\varepsilon}{2(B_\varepsilon)_\varepsilon}\right)' - \frac{(A'^2_\varepsilon)_\varepsilon}{2(A_\varepsilon)_\varepsilon(B_\varepsilon)_\varepsilon} + \frac{(A'_\varepsilon)_\varepsilon}{2(B_\varepsilon)_\varepsilon}\left(\frac{(A'_\varepsilon)_\varepsilon}{2(A_\varepsilon)_\varepsilon} + \frac{(B'_\varepsilon)_\varepsilon}{2(B_\varepsilon)_\varepsilon} + \frac{2}{r}\right) = \\ \frac{1}{2(B_\varepsilon)_\varepsilon}\left((A'') - \frac{(A')_\varepsilon(B')_\varepsilon}{2(B_\varepsilon)_\varepsilon} - \frac{(A'^2_\varepsilon)_\varepsilon}{2(A_\varepsilon)_\varepsilon} + \frac{2(A'_\varepsilon)_\varepsilon}{r}\right),\end{gathered} \quad (1.3.24)$$

and

$$\begin{gathered}(R_{11}(\varepsilon))_\varepsilon = -\left(\log\sqrt{-(g^{\pm}(\varepsilon))_\varepsilon}\right)_{,1,1} + (\Gamma^1_{11,1}(\varepsilon))_\varepsilon - \\ (\Gamma^0_{10}(\varepsilon))^2 - \left((\Gamma^1_{11}(\varepsilon))^2\right)_\varepsilon - \left((\Gamma^2_{21}(\varepsilon))^2\right)_\varepsilon - \\ -\left((\Gamma^3_{31}(\varepsilon))^2\right)_\varepsilon + (\Gamma^1_{11}(\varepsilon))_\varepsilon\left(\log\sqrt{-(g^{\pm}(\varepsilon))_\varepsilon}\right)_{,1} = \\ \frac{1}{2(A_\varepsilon)_\varepsilon}\left(-(A''_\varepsilon) + \frac{(A'_\varepsilon)_\varepsilon(B'_\varepsilon)_\varepsilon}{2(B_\varepsilon)_\varepsilon} + \frac{(A'^2_\varepsilon)_\varepsilon}{2(A_\varepsilon)_\varepsilon} + \frac{2(A_\varepsilon)_\varepsilon(B'_\varepsilon)_\varepsilon}{r(B_\varepsilon)_\varepsilon}\right),\end{gathered} \quad (1.3.25)$$

and

$$\begin{gathered}(R_{22}(\varepsilon))_\varepsilon = -\left(\log\sqrt{-(g^{\pm}(\varepsilon))_\varepsilon}\right)_{,2,2} + ((\Gamma^1_{22,1}(\varepsilon)))_\varepsilon - \\ 2((\Gamma^1_{22}(\varepsilon)))_\varepsilon((\Gamma^2_{21}(\varepsilon)))_\varepsilon - \left((\Gamma^3_{23}(\varepsilon))^2\right)_\varepsilon + \Gamma^1_{22}(\varepsilon)\left(\log\sqrt{-(g^{\pm}(\varepsilon))_\varepsilon}\right)_{,1} = \\ -\frac{d\cot\theta}{d\theta} - \left(\frac{r}{(B_\varepsilon)_\varepsilon}\right)' + \frac{2}{(B_\varepsilon)_\varepsilon} - \cot^2\theta - \frac{r}{(B_\varepsilon)_\varepsilon}\left(\frac{2}{r} + \frac{((A_\varepsilon)_\varepsilon(B_\varepsilon)_\varepsilon)'}{2(A_\varepsilon)_\varepsilon(B_\varepsilon)_\varepsilon}\right).\end{gathered} \quad (1.3.26)$$

Weak distributional limit in $D'(\mathbb{R}^3)$ of the RHS of the Eq.(1.3.18), i.e. $w\text{-}\lim_{\varepsilon\to 0} T_{\mu\nu}(\varepsilon)$ is calculated in our papers [18]-[19], see also Appendix B.

Remark 1.3.6. It turns out that the distributional Schwarzschild metric (1.3.22) has a gravitational singularity with compact support at origin $\{r = 0\}$ [6]-[11] and at Schwarzschild horizon $\mathbf{S}^2 \times \{r = 2m\}$ [18]-[19].

1.4.On the near horizon Colombeau approximation for the

classical singular Schwarzschild black hole geometry.

Let us perform the following coordinate transformation

$$\bar{t} = \frac{t}{4m}, \quad \bar{r}_\varepsilon = \sqrt{8m(r-2m)+\varepsilon^2}, \varepsilon \in (0,1] \tag{1.4.1}$$

to the classical singular Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{1.4.2}$$

we get

$$ds_\varepsilon^2 = -\bar{r}_\varepsilon^2\left(1 + \frac{\bar{r}_\varepsilon^2}{16m^2}\right)^{-1}d\bar{t}^2 + \left(1 + \frac{\bar{r}_\varepsilon^2}{16m^2}\right)d\bar{r}^2 + 4m^2\left(1 + \frac{\bar{r}_\varepsilon^2}{16m^2}\right)^2 d\Omega^2. \tag{1.4.3}$$

In Eq.(1.4.2), m is the central mass, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and $G = c = 1$.Taking the limit $m \to \infty$, the spherical horizon becomes planar and Eq. (1.4.3) leads to the Colombeau type metric

$$(ds_\varepsilon^2)_\varepsilon = -\{(\bar{r}_\varepsilon^2)_\varepsilon\}d\bar{t}^2 + (d\bar{r}_\varepsilon^2)_\varepsilon + 4m^2 d\Omega^2 \tag{1.4.4}$$

which is distributional Rindler's spacetime if we neglect the angular contribution.The condition $m \to \infty$ is equivalent to the "near horizon approximation" for the exterior geometry of a black hole : for $r \approx 2m$ $(r > 2m)$ the line element (1.4.2) appears, indeed, as

$$ds^2 = -\frac{r-2m}{2m}dt^2 + \frac{2m}{r-2m}dr^2 + 4m^2 d\Omega^2. \tag{1.4.5}$$

By using simple coordinate transformations it could be shown that (1.4.5) again becomes the distributional Rindler metric when we take $\theta, \phi = const.$ or $\Delta\theta$ and $\Delta\phi$ are negligible. We stress that the condition $r \approx 2m$ only is not enough to obtain Rindler's spacetime which has no spherical symmetry as Schwarzschild.

Remark 1.4.1.At this stage of consideration, it is already clear that near horizon Schwarzschild black hole geometry has a gravitational singularity at horizon. Notice that in classical handbooks (see for example [26]-[32]) near horizon Schwarzschild black hole geometry mistakenly acepted as regular with the Ricci tensor and the Ricci scalar vanish identically. A.Eddington,D.Finkelstein and G.Lemaïtre abnormal papers [33]-[35] based on misunderstanding and missconception the authors about fundamental notion of the semi-Riemannian geometry.See also a critico-historical notes in [36].

1.5. Colombeau distributional semi-Riemannian geometry.Preliminaries.

1.5.1. The ring of Colombeau generalized numbers $\widetilde{\mathbb{R}}$.

We denote by $\widetilde{\mathbb{R}}$ the ring of real, Colombeau generalized numbers. Recall that [2]-[3] by definition $\widetilde{\mathbb{R}} = \mathbf{E}_{\mathbb{R}}(\mathbb{R})/\mathbf{N}(\mathbb{R})$ where

$$\mathbf{E}_{\mathbb{R}}(\mathbb{R}) = \{(x_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1)} | (\exists a \in \mathbb{R})(\exists \varepsilon_0 \in (0,1))(\forall \varepsilon \le \varepsilon_0)[|x_\varepsilon| \le \varepsilon^{-a}]\},$$
$$\mathbf{N}(\mathbb{R}) = \{(x_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1)} | (\forall a \in \mathbb{R})(\exists \varepsilon_0 \in (0,1))(\forall \varepsilon \le \varepsilon_0)[|x_\varepsilon| \le \varepsilon^{a}]\}. \quad (1.5.1)$$

1.5.2.A real Colombeau vector bundle.

Definition 1.5.1.A real vector bundle consists of:

1.topological spaces X (base space) and E (total space)

2.a continuous surjection $\pi : E \to X$ (bundle projection)

3.for every x in X, the structure of a finite-dimensional vector space over Colombeau ring

$\widetilde{\mathbb{R}}$ on the fiber $\pi^{-1}(\{x\})$

where the following compatibility condition is satisfied: for every point in X, there is an

open neighborhood U, a natural number k, and a homeomorphism

$\varphi : U \times \widetilde{\mathbb{R}}^k \to \pi^{-1}(U)$

such that for all $x \in U$,

$(\pi \circ \varphi)(x,v) = x$ for all vectors v in $\widetilde{\mathbb{R}}^k$, and

the map $v \mapsto \varphi(x,v)$ is a linear isomorphism between the vector spaces $\widetilde{\mathbb{R}}^k$ and $\pi^{-1}(\{x\})$.

The open neighborhood U together with the homeomorphism φ is called a local trivialization of the Colombeau vector bundle. The local trivialization shows that locally the map π "looks like" the projection of $U \times \widetilde{\mathbb{R}}^k$ on U.

The Cartesian product $X \times \widetilde{\mathbb{R}}^k$, equipped with the projection $X \times \widetilde{\mathbb{R}}^k \to X$, is called the trivial bundle of rank k over X.

1.5.3.The basic idea of Colombeau's theory of generalized functions is regularization by sequences (nets) of smooth functions and the use of asymptotic estimates in terms of a regularization parameter $\varepsilon \in (0,1]$. Let $(u_\varepsilon)_\varepsilon \in (0,1]$ with $u_\varepsilon \in C^\infty(M)$ for all $\varepsilon \in (0,1]$ (M a separable, smooth orientable Hausdorff manifold of dimension n).The algebra of Colombeau generalized functions on M is defined as the quotient

$$\mathbf{G}(M) = \mathbf{E}_M(M)/\mathbf{N}(M) \quad (1.5.2)$$

of the space $\mathbf{E}_M(M)$ of sequences of moderate growth modulo the space $\mathbf{N}(M)$ of negligible sequences. More precisely the notions of moderateness resp. negligibility are defined by the following asymptotic estimates ($\widetilde{\mathbf{X}}_{\widetilde{\mathbb{R}}}(M)$ or $\widetilde{\mathbf{X}}(M)$ denoting the space of smooth vector fields on M).

$$\begin{gathered}\mathbf{E}_M(M)=\{(u_\varepsilon)_\varepsilon\in M^{(0,1]}|(\forall K\subset\subset M)(\forall k\in\mathbb{N}_0)(\exists n\in\mathbb{N})\\ \left(\forall\xi_1\in\widetilde{\mathbf{X}}(M),\ldots,\forall\xi_k\in\widetilde{\mathbf{X}}(M)\right)[\sup_{p\in K}|L_{\xi_1}\ldots L_{\xi_n}u_\varepsilon(p)|\leq O(\varepsilon^{-n})]\Big\},\\ \mathbf{N}(M)=\{(u_\varepsilon)_\varepsilon\in M^{(0,1]}|(\forall K\subset\subset M)(\forall k,q\in\mathbb{N}_0)(\exists n\in\mathbb{N})\\ \left(\forall\xi_1\in\widetilde{\mathbf{X}}(M),\ldots,\forall\xi_k\in\widetilde{\mathbf{X}}(M)\right)[\sup_{p\in K}|L_{\xi_1}\ldots L_{\xi_n}u_\varepsilon(p)|\leq O(\varepsilon^{q})]\Big\}.\end{gathered}\tag{1.5.3}$$

Elements of $\mathbf{G}(M)$ are denoted by

$$\bar{u}=\mathbf{cl}[(u_\varepsilon)_\varepsilon]=\overline{[(u_\varepsilon)_\varepsilon]}=(u_\varepsilon)_\varepsilon+\mathbf{N}(M).\tag{1.5.4}$$

With componentwise operations $\mathbf{G}(M)$ is a fine sheaf of differential algebras with respect to the Lie derivative defined by

$$L_\xi\bar{u}:=\mathbf{cl}[(L_\xi u_\varepsilon)_\varepsilon]=\overline{[(L_\xi u_\varepsilon)_\varepsilon]}.\tag{1.5.5}$$

The spaces of moderate resp. negligible sequences and hence the algebra itself may be characterized locally, i.e., $\bar{u}\in\mathbf{G}(M)$ iff $\bar{u}\circ\psi_\alpha\in\mathbf{G}(\psi_\alpha(V_\alpha))$ for all charts (V_α,ψ_α), where on the open set $\psi_\alpha(V_\alpha)\subset\mathbb{R}^n$ in the respective estimates Lie derivatives are replaced by partial derivatives. Smooth functions are embedded into $\mathbf{G}(M)$ simply by the "constant" embedding σ, i.e., $\sigma(f):=\mathbf{cl}[(f)_\varepsilon]$, hence $C^\infty(M)$ is a faithful subalgebra of $\mathbf{G}(M)$. On open sets of $\mathbb{R}^n$ compactly supported distributions are embedded into $\mathbf{G}$ via convolution with a mollifier $\rho\in S(\mathbb{R}^n)$ with unit integral satisfying $\int\rho(x)x^\alpha dx=0$ for all $|\alpha|\geq 1$; more precisely setting $\rho_\varepsilon(x)=(1/\varepsilon^n)\rho(x/\varepsilon)$ we have $\iota(w):=\mathbf{cl}[(w*\rho_\varepsilon)_\varepsilon]$. In case $\mathbf{supp}(w)$ is not compact one uses a sheaf-theoretical construction.

1.5.4. Let $\bar{f}=\mathbf{cl}[(f_\varepsilon(\mathbf{x}))_\varepsilon]=\overline{[(f_\varepsilon(\mathbf{x}))_\varepsilon]}\in\mathbf{G}(\mathbb{R}^n)$, where $f_\varepsilon(\mathbf{x}):\mathbb{R}^n\to\mathbb{R},\varepsilon\in(0,1)$ is a differentiable function and let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. We define the Colombeau directional derivative in the $\mathbf{v}$ direction at a point $\mathbf{x}\in\mathbb{R}^n$ by

$$\begin{gathered}D_{\mathbf{v}}^{Col}[\bar{f}]=D_{\mathbf{v}}\left(\overline{[(f_\varepsilon(\mathbf{x}))_\varepsilon]}\right)=\overline{[(D_{\mathbf{v}}f_\varepsilon(\mathbf{x}))_\varepsilon]}=\\ \overline{\left[\left(\frac{d}{dt}f_\varepsilon(\mathbf{x}+t\mathbf{v})|_{t=0}\right)_\varepsilon\right]}=\overline{\left[\left(\sum_{i=1}^n\mathbf{v}_i\frac{\partial f_\varepsilon(\mathbf{x})}{\partial x_i}\right)_\varepsilon\right]}.\end{gathered}\tag{1.5.6}$$

The Colombeau tangent vector at the point $\mathbf{x}$ may then be defined as

$$\mathbf{v}^{Col}(\bar{f})=\mathbf{v}\left(\overline{[(f_\varepsilon(\mathbf{x}))_\varepsilon]}\right)=\overline{[(D_{\mathbf{v}}f_\varepsilon(\mathbf{x}))_\varepsilon]}.\tag{1.5.7}$$

Let $\bar{f}=\overline{[(f_\varepsilon(\mathbf{x}))_\varepsilon]}\in G(\mathbb{R}^n),\bar{g}=\overline{[(g_\varepsilon(\mathbf{x}))_\varepsilon]}\in G(\mathbb{R}^n)$, where $f_\varepsilon,g_\varepsilon:\mathbb{R}^n\to\mathbb{R},\varepsilon\in(0,1)$ be differentiable functions, let $\mathbf{v},\mathbf{w}$ be tangent vectors in $\mathbb{R}^n$ at $x\in\mathbb{R}^n$ and let $a,b\in\widetilde{\mathbb{R}}$. Then

1. $(a\cdot\mathbf{v}+b\cdot\mathbf{w})^{Col}\left(\bar{f}\right)=(a\cdot\mathbf{v}+b\cdot\mathbf{w})\left(\overline{[(f_\varepsilon)_\varepsilon]}\right)=a\mathbf{v}\left(\overline{[(f_\varepsilon)_\varepsilon]}\right)+b\mathbf{w}\left(\overline{[(f_\varepsilon)_\varepsilon]}\right)=$
   $=a\mathbf{v}^{Col}(\bar{f})+b\mathbf{w}^{Col}(\bar{f})$;
2. $\mathbf{v}^{Col}(a\cdot\bar{f}+b\cdot\bar{g})=\mathbf{v}(a\cdot\overline{[(f_\varepsilon)_\varepsilon]}+b\cdot\overline{[(g_\varepsilon)_\varepsilon]})=a\cdot\mathbf{v}\left(\overline{[(f_\varepsilon)_\varepsilon]}\right)+b\cdot\mathbf{v}\left(\overline{[(g_\varepsilon)_\varepsilon]}\right)=$
   $a\cdot\mathbf{v}^{Col}(\bar{f})+b\cdot\mathbf{v}^{Col}(\bar{g})$;

3.$\mathbf{v}^{Col}(\bar{f}\bullet\bar{g}) = \mathbf{v}(\overline{[(f_\varepsilon \bullet g_\varepsilon)_\varepsilon]}) = \overline{[(f_\varepsilon(\mathbf{x}))_\varepsilon]} \bullet \mathbf{v}\left(\overline{[(g_\varepsilon)_\varepsilon]}\right) + \overline{[(g_\varepsilon(\mathbf{x}))_\varepsilon]} \bullet \mathbf{v}([(f_\varepsilon)_\varepsilon]) =$
$\bar{f}\bullet\mathbf{v}^{Col}(\bar{g}) + \bar{g}\bullet\mathbf{v}^{Col}(\bar{f}).$

1.5.5.Colombeau tangent vector to differentiable manifold M.

Let M be a differentiable manifold and let $\mathbf{G}(M)$ be the algebra of real-valued Colombeau generalized functions on M. Then the tangent vector to M at a point x in the manifold is given by the derivation $D_\mathbf{v} : \mathbf{G}(M) \to \widetilde{\mathbb{R}}$ which shall be linear - i.e., for any $\bar{f} = \overline{[(f_\varepsilon)_\varepsilon]}, \bar{g} = \overline{[(g_\varepsilon)_\varepsilon]} \in \mathbf{G}(M)$ and $a,b \in \widetilde{\mathbb{R}}$ we have

1.$D_\mathbf{v}^{Col}(a\bullet\bar{f} + b\bullet\bar{g}) = D_\mathbf{v}(a\bullet\overline{[(f_\varepsilon)_\varepsilon]} + b\bullet\overline{[(g_\varepsilon)_\varepsilon]}) = a\bullet D_\mathbf{v}(\overline{[(f_\varepsilon)_\varepsilon]}) + b\bullet D_\mathbf{v}(\overline{[(g_\varepsilon)_\varepsilon]}) =$
$= a\bullet D_\mathbf{v}^{Col}(\bar{f}) + b\bullet D_\mathbf{v}^{Col}(\bar{g}).$

Note that the derivation will by definition have the Leibniz property

2.$D_\mathbf{v}^{Col}(\bar{f}\bullet\bar{g}) = D_\mathbf{v}(\overline{[(f_\varepsilon \bullet g_\varepsilon)_\varepsilon]}) = D_\mathbf{v}(\overline{[(f_\varepsilon)_\varepsilon]})\bullet\overline{[(g_\varepsilon(x))_\varepsilon]} + \overline{[(f_\varepsilon(x))_\varepsilon]}\bullet D_\mathbf{v}(\overline{[(g_\varepsilon)_\varepsilon]}) =$
$= D_\mathbf{v}^{Col}(\bar{f})\bullet\bar{g} + \bar{f}\bullet D_\mathbf{v}^{Col}(\bar{g}).$

1.5.6.Colombeau vector fields on distributional manifolds.

Colombeau vector field $\widetilde{\mathbf{X}}_{\widetilde{\mathbb{R}}}$ (denotin often by $\widetilde{\mathbf{X}}$) on a manifold M is a linear map $\widetilde{\mathbf{X}}_{\widetilde{\mathbb{R}}}$: $\mathbf{G}(M) \to \mathbf{G}(M)$ such that for all $\bar{f},\bar{g} \in \mathbf{G}(M)$:

$$\widetilde{\mathbf{X}}_{\widetilde{\mathbb{R}}}(\bar{f}\bullet\bar{g}) = \bar{f}\bullet\widetilde{\mathbf{X}}_{\widetilde{\mathbb{R}}}(\bar{f}\bullet\bar{g}) + \widetilde{\mathbf{X}}_{\widetilde{\mathbb{R}}}(\bar{f})\bullet\bar{g}. \tag{1.5.8}$$

1.5.7.Colombeau tangent space.

Suppose now that M is a C^∞ manifold. A real-valued Colombeau generalized function $(f_\varepsilon)_\varepsilon : M \to \widetilde{\mathbb{R}}, \varepsilon \in (0,1]$ is said to belong to $\mathbf{G}(M)$ if and only if for every coordinate chart $\varphi : U \to \mathbb{R}^n$, the map $f_\varepsilon \circ \varphi^{-1} : \varphi[U] \subseteq \mathbb{R}^n \to \mathbb{R}$ is infinitely differentiable. Note that $\mathbf{G}(M)$ is a real associative algebra with respect to the pointwise product and sum of Colombeau generalized functions. Pick a point $x \in M$. A

derivation at x is defined as a linear map $D : \mathbf{G}(M) \to \widetilde{\mathbb{R}}$ that satisfies the Leibniz identity:

$\bar{f} = \overline{[(f_\varepsilon)_\varepsilon]}, \bar{g} = \overline{[(g_\varepsilon)_\varepsilon]} \in \mathbf{G}(M) : D(\bar{f}\bullet\bar{g}) = D(\bar{f})\bullet\bar{g}(x) + \bar{f}(x)\bullet D(\bar{g}),$

which is modeled on the product rule of calculus.

If we define addition and scalar multiplication on the set of derivations at x by

$(D_1 + D_2)(\bar{f}) = f\bullet D_1(\bar{f}) + D_2(\bar{f})$ and

$(\lambda\bullet D)(\bar{f}) = \bar{f}\bullet\lambda\bullet D(\bar{f}),$

where $\lambda \in \widetilde{\mathbb{R}}$, then we obtain a real vector space over $\widetilde{\mathbb{R}}$, which we define as the Colombeau tangent space $T_x^{Col}M$ of M at x.

1.5.8.We call a separable, smooth Hausdorff manifold M furnished with a generalized pseudo-Riemannian metric $(g_\varepsilon)_\varepsilon$ generalized pseudo-Riemannian manifold or generalized spacetime and denote it by $(M,\bar{g})$ [20]-[22].

1.5.9.Colombeau isometric embedding.

Let $(M,\bar{g})$ and $(N,\bar{h})$ be generalized pseudo-Riemannian manifolds. An isometric Colombeau embedding is a Colombeau generalized function $(f_\varepsilon)_\varepsilon : M \to N$ which preserves the metric in the sense that $(g_\varepsilon)_\varepsilon$ is equal to the pullback of $(h_\varepsilon)_\varepsilon$ by $(f_\varepsilon)_\varepsilon$, i.e. $(g_\varepsilon)_\varepsilon = (f_\varepsilon^* h_\varepsilon)_\varepsilon$. Explicitly, for any two tangent vectors $\mathbf{v},\mathbf{w} \in T_x(M)$ we have $\left(g_\varepsilon(\mathbf{v}\ ,\ \mathbf{w})\right)_\varepsilon = (h_\varepsilon(df_\varepsilon(\mathbf{v}), df_\varepsilon(\mathbf{w})))_\varepsilon$.

2.Distributional Schwarzschild spacetime.

2.1.Distributional Schwarzschild spacetime as Colombeau extension of the Lorentzian manifold with nonregularity conditions on Schwarzschild horizon.

Singular space-times present one of the major challenges in general relativity. Originally it was believed that their singular nature is due to the high degree of symmetry of the well-known examples ranging from the Schwarzschild geometry to the Friedmann-Robertson-Walker cosmological models. However, Penrose and Hawking [37] have shown in their classical singularity theorems that singularities are a phenomenon which is inherent to general relativity. Since the standard approach allows only for smooth space-time metrics, one has to exclude the socalled singular regions from the space-time manifold.In a recent work many authors advocated the use Colombeau distributional techniques [5]-[23] to calculate the energy-momentum tensor of the Schwarzschild geometry.It turns out that it is possible to include the singular region (i.e. the space-like line $r = 0$ with respect to Schwarzschild coordinates) in the space-time which now no longer is a vacuum geometry, and to identify it with the support of the energy-momentum tensor [5],[9],[11]-[13]. The same “physically expected” result for the distributional energy momentum tensor of the Schwarzschild geometry was obtained in papers [12]-[22], i.e.,

$$T_0^0 = 8\pi m\delta\left(\vec{x}\right), \qquad (2.1.1)$$

in a conceptually satisfactory way.

Remark 2.1.1.The result (2.1.1) can be easily obtained by using apropriate nonsmooth regularization of the Schwarzschild singularity at the origin $r = 0$.

The nonsmooth regularization of the Schwarzschild singularity at the origin $r = 0$ originally considered by N. R. Pantoja and H. Rago in paper [12]. Such non smooth regularization of the Schwarzschild singularity is

$$(h_\varepsilon(r))_\varepsilon = -1 + \left(\frac{r_s}{r}\Theta_\varepsilon(r-\varepsilon)\right)_\varepsilon, \varepsilon \in (0,1], r < r_s. \tag{2.1.2}$$

Here $(\Theta_\varepsilon(u))_\varepsilon$ is the generalized Heaviside function,where

$$\Theta_\varepsilon(u) = \begin{cases} (\varepsilon)_\varepsilon & u < 0 \\ \frac{1}{2} & u = 0 \\ 1 & u > 0 \end{cases} \tag{2.1.3}$$

and the limit $\varepsilon \to 0$ is understood in a weak distributional sense.The equation

$$\begin{gathered}(ds_\varepsilon^2)_\varepsilon = (h_\varepsilon(r)(dt)^2)_\varepsilon - (h_\varepsilon^{-1}(r)(dr)^2)_\varepsilon + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2], \\ h_0(r) = -1 + \frac{r_s}{r},\end{gathered} \tag{2.1.4}$$

with $h_\varepsilon, \varepsilon \in (0,1]$, as given in (2.1.4) can be considered as Colombeau version of the Schwarzschild line element in curvature coordinates. From equation (2.1.2), the calculation of the distributional Einstein tensor $(G_t^t(r,\varepsilon))_\varepsilon, (G_r^r(r,\varepsilon))_\varepsilon, (G_\theta^\theta(r,\varepsilon))_\varepsilon, (G_\varphi^\varphi(r,\varepsilon))_\varepsilon$ proceeds in a straighforward manner. By simple calculation one obtains [12]:

$$\begin{gathered}(G_t^t(r,\varepsilon))_\varepsilon = (G_r^r(r,\varepsilon))_\varepsilon = -\left(\frac{h_\varepsilon'(r)}{r}\right)_\varepsilon - \left(\frac{1+h_\varepsilon(r)}{r^2}\right)_\varepsilon = \\ = -r_s\left(\frac{\delta(r-\varepsilon)}{r^2}\right)_\varepsilon = -r_s\frac{\delta(r)}{r^2}\end{gathered} \tag{2.1.5}$$

and

$$\begin{gathered}(G_\theta^\theta(r,\varepsilon))_\varepsilon = (G_\varphi^\varphi(r,\varepsilon))_\varepsilon = -\left(\frac{h_\varepsilon''(r)}{2}\right)_\varepsilon - \left(\frac{h_\varepsilon(r)}{r^2}\right)_\varepsilon = \\ r_s\left(\frac{\delta(r-\varepsilon)}{r^2}\right)_\varepsilon - r_s\left(\frac{\varepsilon}{r^2}\frac{d}{dr}\delta(r-\varepsilon)\right)_\varepsilon \approx -r_s\frac{\delta(r)}{r^2}.\end{gathered} \tag{2.1.6}$$

In papers [10],[28] Colombeau distributional techniques were extended to the general axisymmetric, stationary Kerr and Newman space-time family.This family also contains the Schwarzschild geometry and its charged extension the Reissner-Nordstrøm solution as special cases of spherical symmetry.In the paper [23] was shown that the solutions will satisfy the Einstein equations everywhere if the energy-momentum tensor has an appropriate singular addition of nonelectromagnetic origin. When this addition term is included, the total energy turns out to be finite and equal to mc^2, while the angular momentum for the Kerr and Kerr-Newman solutions is

$mc\mathbf{a}$.

Remark 2.1.2. The nonsmooth regularization of the Schwarzschild singularity at the horizon $r = r_s$ is

$$(h_\varepsilon^+(r))_\varepsilon = -1 + \left(\frac{r_s}{r}\Theta_\varepsilon((r-r_s)-\varepsilon)\right)_\varepsilon, \varepsilon \in (0,1], r \geq r_s. \tag{2.1.7}$$

Here $(\Theta_\varepsilon(u))_\varepsilon$ is the generalized Heaviside function and the limit $\varepsilon \to 0$ is understood in a weak distributional sense.The equation

$$\begin{gathered}(ds_\varepsilon^{+2})_\varepsilon = (h_\varepsilon^+(r)(dt)^2)_\varepsilon - \left([h_\varepsilon^+(r)]^{-1}(dr)^2\right)_\varepsilon + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2],\\ h_0(r) = -1 + \frac{r_s}{r},\end{gathered} \tag{2.1.8}$$

$h_\varepsilon, \varepsilon \in (0,1]$, as given in (2.1.8) can be considered as Colombeau version of the Schwarzschild line element in curvature coordinates above horizon. From equation (2.1.7), the calculation of the distributional Einstein tensor above horizon $(G_t^{+t}(r,\varepsilon))_\varepsilon, (G_r^{+r}(r,\varepsilon))_\varepsilon$, $(G_\theta^{+\theta}(r,\varepsilon))_\varepsilon$, $(G_\varphi^{+\varphi}(r,\varepsilon))_\varepsilon$ proceeds in a straighforward manner. By simple calculation one obtains

$$\begin{gathered}(G_t^{+t}(r,\varepsilon))_\varepsilon = (G_r^{+r}(r,\varepsilon))_\varepsilon = -\left(\frac{h'_\varepsilon((r-r_s)-\varepsilon)}{r}\right)_\varepsilon - \left(\frac{1+h_\varepsilon((r-r_s)-\varepsilon)}{r^2}\right)_\varepsilon = \\ = -r_s\left(\frac{\delta((r-r_s)-\varepsilon)}{r^2}\right)_\varepsilon \approx -r_s\frac{\delta(r-r_s)}{r^2}.\end{gathered} \tag{2.1.9}$$

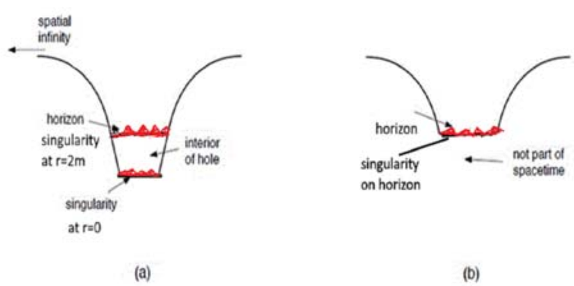

Fig.2.1.1.(a) The picture of a distributional Schwarzschild blackhole,
given by Colombeau generalized object (1.4.3).
Distributional spacetime ends just on the Schwarzschild singularity.
(b) The truncated Schwarzschild distributional geometry,
given by Colombeau generalized object (1.4.12)
Distributional spacetime ends just on the Schwarzschild horizon.

Remark 2.1.3.In a nutshell, there is a widespread but mistaken belief that there exist true gravitational singularities,for example at origin $r = 0$ of the Schwarzschild spacetime, and non principal non gravitational,i.e. purely coordinate singularities, for example at horizon $r = r_g$ of the Schwarzschild spacetime. A coordinate singularity or coordinate degeneracy occurs when an apparent singularity or degeneracy occurs in one coordinate frame, which can be removed by choosing a different frame. Classical example of such mistake is ubnormal deletion of the gravitational singularity,for example from Schwarzschild spacetime

$$\mathbf{Sch} = (S^2 \times \{r \geq 2m\}) \times \mathbb{R}\ , g_{ij}(r,\theta,\phi), \tag{2.1.10}$$

originally defined by singular and degenerate Schwarzschild metric [31],

$$ds^2 = -h(r)(dx^0)^2 + h^{-1}(r)(dr)^2 + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2], h(r) = 1 - \frac{r_g}{r}. \tag{2.1.11}$$

by using apropriate singular coordinate change [28]-[36].

Remark 2.1.4.Note that: (i) metric (2.1.11) is singular and degenerate at Schwarzschild horizon $r = r_g$, and thus metric (2.1.11) beiond canonical rigorous semi-Riemannian geometry.

(ii) however in physical literature (see for example [29]-[31]) singularity and degeneracy at Schwarzschild horizon $r = r_g$ acepted as coordinate singularity and coordinate degeneracy.

Remark 2.1.5. (see [31] section 100,p.296)."In the Schwarzschild metric (97.14), g_{00} goes to zero and g_{11} to infinity at $r = r_g$ (on the "Schwarzschild sphere"). This could give the basis for concluding that there must be a singularity of the space-time metric and that it is therefore impossible for bodies to exist that have a "radius" (for a given mass) that is less than the gravitational radius. Actually,however, this conclusion would be wrong. This is already evident from the fact that the determinant $g(r) = -r^4 \sin^2\theta$ has no singularity at $r = r_g$, so that the condition $g < 0$ (82.3) is not violated. We shall see that in fact we are dealing simply with the impossibility of establishing a suitable reference system for $r < r_g$."

Remark 2.1.5. Notice that consideration above meant the following definition of the gravitational singularity.

Definition 2.1.1.There is no gravitational singularity at $r = \bar{r}$ iff the determinant $g(r,\theta) = \det(g_{ij}(r,\theta))$ has no singularity at $r = \bar{r}$.

Remark 2.1.6. Notice that at singular point $r = r_g$ the determinant $g(r_g)$ is well

defined
only by the limit

$$g(r_g) = \lim_{r \to r_g} \det(g_{ij}(r,\theta)) = -r^4 \sin^2\theta. \quad (2.1.12)$$

however in the limit $r \to r_g = 2m$ the classical Levi-Civitá connection Γ^l_{kj} becomes infinite

$$\Gamma^1_{11}(r)|_{r=2m} = \lim_{r \to 2m} \frac{-m}{r(r-2m)} = \infty, \Gamma^0_{01}(r)|_{r=2m} = \lim_{r \to 2m} \frac{m}{r(r-2m)} = \infty, \quad (2.1.13)$$

and therefore the Definition 2.1.1 is not sound and even does not any sense under canonical semi-Riemannian geometry.

Remark 2.1.7. Notice that:

(i) in order to fixin problem with singularity and degeneracy of the Schwarzschild metric (2.1.11) at Schwarzschild horizon $r = r_g$, in physical literature [28]-[36], many years one considers the abnormal formal change of coordinates obtained by replacing the canonical Schwarzschild time by "retarded time" $v(t,r)$, i.e., Eddington–Finkelstein coordinates, given by

$$\begin{aligned} dv(t,r) &= dt + [h(r)]^{-1} dr, \\ h(r) &= 1 - \frac{r_g}{r}; \end{aligned} \quad (2.1.14)$$

(ii) the change (2.1.14) of Schwarzschild coordinates is singular at Schwarzschild horizon $r = r_g$, as at Schwarzschild horizon $h(r_g) = \infty$ and therefore the change (2.1.14) does not holds on Schwarzschild horizon [36];

(ii) under the singular change (2.1.14) Schwarzschild metric (2.1.11) becomes to well known regular and nondegenerate Eddington-Finkelstein metric [28]-[36]:

$$ds^2_{\mathbf{EF}} = -\left(1 - \frac{2m}{r}\right)dv^2 + 2drdv + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]; \quad (2.1.15)$$

(iii) in physical literature many years exist abnormal belief that by formal singular change (2.1.15) the singular and degenerate Schwarzschild spacetime $(\mathbf{S}^2 \times \{r > 2m\}) \times \mathbb{R}$ was immersed in a larger Eddington-Finkelstein spacetime

$$\mathbf{EF}_{\geq} = (\mathbf{S}^2 \times \{r \geq 2m\} \cup \{0 < r \leq 2m\}) \times \mathbb{R}, g_{\mathbf{EF}_{\geq}}(r,\theta), \quad (2.1.16)$$

with regular and non degenerate metric tensor $g_{\mathbf{EF}_{\geq}}(r,\theta)$, and whose manifold is not covered by the canonical Schwarzschild coordinate with $r \leq 2m,$and therefore singularity and degeneracy on Schwarzschild horizon $r = r_g$ is only coordinate singularity and coordinate degeneracy;

(iv) from statement (iii) it was mistakenly assumed that there is no gravitational singularity at BH horizon.

We remind now canonical definitions.

Definition 2.1.2. Let (M,g) and (N,h) be semi-Riemannian manifolds. An isometric embedding is a smooth embedding $f : M \to N$ which preserves the metric in the sense that g is equal to the pullback of h by $f,$ i.e. $g = f^*h.$ Explicitly, for any two tangent vectors $\mathbf{v},\mathbf{w} \in T_x(M)$ we have

$$g(\mathbf{v},\mathbf{w}) = h(df(\mathbf{v}),df(\mathbf{w})). \tag{2.1.17}$$

Remark 2.1.8.Notice that such isometric embedding is a mathematical definition only and does not meant the equivalence $(M,g) \equiv (f(M),h)$ in absolute sense. Thus it is not alwais apropriate as equivalence of the Lorentzian manifolds (M,g) and (N,h) corresponding to the physical frames $(M_{\mathbf{ph}},g_{\mathbf{ph}})$ and $(N_{\mathbf{ph}},h_{\mathbf{ph}})$.

Definition 2.1.3.[32].In general,a Lorentzian manifold $\left(M',h\right)$ is said to be an extension of a Lorentzian manifold (M,g) if there exists an isometric embedding $i : M \hookrightarrow M'$.

Remark 2.1.9.Notice that such extension is a mathematical definition only and therefore it is not alwais apropriate as extension of the Lorentzian manifolds (M,g) and (M',h) corresponding to the physical frames $(M_{\mathbf{ph}},g_{\mathbf{ph}})$ and $(M'_{\mathbf{ph}},h_{\mathbf{ph}})$.

Remark 2.1.10. In order to obtain example for the statement mentioned and Remark 2.1.8 and Remark 2.1.9 we go to prove below that the geometry of Schwarzschild spacetime $\mathbf{Sch}_{>} \triangleq \{(\mathbf{S}^2 \times \{r > 2m\}) \times \mathbb{R}, g_{\mathbf{Sch}_{>}}\}$ above Schwarzschild horizon, essantially cardinally different in comparizon with the geometry of Eddington-Finkelstein spacetime $\mathbf{EF}_{>} \triangleq \{(\mathbf{S}^2 \times \{r > 2m\}) \times \mathbb{R}, g_{\mathbf{EF}_{>}}\}$ above Eddington-Finkelstein horizon.

We remind now canonical definitions.

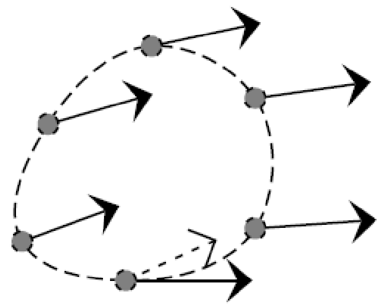

Fig.2.1.2.Paralel displacement along a closed contour Γ in a curved space.

Definition 2.1.3.Let $\Delta_\Gamma A_k$ be the change in a vector $A_i(\hat{x})$ after parallel displacement around closed contour Γ located in BH spacetime as ploted in Fig.2.1.3. This change $\Delta_\Gamma A_k$ can clearly be written in the form $\oint_\Gamma \delta A_k$. Substituting in place of δA_k the canonical expression $\delta A_k = \Gamma^i_{kl}(\hat{x})A_k dx^l$ (see 31],Eq.(85.5)) one obtains

$$\Delta_\Gamma A_k = \oint_\Gamma \delta A_k = \oint_\Gamma \Gamma^i_{kl}(\hat{x})A_k dx^l\,. \tag{2.1.18}$$

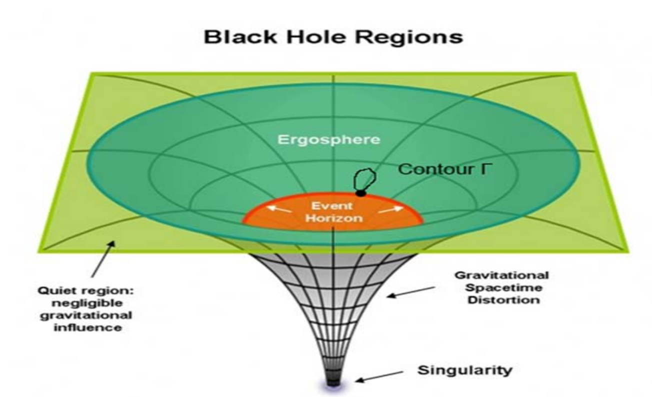

Fig.2.1.3.Paralel displacement along a closed contour Γ in BH spacetime.

Definition 2.1.4.(**I**) Let Σ^g_{Sh} be Schwarzschild horizon, let $\Gamma_{\hat{x}}$ be a contour located in Schwarzschild spacetime as plotted in Fig.2.1.4 and such that (i) $\hat{x} \in \Gamma_{\hat{x}}$, (ii) $\Sigma^g_{\mathbf{Sch}} \cap \Gamma_{\hat{x}} = \hat{x}$, and let $\Gamma^{\hat{x}}$ be a curve $\Gamma^{\hat{x}} = \Gamma_{\hat{x}} \backslash \hat{x}$. Let $\Delta_{\Gamma^{\hat{x}}} A_k$ be the integral

$$\Delta_{\Gamma^{\hat{x}}} A_k = \oint_{\Gamma_{\hat{x}}\backslash\hat{x}} \delta A_k = \oint_{\Gamma_{\hat{x}}\backslash\hat{x}} \Gamma^i_{kl}(\hat{x})A_k dx^l\,. \tag{2.1.19}$$

(**II**) Let $\Sigma_{\mathbf{EF}}^{g}$ be Eddington-Finkelstein horizon, let $\Gamma_{\hat{x}}$ be a contour located in Eddington- Finkelstein spacetime as plotted in Fig.2.1.5 and such that (i) $\hat{x} \in \Gamma_{\hat{x}}$, (ii) $\Sigma_{\mathbf{EF}}^{g} \cap \Gamma_{\hat{x}} = \hat{x}$, and let $\Gamma^{\hat{x}}$ be a curve $\Gamma^{\hat{x}} = \Gamma_{\hat{x}} \backslash \hat{x}$. Let $\Delta_{\Gamma^{\hat{x}}} A_k$ be the integral

$$\Delta_{\Gamma^{\hat{x}}} A_k = \oint\limits_{\Gamma_{\hat{x}} \backslash \hat{x}} \delta A_k = \oint\limits_{\Gamma_{\hat{x}} \backslash \hat{x}} \Gamma_{kl}^{i}(\hat{x}) A_k dx^l \ . \tag{2.1.20}$$

Fig.2.1.4.Paralel displacement $\Delta_{\Gamma^{\hat{x}}} A_k$ along a curve $\Gamma^{\hat{x}}$ in Schwarzschild spacetime such that $\Sigma_{\mathbf{Sch}}^{g} \cap \Gamma_{\hat{x}} = \hat{x}$, then alwais $\Delta_{\Gamma^{\hat{x}}} A_k = \infty$.

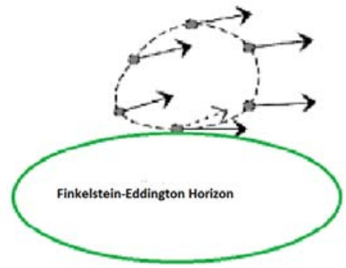

Fig.2.1.5.Paralel displacement along a curve $\Gamma^{\hat{x}}$ in Eddington-Finkelstein spacetime $\Sigma_{\mathbf{EF}}^{g} \cap \Gamma_{\hat{x}} = \hat{x}$, then alwais $\Delta_{\Gamma^{\hat{x}}} A_k < \infty$.

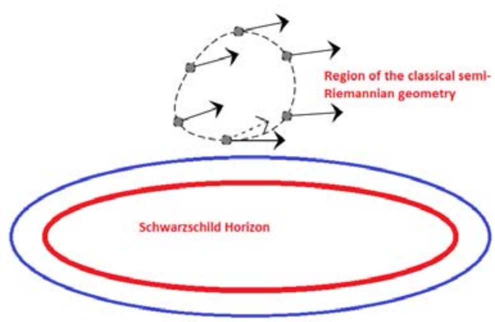

Fig.2.1.6.Paralel displacement along aclosed contour Γ located in region of the classical semi-Riemannian geometry of the Schwarzschild spacetime such that $\Sigma_{\mathbf{Sh}}^{g} \cap \Gamma = \varnothing$, then alwais $\Delta_{\Gamma} A_k < \infty$.

Remark 2.1.11. (**I**) Note that the geometry of Schwarzschild spacetime $\mathbf{Sch}_{>}$

$$\mathbf{Sch}_{>} \triangleq \{(\mathbf{S}^2 \times \{r > 2m\}) \times \mathbb{R}, g_{\mathbf{Sch}_{>}}\} \tag{2.1.21}$$

above Schwarzschild horizon $\Sigma^{g}_{\mathbf{Sch}}$, essantially cardinally different in comparizon with the geometry of Eddington-Finkelstein spacetime $\mathbf{EF}_{>}$

$$\mathbf{EF}_{>} \triangleq \{(\mathbf{S}^2 \times \{r > 2m\}) \times \mathbb{R}, g_{\mathbf{EF}_{>}}\} \tag{2.1.22}$$

above Eddington-Finkelstein horizon $\Sigma^{g}_{\mathbf{EF}}$.

(**II**) Note that Schwarzschild spacetime $\mathbf{Sch}_{>}$ obviously satisfies a very strong nonregularity condition

$$\text{if } \Sigma^{g}_{\mathbf{Sch}} \cap \Gamma_{\hat{x}} = \hat{x}, \text{ then } \Delta_{\Gamma^{\hat{x}}} A_k = \infty. \tag{2.1.23}$$

Thus the geometry of spacetime $\mathbf{Sch}_{>}$ that is nonclassical geometry beyond apparatus of the classical semi-Riemannian geometry. Ofcourse the geometry any part of spacetime $\mathbf{Sch}_{>}$ located above some neighborhood of Schwarzschild horizon as plotted in Fig.2.1.6 that is a classical semi-Riemannian geometry.

Remark 2.1.12. Note that from Remark 2.1.11 it follows that Eddington-Finkelstein spacetime does not holds in regorous mathematical sense as extension of the Schwarzschild spacetime $\mathbf{Sch}_{>} \triangleq \{(\mathbf{S}^2 \times \{r > 2m\}) \times \mathbb{R}, g_{\mathbf{Sch}_{>}}\}$ above Schwarzschild horizon.

Remark 2.1.13. It is clear that nonregularity condition (2.1.23) arises not only from singularity of the function $h^{-1}(r)$ at point $r = r_g$ but from degeneracy of the function $h(r)$ at point $r = r_g$.

Remark 2.1.14. We remind now that the relations (see [31] p.234,Eq.(84.7))

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}} \tag{2.1.24}$$

give the connection between the metric of real space

$$dl^2 = \gamma_{\alpha\beta} dx^{\alpha} dx^{\beta} \tag{2.1.25}$$

and the metric of the four-dimensional space-time

$$ds^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta} + 2g_{0\alpha} dx^0 dx^{\alpha} + g_{00}(dx^0)^2. \tag{2.1.26}$$

For Eddington-Finkelstein metric (2.1.15) metric of the corresponding real space is

$$dl^2_{\mathbf{EF}} = \frac{dr^2}{1 - \frac{2m}{r}} + r^2[(d\theta)^2 + \sin^2\theta (d\phi)^2]. \tag{2.1.27}$$

Remark 2.1.15. Notice that the Eddington-Finkelstein metric (2.1.15) is regular at the horizon and therefore the infalling observer encounters nothing unusual at the horizon.

However from Eq.(2.1.17) it follows that the infalling observer encounters singularity on

horizon.But this is a contradiction.

Remark 2.1.16.Note that in order dealing with singular Schwarzschild metric (2.1.11) using mathematically and logically soundness approach, one applies contemporary distributional geometry based on Colombeau generalized functions [2]-[4].Distributional Schwarzschild geometry and distributional BHs geometry by using Colombeau generalized functions [2]-[4] many developed in papers [4]-[23].By aproporiate regularization $g_{ij,\varepsilon}(r,\theta,\phi),\varepsilon \in (0,1]$ of the singular Schwarzschild metric$g_{ij}(r,\theta,\phi)$ such that:

(i) $g_{ij,0}(r,\theta,\phi) = g_{ij}(r,\theta,\phi)$ and

(ii) for any $\varepsilon \in (0,1]$ metric tensor $g_{ij,\varepsilon}(r,\theta,\phi)$ is regular and nondegenerate,one obtains Colombeau generalized object $[(g_{ij,\varepsilon}(r,\theta,\phi))_\varepsilon] \in \mathbf{G}(\mathbb{R}^3)$ with an representative $(g_{ij,\varepsilon}(r,\theta,\phi))_\varepsilon$,for a more detailed explanation see [11],[18],[19].Using rigorous Colombeau approach one obtains mathematically and logically soundness notion of singularity in Distributional Schwarzschild spacetime.

Remark 2.1.14. Note that in the case of Schwarzschild spacetime the conditions (i) and (ii) mentioned above (see Remark 2.1.13) are satisfied only by using non smooth regularization of the singular and degenerate Schwarzschild metric $g_{ij}(r,\theta,\phi)$ via Schwarzschild horizon [18]-[19].

By apriporiate nonsmooth regularization one obtain Colombeau generalized object modeling the singular Schwarzschild metric above and below horizon [18]-[19]:

$$\begin{aligned}
(ds_\varepsilon^{+2})_\varepsilon &= -(h_\varepsilon^+(r)dt^2)_\varepsilon + \left([h_\varepsilon^+(r)]^{-1}dr^2\right)_\varepsilon + r^2 d\Omega^2, \\
(ds_\varepsilon^{-2})_\varepsilon &= (h_\varepsilon^-(r)dt^2)_\varepsilon - \left([h_\varepsilon^-(r)]^{-1}dr^2\right)_\varepsilon + r^2 d\Omega^2, \\
h_\varepsilon^+(r) &= \frac{\Theta_\varepsilon((r-r_s)-\varepsilon)\sqrt{(r-r_g)^2+\varepsilon^2}}{r}, r \geq r_g, \\
&\varepsilon \in (0,1].
\end{aligned} \tag{2.1.24}$$

Remark 2.1.6. Let us rewrite now the metric (2.1.24) (above horizon) in the form

$$\begin{gathered}
(ds_\varepsilon^{+2})_\varepsilon = -(h_\varepsilon^+(r)dt^2)_\varepsilon + \left([h_\varepsilon^+(r)]^{-1}dr^2\right)_\varepsilon + r^2(d\theta^2 + \sin^2\theta d\varphi^2) = \\
-(h_\varepsilon^+(r))_\varepsilon\left(\left[dt - [h_\varepsilon^+(r)]^{-1}dr\right]\left[dt + [h_\varepsilon^+(r)]^{-1}dr\right]\right)_\varepsilon + r^2(d\theta^2 + \sin^2\theta d\varphi^2),
\end{gathered} \tag{2.1.25}$$

and define a new generalized Colombeau coordinates $((\tau_\varepsilon)_\varepsilon, \bar{r}, \theta, \varphi)$, where $(\tau_\varepsilon(t,r))_\varepsilon \in \mathbf{G}(\mathbb{R}^2)$, by formula

$$(d\tau_\varepsilon(t,r))_\varepsilon = dt + \left([h_\varepsilon^+(r)]^{-1}dr\right)_\varepsilon,$$
$$\bar{r} = r. \tag{2.1.26}$$

Remark 2.1.15. Notice that:
(i) Colombeau generalized coordinates (2.1.26) are the Colombeau extension of the canonical Eddington-Finkelstein coordinates (2.1.14) by Colombeau generalized function.
(ii) In contrast with canonical Eddington-Finkelstein coordinates (2.1.14) (see Remark 2.1.7), Colombeau generalized coordinates (2.1.26) holds at Schwarzschild horizon $r = r_g$ as at Schwarzschild horizon Colombeau generalized function $\left([h_\varepsilon^+(r)]^{-1}\right)_\varepsilon$ become well defined Colombeau generalized number $[h_\varepsilon^+(r_g)]^{-1} \in \widetilde{\mathbb{R}}$.
Rewriting now the metric (2.1.25) in terms of the Colombeau generalized coordinates $((\tau_\varepsilon)_\varepsilon, \bar{r}, \theta, \varphi)$, it then above horizon takes the form

$$(ds_\varepsilon^{+2})_\varepsilon =$$
$$-((h_\varepsilon^+(\bar{r}))_\varepsilon)\left(\left[d\tau_\varepsilon - 2[h_\varepsilon^+(\bar{r})]^{-1}d\bar{r}\right]d\tau_\varepsilon\right)_\varepsilon + \bar{r}^2(d\theta^2 + \sin^2\theta d\varphi^2) = \tag{2.1.27}$$
$$-((h_\varepsilon^+(\bar{r}))_\varepsilon)(d\tau_\varepsilon^2)_\varepsilon + 2d\bar{r}(d\tau_\varepsilon)_\varepsilon + \bar{r}^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

We rewrite now Colombeau metric (2.1.27) in the equivalent form

$$(ds_\varepsilon^{+2})_\varepsilon =$$
$$-(h_\varepsilon^+(r)d\tau^2)_\varepsilon + 2drd\tau + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.1.28}$$

Colombeau metric (2.1.28) define the distributional Eddington-Finkelstein space-time

$$\mathbf{EF}_\geq^+ \triangleq \left\{\left(\widetilde{\mathbf{S}}^2 \times \{\bar{r} \geq 2m\}\right) \times \widetilde{\mathbb{R}}, g_{\mathbf{EF}_\geq}^+\right\} \tag{2.1.29}$$

above the Eddington-Finkelstein horizon $r = 2m$.
Remark 2.1.16. Notice that

$$(h_\varepsilon^+(r))_\varepsilon|_{r=r_g} = r_g^{-1}(\varepsilon)_\varepsilon, \left([h_\varepsilon^+(r)]^{-1}\right)_\varepsilon\Big|_{r=r_g} = r_g \cdot (\varepsilon^{-1})_\varepsilon \in \widetilde{\mathbb{R}},$$
$$(d\tau_\varepsilon)_\varepsilon|_{r=r_g} = dt + ((\varepsilon^{-1})_\varepsilon) \cdot r_g dr,$$
$$(d\tau_\varepsilon^2)_\varepsilon\Big|_{r=r_g} = dt^2 + 2((\varepsilon^{-1})_\varepsilon) \cdot r_g dt dr + ((\varepsilon^{-2})_\varepsilon) r_g^2 dr^2, \tag{2.1.30}$$
$$\varepsilon \in (0,1].$$

Of course at horizon $(h_\varepsilon^+(t,r_g))_\varepsilon \approx 0$, becarse at horizon $h_0^+(t,r_g) = 0$, however it follows from (2.1.24) at horizon the quantities $((h_{\varepsilon}^{+}(r_g))_{\varepsilon})(d\tau_{\varepsilon}^{2}(t,r_g))_{\varepsilon} \approx ((\varepsilon^{-1})_{\varepsilon})r_g dr^2$ and $(d\tau_{\varepsilon})_{\varepsilon} \approx ((\varepsilon^{-1})_{\varepsilon})r_g dr$ are infinite large Colombeau quantities,i.e.,the differential $(d\tau_{\varepsilon})_{\varepsilon}$ is not classical but it is Colombeau differential.

Remark 2.1.17. Note that:

(i) ander coordinate change (2.1.26) distributional curvature

scalars of the distributional Schwarzschild space-time given by metric (2.1.24), does not

changes because these scalars depend only on variable $r = \bar{r}$,

(ii) in contrast with classical Eddington-Finkelstein space-time

$$\mathbf{EF}_{\geq}=(\mathbf{S}^2 \times \{r \geq 2m\} \cup \{0 < r \leq 2m\}) \times \mathbb{R}\ , g_{\mathbf{EF}_{\geq}}(r,\theta),$$

distributional Eddington-Finkelstein space-time has a gravitational singularity at horizon.

Remark 2.1.18. Note that for the case of the distributional space-time the relations (2.1.24) obviously takes the form

$$(\gamma_{\alpha\beta}(\varepsilon))_{\varepsilon} = \left(-g_{\alpha\beta}(\varepsilon) + \frac{g_{0\alpha}(\varepsilon)g_{0\beta}(\varepsilon)}{g_{00}(\varepsilon)}\right)_{\varepsilon} \tag{2.1.30}$$

where (2.1.30) give the connection between the Colombeau metric of the distributional

real space

$$(dl_{\varepsilon}^{2})_{\varepsilon} = ((\gamma_{\alpha\beta}(\varepsilon)dx^{\alpha}dx^{\beta}))_{\varepsilon} \tag{2.1.31}$$

and the Colombeau metric of the four-dimensional distributional space-time

$$\begin{gathered}(ds_{\varepsilon}^{2})_{\varepsilon} = \\ (g_{\alpha\beta}(\varepsilon)dx^{\alpha}dx^{\beta})_{\varepsilon} + 2(g_{0\alpha}(\varepsilon)dx^{0}dx^{\alpha})_{\varepsilon} + \left(g_{00}(\varepsilon)(dx^{0})^{2}\right)_{\varepsilon}.\end{gathered} \tag{2.1.32}$$

For distributional Eddington-Finkelstein metric (2.1.25) metric (above horizon) of the corresponding distributional real space is

$$\left(dl_{\varepsilon,\mathbf{EF}_{\geq}^{+}}^{+2}\right)_{\varepsilon} = (h_{\varepsilon}^{+}(r)dr^{2})_{\varepsilon} + r^{2}[(d\theta)^{2} + \sin^{2}\theta(d\phi)^{2}]. \tag{2.1.27}$$

2.2.Distributional Schwarzschild spacetime and distributional Rindler spacetime with distributional Levi-Cività connection. Generalized Einstein equivalence principle.

2.2.1.Distributional Schwarzschild spacetime with

distributional Levi-Cività connection.

Remark 2.2.1. Note that due to the degeneracy of the metric (2.1.11) on Schwarzschild horizon, the classical Levi-Civit'a connection on whole Schwarzschild spacetime is not available [19],[18],[19] as classical Levi-Civit'a connection on Schwarzschild horizon becomes infinity

$$\Gamma^{1}_{11}(r)|_{r=2m} = \lim_{r\to 2m} \frac{-m}{r(r-2m)} = -\infty, \Gamma^{0}_{01}(r)|_{r=2m} = \lim_{r\to 2m} \frac{m}{r(r-2m)} = \infty, \qquad (2.2.1)$$

Remark 2.2.2. In order to avoid difficultness with classical Levi-Civit'a connection mentioned above in Remark 2.2.1, in papers [18],[19] we have applied the non smooth regularization via Schwarzschild horizon, see Remark 2.1.5 and Eq.(2.1.6). Corresponding Colombeau distributional connections $(\Gamma^{+l}_{kj}(\varepsilon))_{\varepsilon}$ and $(\Gamma^{-l}_{kj}(\varepsilon))$ above and below Schwarzschild horizon are [18]-[19].

Obviously distributional connections $([\Gamma^{+l}_{kj}[h^{+}_{\varepsilon}])_{\varepsilon}, ([\Gamma^{-l}_{kj}[h^{-}_{\varepsilon}])_{\varepsilon}$ coincides, in distributional sense, with the corresponding classical Levi-Cività connections on $\mathbb{R}^3 \backslash \{r = 2m\}$, since $(h^{+}_{\varepsilon})_{\varepsilon} = h^{+}_{0}, (h^{-}_{\varepsilon})_{\varepsilon} = h^{-}_{0}$, and $(g^{+lm}_{\varepsilon})_{\varepsilon} = g^{+lm}_{0}$, $(g^{-lm}_{\varepsilon})_{\varepsilon} = g^{-lm}_{0}$ there. Clearly, connections $\Gamma^{+l}_{kj}(\epsilon), \Gamma^{-l}_{kj}(\epsilon), \epsilon \in (0,1]$ in respect the regularized metric $g^{\pm}_{\epsilon}, \epsilon \in (0,1]$, i.e., $(g^{\pm}_{\epsilon})_{ij;k} = 0$. Proceeding in this manner, we obtain the nonstandard result [23]-[24]:

$$\left[\left([R^{+}_{\epsilon}]^{1}_{1}\right)_{\epsilon}\right] = \left[\left([R^{+}_{\epsilon}]^{0}_{0}\right)_{\varepsilon}\right] = -4\pi m \frac{\delta(r-2m)}{r^2},$$
$$\left[\left([R^{-}_{\epsilon}]^{1}_{1}\right)_{\epsilon}\right] = \left[\left([R^{-}_{\epsilon}]^{0}_{0}\right)_{\varepsilon}\right] = 4\pi m \frac{\delta(r-2m)}{r^2}. \qquad (2.2.3)$$

Remark 2.2.3. As expected, the distributional Ricci tensor as well as the distributional Ricci scalar vanish identically on $\mathbb{R}^3 \backslash \{r = 2m\}$, since $\mathrm{supp}(\delta(r-2m)) = \{r = 2m\}$. This result in a good agrement with canonical result [25]-[31] on $\mathbb{R}^3 \backslash \{r = 2m\}$ since distributional connections (2.2.2) coincides with the corresponding classical Levi-Cività connections on $\mathbb{R}^3 \backslash \{r = 2m\}$ at least in distributional sense. We obtain for $r \asymp 2m$ the nonstandard result [18]-[19]:

$$(\mathbf{R}^{\pm\mu\nu}(\varepsilon)\mathbf{R}^{\pm}_{\mu\nu}(\varepsilon))_{\varepsilon} \asymp \left(\frac{c_1}{16m^2\left[(r-2m)^2+\varepsilon^2\right]}\right)_{\varepsilon} + \ldots,$$
$$(\mathbf{R}^{\pm\rho\sigma\mu\nu}(\varepsilon)\mathbf{R}^{\pm}_{\rho\sigma\mu\nu}(\varepsilon))_{\varepsilon} \asymp \left(\frac{c_2}{16m^2\left[(r-2m)^2+\varepsilon^2\right]}\right)_{\varepsilon} + \ldots, \tag{2.2.4}$$

where $c_1, c_2 = O(1)$.

2.2.2.Distributional Rindler spacetime with distributional Levi-Cività connection.

We remind now that $2D$ Rindler spacetime is a patch of Minkowski spacetime. In $2D$, the Rindler metric is

$$ds^2 = dR^2 - R^2 d\eta^2. \tag{2.2.5}$$

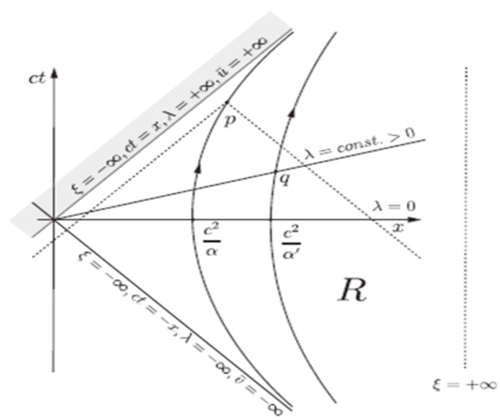

Fig.2.1.Hyperbolic motion in the right Rindler wedge.
$x^2 - c^2t^2 = (c^2/a)^2$.

Remark 2.1.Due to the degeneracy of the metric (2.2.5) at Rindler gorizon $R = 0$, the classical Levi-Cività connection is not available on whole $\mathbb{R}^2$,e.g.,

$$\Gamma^1_{44} = R, \Gamma^4_{14} = \Gamma^4_{41} = R^{-1}, \tag{2.2.6}$$

and all other components being zero.

Remark 2.2.4.Note that in order to avoid this difficultnes, the origin in classical consideration is always excluded from the space $\mathbb{R}^{3.1}$ and we are working on $\mathbb{R}^{3.1}\setminus\{0\}$ following Moller [24] we get

$$G^k_i = \mathbf{R}^k_i - \frac{1}{2}\delta^k_i\mathbf{R}, \mathbf{R} \triangleq \mathbf{R}^i_i \tag{2.2.7}$$

$$\mathbf{G}_2^2 = \mathbf{G}_3^3 = -\frac{1}{2g_{44}}\left[g''_{44} - \frac{(g'_{44})^2}{2g_{44}}\right] = -\frac{1}{2R^2}\left[2 - \frac{(2R)^2}{2R^2}\right] \equiv 0, \quad (2.2.8)$$

where the accents indicate differentiation with respect variable R, and all other components of $\mathbf{G}_i^k$ vanish identically. Thus Rindler metrical tensor satisfy on $\mathbb{R}^{3,1}\backslash\{0\}$ the

$$\mathbf{G}_i^k \triangleq \mathbf{R}_i^k - \tfrac{1}{2}\delta_i^k\mathbf{R} = 0. \quad (2.2.9)$$

Remark 2.2.5. By calculations mentioned above, from Møller's times until nowdays,Rindler metrical tensor was mistakenly considered in physical literature as an vacuum solution of the Einstein's field equations,e.g.,solution for empty space,see Møller [24].

Remark 2.2.6.Note that Levi-Cività connection on whole space $\mathbb{R}^{3.1}$ is available only in Colombeau sense under smooth regularization $R^2 \to R^2 + \varepsilon^2, \varepsilon \in (0,1]$ and therefore we forced to change metric (2.5) by Colombeau object

$$\begin{gathered}[(ds_\varepsilon^2)_\varepsilon] = dR^2 - [(R^2+\varepsilon^2)_\varepsilon]dt^2 = dR^2 - [(g_{44,\varepsilon})_\varepsilon]dt^2 \\ [(g_{44,\varepsilon})_\varepsilon] = [(R^2+\varepsilon^2)_\varepsilon], \varepsilon \in (0,1].\end{gathered} \quad (2.2.10)$$

Then for Einstein distributional tensor [18]-[19],[21]:

$$(\mathbf{G}_{i,\varepsilon}^k)_\varepsilon \triangleq (\mathbf{R}_{i,\varepsilon}^k)_\varepsilon - \tfrac{1}{2}\delta_i^k(\mathbf{R}_\varepsilon)_\varepsilon, (\mathbf{R}_\varepsilon)_\varepsilon \triangleq \mathbf{R}_{i,\varepsilon}^i \quad (2.2.11)$$

we get

$$(\mathbf{G}_{2,\varepsilon}^2(R))_\varepsilon = (\mathbf{G}_{3,\varepsilon}^3(R))_\varepsilon = -\left(\frac{1}{2g_{44,\varepsilon}}\left[g''_{44,\varepsilon} - \frac{(g'_{44,\varepsilon})^2}{2g_{44,\varepsilon}}\right]\right)_\varepsilon = -\left(\frac{\varepsilon^2}{(R^2+\varepsilon^2)^2}\right)_\varepsilon. \quad (2.2.12)$$

Thus

$$\left[(\mathbf{G}_{2,\varepsilon}^2(0))_\varepsilon\right] = \left[(\mathbf{G}_{3,\varepsilon}^3(0))_\varepsilon\right] = [(\varepsilon^{-2})_\varepsilon], \quad (2.2.13)$$

where $[(\varepsilon^{-2})_\varepsilon] \in \widetilde{\mathbb{R}}$ is infinite Colombeau generalized numbers, and therefore $(\mathbf{G}_{2,\varepsilon}^2(R))_\varepsilon$ and $(\mathbf{G}_{3,\varepsilon}^3(R))_\varepsilon$ is nontrivial Colombeau generalized functions and distributional Rindler metric tensor given by (2.2.12) that is non vacuum Colombeau solution of the Einstein field equations.

2.2.3. Generalized Einstein equivalence principle.

We remind that originally Einstein's gravity was formulated by using classical pseudo Riemannian geometry with classical Levi-Civit'a connection.In classical pseudo Riemannian geometry, the Levi-Civita connection is a specific connection on the tangent bundle of a manifold. More specifically, it is the torsion-free metric connection, i.e., the torsion-free connection on the tangent bundle (an affine connection) preserving a given (pseudo-Riemannian) Riemannian metric.The fundamental theorem of classical Riemannian geometry states that there is a unique connection which satisfies these properties.

Remark 2.3.1.Note that classical Einstein "Equivalence Principle" asserts the equivalence between inertial and gravitational forces of acceleration. The classical Einstein equivalence principle is the heart and soul of gravitational theory, for it is possible to argue convincingly that if EEP is valid, then gravitation must be a "curved spacetime" phenomenon, in other words, gravity must be governed by a "metric theory of gravity", whose postulates are:

1. Spacetime is endowed with a symmetric Lorentzian metric.
2. The trajectories of freely falling test bodies are geodesics of that metric.
3. In local freely falling reference frames, the non-gravitational laws of physics are those written in the language of special relativity.

In order to obtain appropriate generalization of EEP based on distributional Colombeau geometry [4]-[7] we claim the following generalized equivalence principle (GEEP):

1. Spacetime in general case is endowed with a symmetric distributional Lorentzian metric.
2. The trajectories of freely falling test bodies are geodesics of that distributional metric.
3. In local freely falling distributional reference frames, the non-gravitational laws of physics are those written in the language of special relativity.

3. Quantum scalar field in curved distributional spacetime. Unruh effect revisited.

3.1.Canonical quantization in curved distributional spacetime

In a recent work [19] the authors advocated the use De Witt-Schwinger approach [31]-[33] in order to establish QFT in general ditributional curved spacetime. The vacuum energy density of free scalar quantum field Φ with a distributional background spacetime is considered successfully. It has been widely believed that, except in very extreme situations, the influence of gravity on quantum fields should amount to just small, sub-dominant contributions. Here we argue that this belief is false by showing that there exist well-behaved spacetime evolutions where the vacuum energy density of free quantum fields is forced, by the very same background distributional spacetime such BHs, to become dominant over any classical energydensity component. This semiclassical gravity effect finds its roots in the singular behavior of quantum fields on curved distributional spacetimes. In particular we obtain that the vacuum fluctuations $\langle\Phi^2\rangle$ has a singular behavior on BHs horizon r_+ : $\langle\Phi^2(r)\rangle \sim |r-r_+|^{-2}$.

Much of formalism can be explained with Colombeau generalized scalar field

[19].The basic concepts and methods extend straightforwardly to distributional tensor and distributional spinor fields. To being with let's take a spacetime of arbitrary dimension D, with a metric $g_{\mu\nu}$ of signature$(+-\ldots-)$. The action for the Colombeau generalized scalar field $(\varphi_\varepsilon)_\varepsilon \in \mathbf{G}(M)$ is

$$(S_\varepsilon)_\varepsilon = \left(\int\limits_M d^D x \frac{1}{2}\sqrt{|g_\varepsilon|}\,(g_\varepsilon^{\mu\nu}\partial_\mu\varphi_\varepsilon\partial_\nu\varphi_\varepsilon) - (m^2 + \xi R_\varepsilon)\varphi_\varepsilon^2\right)_\varepsilon. \quad (3.1.1)$$

Here ξ is a coupling constant (see [41] chapter 3). The corresponding equation of motion is

$$([\Box_{\varepsilon,x} + m^2 + \xi R_\varepsilon]\varphi_\varepsilon)_\varepsilon, \varepsilon \in (0,1]. \quad (3.1.2)$$

Here

$$(\Box_{\varepsilon,x}\varphi_\varepsilon)_\varepsilon = \left(|g_\varepsilon|^{-1/2}\partial_\mu|g_\varepsilon|^{1/2}g_\varepsilon^{\mu\nu}\partial_\mu\varphi_\varepsilon\right)_\varepsilon. \quad (3.1.3)$$

With $\hbar$ explicit, the mass m should be replaced by $m/\hbar$. Separating out a time coordinate x^0, $x^\mu = (x^0, x^i), i = 1,2,3$ we can write the action as

$$(S_\varepsilon)_\varepsilon = \left(\int dx^0 L_\varepsilon\right)_\varepsilon, (L_\varepsilon)_\varepsilon = \left(\int d^{D-1}x\mathcal{L}_\varepsilon\right)_\varepsilon. \quad (3.1.4)$$

The canonical momentum at a time x^0 is given by

$$(\pi_\varepsilon(\underline{x}))_\varepsilon = (\delta L_\varepsilon/\delta(\partial_0\varphi_\varepsilon(\underline{x})))_\varepsilon = (|h_\varepsilon|^{1/2}n^\mu\partial_\mu\varphi_\varepsilon(\underline{x}))_\varepsilon, \quad (3.1.5)$$

where $\underline{x}$ labels a point on a surface of constant x^0, the x^0 argument of $(\varphi_\varepsilon)_\varepsilon$ is suppressed, n^μ is the unit normal to the surface, and $(|h_\varepsilon|)_\varepsilon$ is the determinant of the induced spatial metric $(h_{ij}(\varepsilon))_\varepsilon$. In order to quantize, the Colombeau generalized field $(\varphi_\varepsilon)_\varepsilon$ and its conjugate momentum $(\pi_\varepsilon(\underline{x}))_\varepsilon$ are now promoted to hermitian operators and required to satisfy the canonical commutation relation,

$$([\varphi_\varepsilon(\underline{x}), \pi_\varepsilon(\underline{y})])_\varepsilon = i\hbar\delta^{D-1}(\underline{x},\underline{y}), \varepsilon \in (0,1]. \quad (3.1.6)$$

Here $\int d^{D-1}y\delta^{D-1}(\underline{x},\underline{y})f(\underline{y}) = f(\underline{x})$ for any scalar function $f \in D(\mathbb{R}^3)$, without the use of a metric volume element. We form now a conserved bracket from two complex Colombeau solutions to the scalar wave equation (3.1.2) by [19]:

$$(\langle\varphi_\varepsilon, \phi_\varepsilon\rangle)_\varepsilon = \left(\int\limits_\Sigma d\Sigma_\mu j_\varepsilon^\mu\right)_\varepsilon, \varepsilon \in (0,1], \quad (3.1.7)$$

where

$$(j_\varepsilon^\mu(\varphi_\varepsilon, \phi_\varepsilon))_\varepsilon = (i/\hbar)\left(|g_\varepsilon|^{1/2}g_\varepsilon^{\mu\nu}(\overline{\varphi}_\varepsilon\partial_\nu\phi_\varepsilon - \varphi_\varepsilon\partial_\nu\overline{\phi}_\varepsilon)\right)_\varepsilon. \quad (3.1.8)$$

Using equation of motion Eq.(3.1.2) one obtains corresponding Colombeau generalization of the canonical Green functions equations. In particular for the Colombeau distributional propagator

$$i(G_\varepsilon^\pm(x,x'))_\varepsilon = (\langle 0|T(\varphi_\varepsilon^\pm(x)\varphi_\varepsilon^\pm(x'))|0\rangle)_\varepsilon, \varepsilon \in (0,1], \quad (3.1.9)$$

one obtains directly

$$([\Box_{\varepsilon,x}+m^2+\xi\mathbf{R}^{\pm}(x,\varepsilon)]G^{\pm}_{\varepsilon}(x,x'))_{\varepsilon}=-\Big([-g^{\pm}(x,\varepsilon)]^{-1/2}\Big)_{\varepsilon}\delta^n(x-x'). \qquad (3.1.10)$$

We obtan now an adiabatic expansion of $(G^{\pm}_{\varepsilon}(x,x'))_{\varepsilon}$ [19]. Introducing Riemann normal coordinates y^{μ} for the point $x,$ with origin at the point x' one obtans

$$\begin{aligned}(g^{\pm}_{\mu\upsilon}(x,\varepsilon))_{\varepsilon}&=\eta_{\mu\upsilon}+\frac{1}{3}\Big[(\mathbf{R}^{\pm}_{\mu\alpha\upsilon\beta}(\varepsilon))_{\varepsilon}\Big]y^{\alpha}y^{\beta}-\frac{1}{6}\Big[(\mathbf{R}^{\pm}_{\mu\alpha\upsilon\beta;\gamma}(\varepsilon))_{\varepsilon}\Big]y^{\alpha}y^{\beta}y^{\gamma}+\\&+\Big[\frac{1}{20}(\mathbf{R}^{\pm}_{\mu\alpha\upsilon\beta;\gamma\delta}(\varepsilon))_{\varepsilon}+\frac{2}{45}\Big[(\mathbf{R}^{\pm}_{\alpha\mu\beta\lambda}(\varepsilon))_{\varepsilon}\Big](\mathbf{R}^{\pm\lambda}_{\gamma\upsilon\delta}(\varepsilon))_{\varepsilon}\Big]y^{\alpha}y^{\beta}y^{\gamma}y^{\delta}+\ldots\end{aligned} \qquad (3.1.11)$$

where $\eta_{\mu\upsilon}$ is the Minkowski metric tensor, and the coefficients are all evaluated at $y=0.$ Defining now

$$(\mathcal{L}^{\pm}_{\varepsilon}(x,x'))_{\varepsilon}=\Big[\Big((-g^{\pm}_{\mu\upsilon}(x,\varepsilon))^{1/4}\Big)_{\varepsilon}\Big](G^{\pm}_{\varepsilon}(x,x'))_{\varepsilon} \qquad (3.1.12)$$

and its Colombeau-Fourier transform $(\mathcal{L}^{\pm}_{\varepsilon}(k))_{\varepsilon}$ by

$$(\mathcal{L}^{\pm}_{\varepsilon}(x,x'))_{\varepsilon}=(2\pi)^{-n}\Big(\int d^nke^{-ik\cdot y}\mathcal{L}^{\pm}_{\varepsilon}(k)\Big)_{\varepsilon} \qquad (3.13)$$

where $k\cdot y=\eta^{\alpha\beta}k_{\alpha}y_{\beta},$ one can work in a sort of localized momentum space. Expanding (3.1.10) in normal coordinates and converting to k-space, $(\mathcal{L}^{\pm}_{\varepsilon}(k))_{\varepsilon}$ can readily be solved by iteration to any adiabatic order. The result to adiabatic order four (i.e., four derivatives of the metric) is

$$\begin{aligned}(\mathcal{L}^{\pm}_{\varepsilon}(k))_{\varepsilon}&=(k^2-m^2)^{-1}-\Big(\frac{1}{6}-\xi\Big)(k^2-m^2)^{-2}(\mathbf{R}^{\pm}(\varepsilon))_{\varepsilon}+\\&+\frac{i}{2}\Big(\frac{1}{6}-\xi\Big)\partial^{\alpha}(k^2-m^2)^{-2}(\mathbf{R}^{\pm}_{;\alpha}(\varepsilon))_{\varepsilon}-\\&-\frac{1}{3}\Big[(a^{\pm}_{\alpha\beta}(\varepsilon))_{\varepsilon}\Big]\partial^{\alpha}\partial^{\beta}(k^2-m^2)^{-2}+\\&\Big[\Big(\frac{1}{6}-\xi\Big)^2(\mathbf{R}^{\pm2}(\varepsilon))_{\varepsilon}+\frac{2}{3}\Big(a^{\pm\lambda}_{\ \lambda}(\varepsilon)\Big)_{\varepsilon}\Big](k^2-m^2)^{-3},\end{aligned} \qquad (3.1.14)$$

where $\partial_{\alpha}=\partial/\partial k^{\alpha}$,

$$\begin{aligned}(a^{\pm}_{\alpha\beta}(\varepsilon))_{\varepsilon}&\asymp\Big(\frac{1}{2}-\xi\Big)(\mathbf{R}^{\pm}_{;\alpha\beta}(\varepsilon))_{\varepsilon}+\frac{1}{120}(\mathbf{R}^{\pm}_{;\alpha\beta}(\varepsilon))_{\varepsilon}-\frac{1}{140}\Big(\mathbf{R}^{\pm}_{\alpha\beta;\lambda}{}^{\lambda}(\varepsilon)\Big)_{\varepsilon}-\\&-\frac{1}{30}\Big[\Big(\mathbf{R}^{\pm\lambda}_{\alpha}(\varepsilon)\Big)_{\varepsilon}\Big](\mathbf{R}^{\pm}_{\lambda\beta}(\varepsilon))_{\varepsilon}+\frac{1}{60}\Big[\Big(\mathbf{R}^{\pm\kappa\ \lambda}_{\ \alpha\ \beta}(\varepsilon)\Big)_{\varepsilon}\Big](\mathbf{R}^{\pm}_{\kappa\lambda}(\varepsilon))_{\varepsilon}+\\&+\frac{1}{60}\Big[\Big(\mathbf{R}^{\pm\lambda\mu\kappa}{}_{\alpha}(\varepsilon)\Big)_{\varepsilon}\Big](\mathbf{R}^{\pm}_{\lambda\mu\kappa\beta}(\varepsilon))_{\varepsilon},\end{aligned} \qquad (3.1.15)$$

and we are using the symbol $\asymp$ to indicate that this is an asymptotic expansion. One ensures that Eq.(3.1.13) represents a time-ordered product by performing the k^0 integral along the appropriate contour in Fig.3.1.1.This is equivalent to replacing m^2 by $m^2-i\epsilon.$ Similarly, the adiabatic expansions of other Green functions can be obtained by using the other contours in Fig.3.1.1.Substituting Eq.(3.1.14) into Eq.(3.1.13) gives [19]

$$
\begin{aligned}
&(\mathcal{L}_{\varepsilon}^{\pm}(x,x'))_{\varepsilon} = \\
&(2\pi)^{-n} \times \left(\int d^n k e^{-iky} (k^2 - m^2)^{-1} \left[a_0^{\pm}(x,x';\varepsilon) + a_1^{\pm}(x,x';\varepsilon) \left(-\frac{\partial}{\partial m^2} \right) + \right. \right. \\
&\left. \left. a_2^{\pm}(x,x';\varepsilon) \left(\frac{\partial}{\partial m^2} \right)^2 \right] \right)_{\varepsilon},
\end{aligned}
\tag{3.1.16}
$$

where $(a_0^{\pm}(x,x';\varepsilon))_{\varepsilon} = 1$ and, to adiabatic order 4,

$$
\begin{cases}
(a_1^{\pm}(x,x';\varepsilon))_{\varepsilon} = \\
\left(\frac{1}{6} - \xi\right)(\mathbf{R}^{\pm}(\varepsilon))_{\varepsilon} - \frac{i}{2}\left(\frac{1}{6} - \xi\right)[(\mathbf{R}^{\pm}_{;\alpha}(\varepsilon))_{\varepsilon}]y^{\alpha} - \frac{1}{3}\left[(a^{\pm}_{\alpha\beta}(\varepsilon))_{\varepsilon}\right]y^{\alpha}y^{\beta} \\
(a_2^{\pm}(x,x';\varepsilon))_{\varepsilon} = \frac{1}{2}\left(\frac{1}{6} - \xi\right)(\mathbf{R}^{\pm 2}(\varepsilon))_{\varepsilon} + \frac{1}{3}\left(a^{\pm\lambda}_{\ \ \lambda}(\varepsilon)\right)_{\varepsilon}
\end{cases}
\tag{3.1.17}
$$

with all geometric quantities on the right-hand side of Eq.(3.1.17) evaluated at x'.

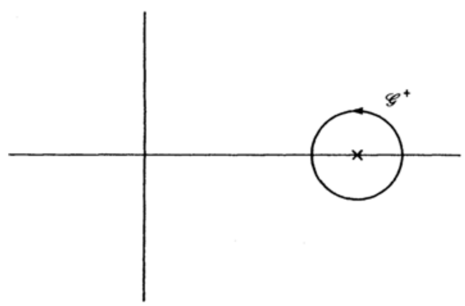

Fig.3.1.1.The contour in the complex k^0 plane $\mathbb{C}$ to be used in the evaluation of the integral giving $\mathcal{L}^{+}$.The cross indicates the pole at $k^0 = (|\mathbf{k}|^2 + m^2)^{1/2}$.

in Eq.(3.16), then the $d^n k$ integration may be interchanged with the ds integration, and performed explicitly to yield (dropping the $i\epsilon$)

$$
\begin{aligned}
&(\mathcal{L}_{\varepsilon}^{\pm}(x,x'))_{\varepsilon} = -i(4\pi)^{-n/2}\left(\int_0^{\infty} ids(is)^{-n/2}\exp\left[-im^2 s + \frac{\sigma(x,x')}{2is}\right]\mathcal{F}_{\varepsilon}^{\pm}(x,x';is)\right)_{\varepsilon} \\
&\sigma(x,x') = \frac{1}{2}y_{\alpha}y^{\alpha}.
\end{aligned}
\tag{3.1.18}
$$

The function $\sigma(x,x')$ which is one-half of the square of the proper distance between x and x', while the function $(\mathcal{F}_{\varepsilon}(x,x';is))_{\varepsilon}$ has the following asymptotic adiabatic expansion

$$
(\mathcal{F}_{\varepsilon}^{\pm}(x,x';is))_{\varepsilon} \asymp (a_0^{\pm}(x,x';\varepsilon))_{\varepsilon} + is(a_1^{\pm}(x,x';\varepsilon))_{\varepsilon} + (is)^2(a_2^{\pm}(x,x';\varepsilon))_{\varepsilon} + \ldots \tag{3.1.19}
$$

Using Eq.(3.1.12), equation (3.1.18) gives a representation of $(G_\varepsilon^\pm(x,x'))_\varepsilon$:

$$(G_\varepsilon^\pm(x,x'))_\varepsilon = -i(4\pi)^{-n/2}\left(\left[\left(\Delta_\pm^{1/2}(x,x';\varepsilon)\right)_\varepsilon\right]\int_0^\infty ids(is)^{-n/2}\exp\left[-im^2s+\frac{\sigma(x,x')}{2is}\right]\mathcal{F}_\varepsilon(x,x';is)\right)_\varepsilon \quad (3.1.20)$$

where $(\Delta_\pm(x,x';\varepsilon))_\varepsilon$ is the distributional Van Vleck determinant

$$(\Delta_\pm(x,x';\varepsilon))_\varepsilon = -\det[\partial_\mu\partial_\nu\sigma(x,x')]\left([g^\pm(x,\varepsilon)g^\pm(x',\varepsilon)]^{-1/2}\right)_\varepsilon. \quad (3.1.21)$$

In the normal coordinates about x' that we are currently using, $(\Delta_\pm(x,x';\varepsilon))_\varepsilon$ reduces to $\left([-g^\pm(x,\varepsilon)]^{-1/2}\right)_\varepsilon$. The full asymptotic expansion of $(\mathcal{F}_\varepsilon^\pm(x,x';is))_\varepsilon$ to all adiabatic orders are

$$(\mathcal{F}_\varepsilon^\pm(x,x';is))_\varepsilon \asymp \sum_{j=0}^\infty (is)^j(a_2^\pm(x,x';\varepsilon))_\varepsilon \quad (3.1.22)$$

with $(a_0^\pm(x,x';\varepsilon))_\varepsilon = 1$, the other $(a_j^\pm(x,x';\varepsilon))_\varepsilon$ being given by canonical recursion relations which enable their adiabatic expansions to be obtained.

Remark 3.1.1.Note that the expansions (3.1.19) and (3.1.22) are, however, only asymptotic approximations in the limit of large adiabatic parameter T.

If (3.1.22) is substituted into (3.1.20) the integral can be performed to give the adiabatic expansion of the Feynman propagator in coordinate space:

$$(G_\varepsilon^\pm(x,x'))_\varepsilon \asymp -(4\pi i)^{-n/2}\left(\Delta_\pm^{1/2}(x,x';\varepsilon)\sum_{j=0}^\infty a_j^\pm(x,x';\varepsilon)\left(-\frac{\partial}{\partial m^2}\right)^j \times \times\left[\left(-\frac{2m^2}{\sigma}\right)^{\frac{n-2}{4}}H_{(n-2)/2}^{(2)}\left((2m^2\sigma)^{\frac{1}{2}}\right)\right]\right)_\varepsilon \quad (3.1.23)$$

which, strictly, a small imaginary part $i\epsilon$ should be subtracted from σ.

Remark 3.1.2.Since we have not imposed global boundary conditions on the distributional Green function Colombeau solution of (3.1.10), the expansion (3.1.23) does not determine the particular vacuum state in (3.1.9). In particular, the "$i\epsilon$" in the expansion of $(G_\varepsilon^\pm(x,x'))_\varepsilon$ only ensures that (3.1.23) represents the expectation value, in some set of states, of a time-ordered product of fields. Under some circumstances the use of "$i\epsilon$" in the exact representation (3.1.20) may give additional information concerning the global nature of the states.

3.2.Effective action for the quantum matter fields in curved distributional spcetime

As in classical case one can obtain Colombeau generalized quantity $(W_\varepsilon)_\varepsilon$, called the effective action for the quantum matter fields in curved distributional spcetime, which, when functionally differentiated,yields

$$\left(\frac{2}{(-g(\varepsilon))^{\frac{1}{2}}}\frac{\delta W_{\varepsilon}}{\delta g^{\mu\nu}(\varepsilon)}\right)_{\varepsilon} = (\langle \mathbf{T}_{\mu\nu}(\varepsilon)\rangle)_{\varepsilon} \tag{3.2.1}$$

Note that the generating functional

$$(Z_{\varepsilon}[\mathbf{J}_{\varepsilon}])_{\varepsilon} = \left(\int D[\varphi_{\varepsilon}]\exp\left\{iS_{\mathbf{m}}(\varepsilon) + i\int \mathbf{J}_{\varepsilon}(x)\varphi_{\varepsilon}(x)d^{n}x\right\}\right)_{\varepsilon} \tag{3.2.2}$$

was interpreted physically as the vacuum persistence amplitude $(\langle \mathbf{out}_{\varepsilon},0|0,\mathbf{in}_{\varepsilon}\rangle)_{\varepsilon}$. The presence of the external distributional current density $(\mathbf{J}_{\varepsilon})_{\varepsilon}$ can cause the initial vacuum state $(|0,\mathbf{in}_{\varepsilon}\rangle)_{\varepsilon}$ to be unstable, i.e., it can bring about the production of particles.

Following canonical calculation one obtains [19]

$$(Z_{\varepsilon}^{\pm}[0])_{\varepsilon} \propto \left([\det(-G_{\varepsilon}^{\pm}(x,x'))]^{\frac{1}{2}}\right)_{\varepsilon} \tag{3.2.3}$$

where the proportionality constant is metric-independent and can be ignored. Thus we obtain

$$(W_{\varepsilon}^{\pm})_{\varepsilon} = -i(\ln Z_{\varepsilon}^{\pm}[0])_{\varepsilon} = -\frac{i}{2}\left(\mathbf{tr}\left[\ln\left(-\hat{G}_{\varepsilon}^{\pm}\right)\right]\right)_{\varepsilon}. \tag{3.2.4}$$

In (3.2.4) $\left(\hat{G}_{\varepsilon}^{\pm}\right)_{\varepsilon}$ is to be interpreted as an Colombeau generalized operator which acts on an linear space $\Im$ of generalized vectors $|x,\varepsilon\rangle, \varepsilon \in (0,1]$ normalized by

$$(\langle x,\varepsilon|x',\varepsilon\rangle)_{\varepsilon} = \delta(x-x')\left([-g^{\pm}(x,\varepsilon)]^{-\frac{1}{2}}\right)_{\varepsilon} \tag{3.2.5}$$

in such a way that

$$(G_{\varepsilon}^{\pm}(x,x'))_{\varepsilon} = \left(\langle x,\varepsilon|\hat{G}_{\varepsilon}^{\pm}|x',\varepsilon\rangle\right)_{\varepsilon}. \tag{3.2.6}$$

Remark 3.2.1.Note that the trace $(\mathbf{tr}[\cdot])_{\varepsilon}$ of an Colombeau generalized operator $(\Re_{\varepsilon})_{\varepsilon}$ which acts on a linear space $\Im$, is defined by

$$(\mathbf{tr}[\Re_{\varepsilon}])_{\varepsilon} = \left(\int d^{n}x[-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}}\Re_{xx;\varepsilon}\right)_{\varepsilon} = \left(\int d^{n}x[-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}}\langle x|\Re_{xx;\varepsilon}|x'\rangle\right)_{\varepsilon}. \tag{3.2.7}$$

Writing now the Colombeau generalized operator $\left(\hat{G}_{\varepsilon}^{\pm}\right)_{\varepsilon}$ as

$$\left(\hat{G}_{\varepsilon}^{\pm}\right)_{\varepsilon} = -(\mathcal{F}_{\varepsilon}^{\pm-1})_{\varepsilon} = -i\left(\int_{0}^{\infty} ds\exp[-s\mathcal{F}_{\varepsilon}^{\pm}]\right)_{\varepsilon}, \tag{3.2.8}$$

by Eq.(3.1.20) we obtain

$$\begin{gathered}(\langle x|\exp[-s\mathcal{F}_{\varepsilon}^{\pm}]|x'\rangle) = \\ i(4\pi)^{-n/2}\left[\left(\Delta_{\pm}^{1/2}(x,x';\varepsilon)\right)_{\varepsilon}\right]\exp\left[-im^{2}s + \frac{\sigma(x,x')}{2is}\right]\mathcal{F}_{\varepsilon}^{\pm}(x,x';is)(is)^{-n/2}.\end{gathered} \tag{3.2.9}$$

Proceeding in standard manner we get [19]

$$(W^{\pm}_{\varepsilon})_{\varepsilon} = \frac{i}{2}\left[\left(\int d^n x[-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}}\right)_{\varepsilon}\right]\left(\lim_{x\to x'}\int_{m^2}^{\infty} G^{\pm}_{\varepsilon}(x,x';m^2)dm^2\right)_{\varepsilon}. \quad (3.2.10)$$

Interchanging now the order of integration and taking the limit $x \to x'$ one obtains

$$(W^{\pm}_{\varepsilon})_{\varepsilon} = \frac{i}{2}\left(\int_{m^2}^{\infty} dm^2 \int d^n x[-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}} G^{\pm}_{\varepsilon}(x,x;m^2)\right)_{\varepsilon}. \quad (3.2.11)$$

Colombeau generalized quantity $(W^{\pm}_{\varepsilon})_{\varepsilon}$ is colled as the one-loop effective action. In the case of fermion effective actions, there would be a remaining trace over spinorial indices. From Eq.(3.2.11) we may define an effective Lagrangian density $(L^{\pm}_{\varepsilon;\mathbf{eff}}(x))_{\varepsilon}$ by

$$(W^{\pm}_{\varepsilon})_{\varepsilon} = \left(\int d^n x[-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}} L^{\pm}_{\varepsilon;\mathbf{eff}}(x)\right)_{\varepsilon} \quad (3.2.12)$$

whence one get

$$(L^{\pm}_{\varepsilon}(x))_{\varepsilon} = \left([-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}} \mathcal{L}^{\pm}_{\varepsilon;\mathbf{eff}}(x)\right)_{\varepsilon} = \frac{i}{2}\left(\lim_{x\to x'}\int_{m^2}^{\infty} dm^2 G^{\pm}_{\varepsilon}(x,x';m^2)\right)_{\varepsilon}. \quad (3.2.13)$$

3.3. Stress-tensor renormalization

Note that $(L^{\pm}_{\varepsilon}(x))_{\varepsilon}$ diverges at the lower end of the s integral because the $\sigma/2s$ damping factor in the exponent vanishes in the limit $x \to x'$. (Convergence at the upper end is guaranteed by the $-i\epsilon$ that is implicitly added to m^2 in the De Witt-Schwinger representation of $(L^{\pm}_{\varepsilon}(x))_{\varepsilon}$. In four dimensions, the potentially divergent terms in the DeWitt- Schwinger expansion of $(L^{\pm}_{\varepsilon}(x))_{\varepsilon}$ are

$$\begin{aligned}(L^{\pm}_{\varepsilon;\mathrm{div}}(x))_{\varepsilon} &= \\ -(32\pi^2)^{-1}&\left(\lim_{x\to x'}\left[\left(\Delta^{1/2}_{\pm}(x,x';\varepsilon)\right)_{\varepsilon}\right]\int_0^{\infty}\frac{ds}{s^3}\exp\left[-im^2 s + \frac{\sigma(x,x')}{2is}\right]\times\right. \\ &\left.\times\left[a^{\pm}_0(x,x';\varepsilon) + isa^{\pm}_1(x,x';\varepsilon) + (is)^2 a^{\pm}_2(x,x';\varepsilon)\right]\right)_{\varepsilon}\end{aligned} \quad (3.3.1)$$

where the coefficients $a^{\pm}_0$, $a^{\pm}_1$ and $a^{\pm}_2$ are given by Eq.(3.1.17).The remaining terms in this asymptotic expansion, involving $a^{\pm}_3$ and higher, are finite in the limit $x \to x'$.

Let us determine now the precise form of the geometrical $(L^{\pm}_{\varepsilon;\mathrm{div}}(x))_{\varepsilon}$ terms, to compare them with the distributional generalization of the gravitational Lagrangian that appears in [19]. This is a delicate matter because (3.3.1) is, of course, infinite. What we require is to display the divergent terms in the form $\infty \times$ [geometrical object]. This can be done in a variety of ways. For example, in n dimensions, the asymptotic (adiabatic) expansion of $(L^{\pm}_{\varepsilon;\mathbf{eff}}(x))_{\varepsilon}$ is

$$
\begin{aligned}
&(L^{\pm}_{\varepsilon;\mathbf{eff}}(x))_{\varepsilon} \asymp \\
&2^{-1}(4\pi)^{-n/2}\left(\lim_{x\to x'}\left[\left(\Delta^{1/2}_{\pm}(x,x';\varepsilon)\right)_{\varepsilon}\right]\sum_{j=0}^{\infty} a_j(x,x';\varepsilon)\times\right. \\
&\left.\times\int_0^{\infty} ids(is)^{j-1-n/2}\exp\left[-im^2 s+\frac{\sigma(x,x')}{2is}\right]\right)_{\varepsilon}
\end{aligned} \tag{3.3.2}
$$

of which the first $n/2+1$ terms are divergent as $\sigma \to 0$. If n is treated as a variable which can be analytically continued throughout the complex plane, then we may take the $x \to x'$ limit

$$
\begin{aligned}
&(L^{\pm}_{\varepsilon;\mathbf{eff}}(x))_{\varepsilon} \asymp 2^{-1}(4\pi)^{-n/2}\left(\sum_{j=0}^{\infty} a_j(x;\varepsilon)\int_0^{\infty} ids(is)^{j-1-n/2}\exp[-im^2 s]\right)_{\varepsilon} = \\
&2^{-1}(4\pi)^{-n/2}\sum_{j=0}^{\infty} a_j(x;\varepsilon)(m^2)^{n/2-j}\Gamma\left(j-\frac{n}{2}\right), a_j(x;\varepsilon) = a_j(x,x;\varepsilon).
\end{aligned} \tag{3.3.3}
$$

From Eq.(3.3.3) follows we shall wish to retain the units of $L^{\pm}_{\varepsilon;\mathbf{eff}}(x)$ as (length)$^{-4}$, even when $n \neq 4$. It is therefore necessary to introduce an arbitrary mass scale μ and to rewrite Eq.(3.3.3) as

$$
(L^{\pm}_{\varepsilon;\mathbf{eff}}(x))_{\varepsilon} \asymp 2^{-1}(4\pi)^{-n/2}\left(\frac{m}{\mu}\right)^{n-4}\left(\sum_{j=0}^{\infty} a_j(x;\varepsilon)(m^2)^{4-2j}\Gamma\left(j-\frac{n}{2}\right)\right)_{\varepsilon}. \tag{3.3.4}
$$

If $n \to 4$, the first three terms of Eq.(3.3.4) diverge because of poles in the Γ- functions:

$$
\begin{aligned}
\Gamma\left(-\frac{n}{4}\right) &= \frac{4}{n(n-2)}\left(\frac{2}{4-n}-\gamma\right)+O(n-4), \\
\Gamma\left(1-\frac{n}{2}\right) &= \frac{4}{(2-n)}\left(\frac{2}{4-n}-\gamma\right)+O(n-4), \\
\Gamma\left(2-\frac{n}{2}\right) &= \frac{2}{4-n}-\gamma+O(n-4).
\end{aligned} \tag{3.3.5}
$$

Denoting these first three terms by $(L^{\pm}_{\varepsilon;\mathbf{div}}(x))_{\varepsilon}$, we have

$$
\begin{aligned}
&(L^{\pm}_{\varepsilon;\mathbf{div}}(x))_{\varepsilon} = (4\pi)^{-n/2}\left\{\frac{1}{n-4}+\frac{1}{2}\left[\gamma+\ln\left(\frac{m^2}{\mu^2}\right)\right]\right\}\times \\
&\left(\left[\frac{4m^4 a_0(x;\varepsilon)}{n(n-2)}-\frac{2m^2 a_1(x;\varepsilon)}{n-2}+a_2(x;\varepsilon)\right]\right)_{\varepsilon}.
\end{aligned} \tag{3.3.6}
$$

The functions $a_0(x;\varepsilon), a_1(x;\varepsilon)$ and $a_2(x;\varepsilon)$ are given by taking the coincidence limits of (3.1.17)

$$(a_0^{\pm}(x;\varepsilon))_{\varepsilon} = 1, (a_1^{\pm}(x;\varepsilon))_{\varepsilon} = \left(\frac{1}{6} - \xi\right)(\mathbf{R}^{\pm}(\varepsilon))_{\varepsilon},$$

$$(a_2^{\pm}(x;\varepsilon))_{\varepsilon} = \frac{1}{180}(\mathbf{R}^{\pm}_{\alpha\beta\gamma\delta}(x,\varepsilon)\mathbf{R}^{\pm\alpha\beta\gamma\delta}(x,\varepsilon))_{\varepsilon} - \frac{1}{180}(\mathbf{R}^{\pm\alpha\beta}(x,\varepsilon)\mathbf{R}^{\pm}_{\alpha\beta}(x,\varepsilon))_{\varepsilon} - \tag{3.3.7}$$

$$-\frac{1}{6}\left(\frac{1}{5} - \xi\right)(\square_{\varepsilon,x}\mathbf{R}^{\pm}(x,\varepsilon))_{\varepsilon} + \frac{1}{2}\left(\frac{1}{6} - \xi\right)(\mathbf{R}^{\pm 2}(x,\varepsilon))_{\varepsilon}.$$

Finally one obtains [19]

$$(L^{\pm}_{\varepsilon;\mathbf{ren}}(x))_{\varepsilon} \asymp -\frac{1}{64\pi^2}\left(\int_0^{\infty} ids\ln(is)\frac{\partial^3}{\partial(is)^3}\left[\mathcal{F}^{\pm}_{\varepsilon}(x,x;is)e^{-ism^2}\right]\right)_{\varepsilon}. \tag{3.3.8}$$

Remark 3.3.1.All the higher order $(j > 2)$ terms in the DeWitt-Schwinger expansion of the effective Lagrangian (3.3.4) are infrared divergent at $n = 4$ as $m \to 0$, we can still use this expansion to yield the ultraviolet divergent terms arising from $j = 0,1,$ and 2 in the four-dimensional case. We may put $m = 0$ immediately in the $j = 0$ and 1 terms in the expansion, because they are of positive power for $n \sim 4$. These terms therefore vanish. The only nonvanishing potentially ultraviolet divergent term is therefore $j = 2$:

$$2^{-1}(4\pi)^{-n/2}\left(\frac{m}{\mu}\right)^{n-4} a_2(x,\varepsilon)\Gamma\left(2 - \frac{n}{2}\right), \tag{3.3.9}$$

which must be handled carefully. Substituting for $a_2(x)$ with $\xi = \xi(n)$ from (3.3.7), and rearranging terms, we may write the divergent term in the effective action arising from (3.3.9) as follows

$$(W^{\pm}_{\varepsilon,\mathbf{div}})_{\varepsilon} = 2^{-1}(4\pi)^{-n/2}\left(\frac{m}{\mu}\right)^{n-4}\Gamma\left(2 - \frac{n}{2}\right)\left(\int d^n x[-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}} a_2(x,\varepsilon)\right)_{\varepsilon} =$$

$$2^{-1}(4\pi)^{-n/2}\left(\frac{m}{\mu}\right)^{n-4}\Gamma\left(2 - \frac{n}{2}\right) \times \tag{3.3.10}$$

$$\left(\int d^n x[-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}}\left[\widetilde{\alpha}\mathsf{F}^{\pm}_{\varepsilon}(x) + \widetilde{\beta}G^{\pm}_{\varepsilon}(x)\right]\right)_{\varepsilon} + O(n-4),$$

where

$$(\mathsf{F}_{\varepsilon}(x))_{\varepsilon} = (\mathbf{R}^{\pm\alpha\beta\gamma\delta}(x,\varepsilon)\mathbf{R}^{\pm}_{\alpha\beta\gamma\delta}(x,\varepsilon))_{\varepsilon} - 2(\mathbf{R}^{\pm\alpha\beta}(x,\varepsilon)\mathbf{R}^{\pm}_{\alpha\beta}(x,\varepsilon))_{\varepsilon} + \frac{1}{3}(\mathbf{R}^{\pm 2}(x,\varepsilon))_{\varepsilon},$$

$$(G^{\pm}_{\varepsilon}(x))_{\varepsilon} = (\mathbf{R}^{\pm\alpha\beta\gamma\delta}(x,\varepsilon)\mathbf{R}^{\pm}_{\alpha\beta\gamma\delta}(x,\varepsilon))_{\varepsilon}, \tag{3.3.11}$$

$$\widetilde{\alpha} = \frac{1}{120}, \widetilde{\beta} = -\frac{1}{360}.$$

Finally we obtain [19]

$$\left(\langle T^{\mu}_{\mu}(x,\varepsilon)\rangle_{\mathbf{ren}}\right)_{\varepsilon} = -(1/2880\pi^2)\left[\widetilde{\alpha}\left(\mathsf{F}_{\varepsilon}(x) - \tfrac{2}{3}\square_{\varepsilon,x}\mathbf{R}^{\pm}(x,\varepsilon)\right)_{\varepsilon} + \widetilde{\beta}(G^{\pm}_{\varepsilon}(x))_{\varepsilon}\right] =$$

$$-(1/2880\pi^2) \times \tag{3.3.12}$$

$$\left[(\mathbf{R}^{\pm}_{\alpha\beta\gamma\delta}(x,\varepsilon)\mathbf{R}^{\pm\alpha\beta\gamma\delta}(x,\varepsilon))_{\varepsilon} - (\mathbf{R}^{\pm}_{\alpha\beta}(x,\varepsilon)\mathbf{R}^{\pm\alpha\beta}(x,\varepsilon))_{\varepsilon} - (\square_{\varepsilon,x}\mathbf{R}^{\pm}(x,\varepsilon))_{\varepsilon}\right].$$

Therefore for the case of the distributional Schwarzchild spesetime using Eq.(2.2.4) and Eq.(3.3.12) for $r \to 2m$ we obtain

$$\left(\langle T^{\mu}_{\mu}(r,\varepsilon)\rangle_{\mathbf{ren}}\right)_{\varepsilon} \asymp -(2880 \cdot \pi^2)^{-1}\Big[\Big(\big[16^{-1}m^2(r-2m)^2+\varepsilon^2\big]^{-1}\Big)_{\varepsilon} + \ldots\Big] \approx$$
$$\approx -O(1)(2880 \cdot 16 \cdot \pi^2)^{-1}m^{-2}(r-2m)^{-2}. \tag{3.3.13}$$

Remark 3.2.2. Thus QFT in ditributional curved spacetime predict that the infalling observer burns up at the BH horizon.

Remark 3.2.3. In order avoid singularity at horizon $r = 2m$ in Eq.(3.3.13) one have applied

the Loop Quantum Gravity approach [45].The first one concerns the requirement of

selfadjointness to the metric components. For instance, the classical quantity

$$g_{tx} = -\frac{(E^x)' K_{\varphi}}{2\sqrt{E^x}\sqrt{1+K_{\varphi}^2-\frac{2Gm}{\sqrt{E^x}}}}, \tag{3.3.14}$$

defined as an evolving constant (i.e. a Dirac observable), must correspond to a selfadjoint operator at the quantum level. Classically, K_{φ} and E^x are pure gauge, and g_{tx} is just a function of the observable m. In the interior of the horizon, if $\widehat{g}_{tx}$ is a selfadjoint operator, a necessary condition will be [46]

$$1+K_{\varphi}^2-\frac{2Gm}{l_{\mathbf{P}}\sqrt{k_j}} \geq 0. \tag{3.3.15}$$

At the singularity, i.e. $j = 1$, and owing to the bounded nature of $K_{\varphi}^2 < \infty$,

$$\sqrt{k_1} \geq \frac{2Gm}{l_{\mathbf{P}}(1+K_{\varphi}^2)} > 0. \tag{3.3.16}$$

Therefore, this argument strongly suggests that the classical singularity will be resolved at the quantum level since k_1 must be a non-vanishing integer.

3.4. Unruh effect revisited

We remind now that a black holes have an approximate Rindler region near the Schwarzschild horizon. For the the distributional Schwarzschild solution (2.1.8) by coordinate transformation

$$r = 2m(1+(\delta^2+\epsilon^2)), \epsilon \in (0,1], \tag{3.4.1}$$

where $\varepsilon \ll \epsilon$,we obtain

$$(ds^2_{\epsilon})_{\varepsilon} = -\Big((\delta^2+\epsilon^2)_{\epsilon}\Big)dt^2 + 16m^2 d\delta^2 + 4m^2 d\Omega_2^2 + O(\varepsilon^2/\epsilon^2)\ldots \tag{3.4.2}$$

The (t,δ) piece of this metric (3.4.2) is Rindler space (we can rescale t ,δ and ϵ to make it look exactly like (2.2.10) for $\varepsilon \to 0$. Thus from (3.3.13) using (3.4.1) we obtain directly for $\delta \asymp 0$

$$\left(\langle T^{\mu}_{\mu}(\delta,\varepsilon)\rangle_{\mathbf{ren}}\right)_{\varepsilon} \asymp \delta^{-4}. \quad (3.4.3)$$

Therefore sufficiently strongly accelerated observer burns up near the Rindler horizon. Thus Polchinski's account doesn't violation of the Einstein equivalence principle.

Remark 3.4.1.Note that by using Eq.(A.8) and Eq.(A.9) (see appendix A) one obtains

Eq.(3.4.3) directly from distributionel Rindler metric.

4.Conclusion

On a Riemannian or a semi-Riemannian manifold, the metric determines invariants like the Levi-Civita connection and the Riemann curvature. If the metric becomes degenerate (as in singular semi-Riemannian geometry), these constructions no longer work, because they are based on the inverse of the metric, and on related operations like the contraction between covariant indices. In order to avoid these difficultnes distribytional geometry by using Colombeau generalized functions [2]-[10].In authors papers [23]-[24] appropriate generalization of classical GR based on Colombeau generalized functions is proposed.

Such generalization of classical GR based on appropriate generalization of the Einstein equivalence principle (GEEP) mentioned above in subsection 2.3. Under this GEEP Unruh effect revisited. We pointed out that GEEP avoid the contradiction mentioned by Z.Merali in paper [36].

Appendix A.

Let us introduce now Colombeau generalized metric which has the form

$$\begin{cases} (ds^2_{\epsilon})_{\epsilon} = -(A_{\epsilon}(r)(dx^0)^2)_{\epsilon} - 2(D_{\epsilon}(r)dx^0dr)_{\epsilon} + ((B_{\epsilon}(r)+C_{\epsilon}(r))(dr)^2)_{\epsilon} \\ \qquad +(B_{\epsilon}(r)r^2[(d\theta)^2+\sin^2\theta(d\phi)^2])_{\epsilon}\ . \end{cases} \quad (A.1)$$

Expressions for the Colombeau quantities $(\mathbf{R}(r,\epsilon))_{\epsilon}$, $(\mathbf{R}^{\mu\nu}(r,(\epsilon))\mathbf{R}_{\mu\nu}(r,(\epsilon)))_{\epsilon}$ and $\left(\mathbf{R}^{\rho\sigma\mu\nu}(r,(\epsilon))\mathbf{R}_{\rho\sigma\mu\nu}(r,(\epsilon))\right)_{\epsilon}$ in terms of $(A_{\epsilon})_{\epsilon},(B_{\epsilon})_{\epsilon},(C_{\epsilon})_{\epsilon}$ and $(D_{\epsilon})_{\epsilon},\epsilon \in (0,1]$ are:

The Colombeau scalars $(\mathbf{R}(r,\epsilon))_{\epsilon},(\mathbf{R}^{\mu\nu}(r,\epsilon)\mathbf{R}_{\mu\nu}(\epsilon))_{\epsilon}$ and $(R^{\rho\sigma\mu\nu}(r,\epsilon)R_{\rho\sigma\mu\nu}(r,\epsilon))_{\epsilon}$, in terms of Colombeau generalized functions $(A_{\epsilon}(r))_{\epsilon},(B_{\epsilon}(r))_{\epsilon},(C_{\epsilon}(r))_{\epsilon},(D_{\epsilon}(r))_{\epsilon}$ are expressed as

$$
\begin{aligned}
(\mathbf{R}(r,\epsilon))_\epsilon = &\left(\frac{A_\epsilon}{\Delta_\epsilon}\left[\frac{2}{r}\left(-2\frac{A'_\epsilon}{A_\epsilon} - 3\frac{B'_\epsilon}{B_\epsilon} + \frac{\Delta'_\epsilon}{\Delta_\epsilon}\right) + \frac{2}{r^2}\frac{A_\epsilon C_\epsilon + D_\epsilon^2}{A_\epsilon B_\epsilon} - \frac{A''_\epsilon}{A_\epsilon} - 2\frac{B''_\epsilon}{B_\epsilon}\right.\right.\\
&\left.\left.+\frac{1}{2}\left(\frac{B'_\epsilon}{B_\epsilon}\right)^2 - 2\frac{A'_\epsilon B'_\epsilon}{A_\epsilon B_\epsilon} + \left(\frac{1}{2}\frac{A'_\epsilon}{A_\epsilon} + \frac{B'_\epsilon}{B_\epsilon}\right)\frac{\Delta'_\epsilon}{\Delta_\epsilon}\right]\right)_\epsilon,\\
(\mathbf{R}^{\mu\nu}(r,\epsilon)\mathbf{R}_{\mu\nu}(\epsilon))_\epsilon = &\left(\frac{A_\epsilon^2}{\Delta_\epsilon^2}\left(\frac{1}{2}\frac{A''_\epsilon}{A_\epsilon} - \frac{1}{4}\frac{A'_\epsilon\Delta'_\epsilon}{A\Delta_\epsilon} + \frac{1}{2}\frac{A'_\epsilon B'_\epsilon}{A_\epsilon B_\epsilon} + \frac{1}{r}\frac{A'_\epsilon}{A_\epsilon}\right)^2\right)_\epsilon +\\
&+2\left(\frac{A_\epsilon^2}{\Delta_\epsilon^2}\left[\frac{1}{r}\left(\frac{1}{2}\frac{\Delta'_\epsilon}{\Delta_\epsilon} - \frac{A'_\epsilon}{A_\epsilon} - 2\frac{B'_\epsilon}{B_\epsilon}\right) + \frac{1}{r^2}\frac{A_\epsilon C_\epsilon + D_\epsilon^2}{A_\epsilon B_\epsilon} - \frac{1}{2}\frac{A'_\epsilon B'_\epsilon}{A_\epsilon B_\epsilon} -\right.\right.\\
&\left.\left.-\frac{1}{2}\frac{B''_\epsilon}{B_\epsilon} + \frac{1}{4}\frac{B'_\epsilon\Delta'_\epsilon}{B_\epsilon\Delta_\epsilon}\right]^2\right)_\epsilon +\\
&\left(\frac{A_\epsilon^2}{\Delta_\epsilon^2}\left[\frac{1}{2}\frac{A''_\epsilon}{A_\epsilon} - \frac{1}{4}\frac{A'_\epsilon\Delta'_\epsilon}{A_\epsilon\Delta_\epsilon} + \frac{1}{2}\frac{A'_\epsilon B'_\epsilon}{A_\epsilon B_\epsilon} + \frac{B''_\epsilon}{B_\epsilon} - \frac{1}{2}\left(\frac{B'_\epsilon}{B_\epsilon}\right)^2\right.\right.\\
&\left.\left.-\frac{1}{2}\frac{B'_\epsilon\Delta'_\epsilon}{B_\epsilon\Delta_\epsilon} + \frac{1}{r}\left(\frac{A'_\epsilon}{A_\epsilon} - \frac{\Delta'_\epsilon}{\Delta_\epsilon} + 2\frac{B'_\epsilon}{B_\epsilon}\right)\right]^2\right)_\epsilon, \qquad (A.2)\\
(\mathbf{R}^{\rho\sigma\mu\nu}(r,\epsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r,\epsilon))_\epsilon = &\\
&\left(\frac{A_\epsilon^2}{\Delta_\epsilon^2}\left(\frac{A''_\epsilon}{A_\epsilon} - \frac{1}{2}\frac{A'_\epsilon\Delta'_\epsilon}{A_\epsilon\Delta_\epsilon}\right)^2 + 2\frac{A_\epsilon^2}{\Delta_\epsilon^2}\left(\frac{1}{r}\frac{A'_\epsilon}{A_\epsilon} + \frac{1}{2}\frac{A'_\epsilon B'_\epsilon}{A_\epsilon B_\epsilon}\right)^2\right.\\
&+4\frac{A_\epsilon^2}{\Delta_\epsilon^2}\left[\frac{1}{r}\frac{B'_\epsilon}{B_\epsilon} - \frac{1}{r^2}\frac{A_\epsilon C_\epsilon + D_\epsilon^2}{A_\epsilon B_\epsilon} + \frac{1}{4}\left(\frac{B'_\epsilon}{B_\epsilon}\right)^2\right]^2 +\\
&+2\frac{A^2}{\Delta_\epsilon^2}\left[\frac{1}{r}\left(\frac{A'_\epsilon}{A_\epsilon} + 2\frac{B'_\epsilon}{B_\epsilon} - \frac{\Delta'_\epsilon}{\Delta_\epsilon}\right) + \frac{1}{2}\frac{A'_\epsilon B'_\epsilon}{A_\epsilon B_\epsilon} + \frac{B''_\epsilon}{B_\epsilon}\right.\\
&\left.\left.-\frac{1}{2}\left(\frac{B'_\epsilon}{B_\epsilon}\right)^2 - \frac{1}{2}\frac{B'_\epsilon\Delta'_\epsilon}{B_\epsilon\Delta_\epsilon}\right]^2\right)_\epsilon.\\
\Delta_\epsilon = &\ A_\epsilon(B_\epsilon + C_\epsilon) + D_\epsilon^2
\end{aligned}
$$

The distributional Møller’s metric is

$$
\begin{aligned}
(d_\varepsilon s^2)_\varepsilon &= -(A_\varepsilon(x)dt^2)_\varepsilon + dx^2 + dy^2 + dz^2,\\
A_\varepsilon(x) &= \left[(a+gx)^2 + \varepsilon^2\right], \varepsilon \in (0,1].
\end{aligned} \qquad (A.3)
$$

In spherical coordinates we get

$$
\begin{aligned}
(d_\varepsilon s^2)_\varepsilon &= -(A_\varepsilon(r)dt^2)_\varepsilon + dr^2 + r^2 d\Omega^2,\\
A_\varepsilon(r) &= \left[(a+gr)^2+\varepsilon^2\right], \varepsilon \in (0,1],\\
A'_\varepsilon(r) &= 2g(a+gr), A''_\varepsilon(r) = 2g^2.
\end{aligned}
\tag{A.4}
$$

We choose now $B_\varepsilon(r) + C_\varepsilon(r) = 1, B_\varepsilon(r) = 2, C_\varepsilon(r) = -1,$ and rewrite Eq.(A.3) in the following equivalent form

$$
\begin{aligned}
(d_\varepsilon s^2)_\varepsilon &= -(A_\varepsilon(r)dt^2)_\varepsilon + dr^2 + r^2 d\Omega'^2,\\
\Omega' &= \Omega\sqrt{2}.
\end{aligned}
\tag{A.5}
$$

Note that

$$
\Delta_\varepsilon = A_\varepsilon(B_\varepsilon + C_\varepsilon) = A_\varepsilon. \tag{A.6}
$$

From Eq.(A.4)-Eq.(A.6) by formulae (A.2) we get

$$
\begin{aligned}
(\mathbf{R}(r,\varepsilon))_\varepsilon = &\left(\frac{A_\varepsilon}{\Delta_\varepsilon}\left[\frac{2}{r}\left(-2\frac{A'_\varepsilon}{A_\varepsilon} - 3\frac{B'_\epsilon}{B_\varepsilon} + \frac{\Delta'_\varepsilon}{\Delta_\varepsilon}\right) + \frac{2}{r^2}\frac{A_\varepsilon C_\varepsilon + D_\varepsilon^2}{A_\varepsilon B_\varepsilon} - \frac{A''_\varepsilon}{A_\varepsilon} - 2\frac{B''_\varepsilon}{B_\varepsilon}\right.\right.\\
&\left.\left. + \frac{1}{2}\left(\frac{B'_\varepsilon}{B_\varepsilon}\right)^2 - 2\frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} + \left(\frac{1}{2}\frac{A'_\varepsilon}{A_\varepsilon} + \frac{B'_\varepsilon}{B_\varepsilon}\right)\frac{\Delta'_\varepsilon}{\Delta_\varepsilon}\right]\right)_\varepsilon =\\
&\left(-\frac{2}{r}\frac{A'_\varepsilon}{A_\varepsilon} - \frac{1}{r^2} - \frac{A''_\varepsilon}{A_\varepsilon} + \frac{1}{2}\frac{A'^2_\varepsilon}{A^2_\varepsilon}\right)_\varepsilon =\\
&-\frac{4g(a+gr)}{r\left[(a+gr)^2+\varepsilon^2\right]} - \frac{1}{r^2} - \frac{2g^2}{(a+gr)^2+\varepsilon^2} + \frac{2g^2(a+gr)^2}{\left[(a+gr)^2+\varepsilon^2\right]^2} =\\
&2g^2\frac{(a+gr)^2+\varepsilon^2-\varepsilon^2}{\left[(a+gr)^2+\varepsilon^2\right]^2} = 2g^2 + \frac{-2g^2\varepsilon^2}{\left[(a+gr)^2+\varepsilon^2\right]^2}.
\end{aligned}
\tag{A.7}
$$

From Eq.(A.4)-Eq.(A.6) by formulae (A.2) we get

$$
\begin{aligned}
&(\mathbf{R}^{\mu\nu}(r,\varepsilon)\mathbf{R}_{\mu\nu}(\varepsilon))_{\varepsilon} = \left(\frac{A_{\varepsilon}^{2}}{\Delta_{\varepsilon}^{2}}\left(\frac{1}{2}\frac{A_{\varepsilon}''}{A_{\varepsilon}} - \frac{1}{4}\frac{A_{\varepsilon}'\Delta_{\varepsilon}'}{A_{\varepsilon}\Delta_{\varepsilon}} + \frac{1}{2}\frac{A_{\varepsilon}'B_{\varepsilon}'}{A_{\varepsilon}B_{\varepsilon}} + \frac{1}{r}\frac{A_{\varepsilon}'}{A_{\varepsilon}}\right)^{2}\right)_{\varepsilon} + \\
&+2\left(\frac{A_{\varepsilon}^{2}}{\Delta_{\varepsilon}^{2}}\left[\frac{1}{r}\left(\frac{1}{2}\frac{\Delta_{\varepsilon}'}{\Delta_{\varepsilon}} - \frac{A_{\varepsilon}'}{A_{\varepsilon}} - 2\frac{B_{\varepsilon}'}{B_{\varepsilon}}\right) + \frac{1}{r^{2}}\frac{A_{\varepsilon}C_{\varepsilon} + D_{\epsilon}^{2}}{A_{\varepsilon}B_{\varepsilon}} - \frac{1}{2}\frac{A_{\varepsilon}'B_{\varepsilon}'}{A_{\varepsilon}B_{\varepsilon}} - \right.\right. \\
&\left.\left. -\frac{1}{2}\frac{B_{\varepsilon}''}{B_{\varepsilon}} + \frac{1}{4}\frac{B_{\varepsilon}'\Delta_{\varepsilon}'}{B_{\varepsilon}\Delta_{\varepsilon}}\right]^{2}\right)_{\varepsilon} + \\
&\left(\frac{A_{\varepsilon}^{2}}{\Delta_{\varepsilon}^{2}}\left[\frac{1}{2}\frac{A_{\varepsilon}''}{A_{\varepsilon}} - \frac{1}{4}\frac{A_{\varepsilon}'\Delta_{\varepsilon}'}{A_{\varepsilon}\Delta_{\varepsilon}} + \frac{1}{2}\frac{A_{\varepsilon}'B_{\varepsilon}'}{A_{\varepsilon}B_{\varepsilon}} + \frac{B_{\varepsilon}''}{B_{\varepsilon}} - \frac{1}{2}\left(\frac{B_{\varepsilon}'}{B_{\varepsilon}}\right)^{2}\right.\right. \\
&\left.\left. -\frac{1}{2}\frac{B_{\varepsilon}'\Delta_{\varepsilon}'}{B_{\varepsilon}\Delta_{\varepsilon}} + \frac{1}{r}\left(\frac{A_{\varepsilon}'}{A_{\varepsilon}} - \frac{\Delta_{\varepsilon}'}{\Delta_{\varepsilon}} + 2\frac{B_{\varepsilon}'}{B_{\varepsilon}}\right)\right]^{2}\right)_{\varepsilon} = \\
&\left(\left(\frac{1}{2}\frac{A_{\varepsilon}''}{A_{\varepsilon}} - \frac{1}{4}\frac{A_{\varepsilon}'\Delta_{\varepsilon}'}{A_{\varepsilon}\Delta_{\varepsilon}} + \frac{1}{r}\frac{A_{\varepsilon}'}{A_{\varepsilon}}\right)^{2}\right)_{\varepsilon} + 2\left(\left[\frac{1}{r}\left(\frac{1}{2}\frac{\Delta_{\varepsilon}'}{\Delta_{\varepsilon}} - \frac{A_{\varepsilon}'}{A_{\varepsilon}}\right) - \frac{1}{r^{2}}\right]^{2}\right)_{\varepsilon} + \\
&\left(\left[\frac{1}{2}\frac{A_{\varepsilon}''}{A_{\varepsilon}} - \frac{1}{4}\frac{A_{\varepsilon}'\Delta_{\varepsilon}'}{A_{\varepsilon}\Delta_{\varepsilon}} + \frac{1}{r}\left(\frac{A_{\varepsilon}'}{A_{\varepsilon}} - \frac{\Delta_{\varepsilon}'}{\Delta_{\varepsilon}}\right)\right]^{2}\right)_{\varepsilon} = \\
&\left(\left[\frac{1}{2}\frac{A_{\varepsilon}''}{A_{\varepsilon}} - \frac{1}{4}\frac{A_{\varepsilon}'^{2}}{A_{\varepsilon}^{2}} + \frac{1}{r}\frac{A_{\varepsilon}'}{A_{\varepsilon}}\right]^{2}\right)_{\varepsilon} + 2\left(\left[-\frac{1}{2r}\frac{A_{\varepsilon}'}{A_{\varepsilon}} - \frac{1}{r^{2}}\right]^{2}\right)_{\varepsilon} + \\
&+\left(\left[\frac{1}{2}\frac{A_{\varepsilon}''}{A_{\varepsilon}} - \frac{1}{4}\frac{A_{\varepsilon}'^{2}}{A_{\varepsilon}^{2}}\right]^{2}\right)_{\varepsilon} = \\
&\left(\left[\frac{1}{2}\frac{4g^{2}}{(a+gr)^{2}+\varepsilon^{2}} - \frac{1}{4}\frac{4g^{2}(a+gr)^{2}}{\left[(a+gr)^{2}+\varepsilon^{2}\right]^{2}} + \frac{1}{r}\frac{2g(a+gr)}{\left[(a+gr)^{2}+\varepsilon^{2}\right]}\right]^{2}\right)_{\varepsilon} + \\
&+2\left(\left[-\frac{1}{2r}\frac{2g(a+gr)}{(a+gr)^{2}+\varepsilon^{2}} - \frac{1}{r^{2}}\right]^{2}\right)_{\varepsilon} + \\
&\left(\left[\frac{1}{2}\frac{4g^{2}}{(a+gr)^{2}+\varepsilon^{2}} - \frac{1}{4}\frac{4g^{2}(a+gr)^{2}}{\left[(a+gr)^{2}+\varepsilon^{2}\right]^{2}}\right]^{2}\right)_{\varepsilon}.
\end{aligned}
\tag{A.8}
$$

We assume now that $a + gr \asymp 0$, then from (A.8) we get

$$
(\mathbf{R}^{\mu\nu}(r,\varepsilon)\mathbf{R}_{\mu\nu}(\varepsilon))_{\varepsilon} \asymp \frac{4g^{4}O(1)}{\left[(a+gr)^{2}+\varepsilon^{2}\right]^{2}}. \tag{A.9}
$$

From Eq.(A.4)-Eq.(A.6) by formulae (A.2) we get

$$
\begin{gathered}
\left(\mathbf{R}^{\rho\sigma\mu\nu}(r,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r,\varepsilon)\right)_{\epsilon} = \\
\left(\frac{A_{\varepsilon}^{2}}{\Delta_{\varepsilon}^{2}}\left(\frac{A_{\varepsilon}''}{A_{\varepsilon}}-\frac{1}{2}\frac{A_{\epsilon}'\Delta_{\varepsilon}'}{A_{\epsilon}\Delta_{\varepsilon}}\right)^{2}+2\frac{A_{\varepsilon}^{2}}{\Delta_{\varepsilon}^{2}}\left(\frac{1}{r}\frac{A_{\varepsilon}'}{A_{\varepsilon}}+\frac{1}{2}\frac{A_{\varepsilon}'B_{\varepsilon}'}{A_{\varepsilon}B_{\varepsilon}}\right)^{2}\right. \\
+4\frac{A_{\varepsilon}^{2}}{\Delta_{\varepsilon}^{2}}\left[\frac{1}{r}\frac{B_{\varepsilon}'}{B_{\varepsilon}}-\frac{1}{r^{2}}\frac{A_{\varepsilon}C_{\varepsilon}+D_{\varepsilon}^{2}}{A_{\varepsilon}B_{\varepsilon}}+\frac{1}{4}\left(\frac{B_{\varepsilon}'}{B_{\varepsilon}}\right)^{2}\right]^{2}+ \\
+2\frac{A_{\varepsilon}^{2}}{\Delta_{\varepsilon}^{2}}\left[\frac{1}{r}\left(\frac{A_{\varepsilon}'}{A_{\varepsilon}}+2\frac{B_{\varepsilon}'}{B_{\varepsilon}}-\frac{\Delta_{\varepsilon}'}{\Delta_{\varepsilon}}\right)+\frac{1}{2}\frac{A_{\varepsilon}'B_{\varepsilon}'}{A_{\varepsilon}B_{\varepsilon}}+\frac{B_{\varepsilon}''}{B_{\varepsilon}}\right. \\
\left.\left.-\frac{1}{2}\left(\frac{B_{\varepsilon}'}{B_{\varepsilon}}\right)^{2}-\frac{1}{2}\frac{B_{\varepsilon}'\Delta_{\varepsilon}'}{B_{\varepsilon}\Delta_{\varepsilon}}\right]^{2}\right)_{\varepsilon} = \\
\left(\left(\frac{A_{\varepsilon}''}{A_{\varepsilon}}-\frac{1}{2}\frac{A_{\varepsilon}'^{2}}{A_{\varepsilon}^{2}}\right)^{2}\right)_{\varepsilon}+2\frac{1}{r}\left(\frac{A_{\varepsilon}'^{2}}{A_{\varepsilon}^{2}}\right)_{\varepsilon}+\frac{1}{r^{4}} = \\
\left(\left(\frac{2g^{2}}{(a+gr)^{2}+\varepsilon^{2}}-\frac{2g^{2}(a+gr)^{2}}{\left[(a+gr)^{2}+\varepsilon^{2}\right]^{2}}\right)^{2}\right)_{\varepsilon}+\frac{1}{r}\left(\frac{8g^{2}(a+gr)^{2}}{\left[(a+gr)^{2}+\varepsilon^{2}\right]^{2}}\right)_{\varepsilon}.
\end{gathered}
\tag{A.10}
$$

We assume now that $a+gr \asymp 0$, then from (A.9) we get

$$
\left(\mathbf{R}^{\rho\sigma\mu\nu}(r,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r,\varepsilon)\right)_{\epsilon} \asymp \frac{4g^{4}O(1)}{\left[(a+gr)^{2}+\varepsilon^{2}\right]^{2}}. \tag{A.11}
$$

Appendix B.

We calculate now the distributional curvature at Schwarzschild horizon. In the usual Schwarzschild coordinates $(t, r>0, \theta, \phi), r \neq 2m$ the metric is

$$
\begin{cases}
ds^{2} = h(r)dt^{2} - h(r)^{-1}dr^{2} + r^{2}d\Omega^{2}, \\
h(r) = -1 + \frac{2m}{r}.
\end{cases}
\tag{B.1}
$$

Metric takes the form above horizon $r>2m$ and below horizon $r<2m$ correspondingly

$$
\begin{cases}
\text{above horizon } r>2m: \\
ds^{+2} = h^{+}(r)dt^{2} - [h^{+}(r)]^{-1}dr^{2} + r^{2}d\Omega^{2}, \\
h^{+}(r) = -1 + \frac{2m}{r} = -\frac{r-2m}{r} \\
\text{below horizon } r<2m: \\
ds^{-2} = h^{-}(r)dt^{2} - h^{-}(r)^{-1}dr^{2} + r^{2}d\Omega^{2}, \\
h^{-}(r) = -1 + \frac{2m}{r} = \frac{2m-r}{r}
\end{cases}
\tag{B.2}
$$

Remark A.1. Following the above discussion we consider the metric coefficients

$h^{+}(r), [h^{+}(r)]^{-1}\ h^{-}(r),$ and $[h^{-}(r)]^{-1}$ as an element of $D'(\mathbb{R}^3)$ and embed it into $\mathbf{D}\left(\widetilde{\mathbb{R}}^3\right)$ by replacement above horizon $r \geqslant 2m$ and below horizon $r \leqslant 2m$ correspondingly

$$\begin{aligned} r \geqslant 2m &: r - 2m \mapsto \sqrt{(r-2m)^2 + \epsilon^2}\,, \\ r \leqslant 2m &: 2m - r \mapsto \sqrt{(2m-r)^2 + \epsilon^2}\,. \end{aligned} \tag{B.3}$$

Note that, accordingly, we have fixed the differentiable structure of the manifold: the Cartesian coordinates associated with the spherical Schwarzschild coordinates in (B.1) are extended

$$h(r) = \left\{ \begin{array}{c} -\frac{r-2m}{r} \text{ if } r \geqslant 2m \\ 0 \text{ if } r \leqslant 2m \end{array} \right\} \mapsto (h^{+}_{\epsilon}(r))_{\epsilon} = \left(-\frac{\sqrt{(r-2m)^2+\epsilon^2}}{r} \right)_{\epsilon},$$

where $(h^{+}_{\epsilon}(r))_{\epsilon} \in \mathbf{G}(\mathbb{R}^3, B^{+}(2m,R)), B^{+}(2m,R) = \{x \in \mathbb{R}^3 | 2m \leqslant \|x\| \leqslant R\}$.

$$h^{-1}(r) = \left\{ \begin{array}{c} -\frac{r}{r-2m}, r > 2m \\ \infty, r = 2m \end{array} \right\} \mapsto (h^{+}_{\epsilon})^{-1}(r) =$$

$$\begin{aligned} h^{-}(r) &= \left\{ \begin{array}{c} -\frac{r-2m}{r} \text{ if } r \leqslant 2m \\ 0 \text{ if } r \geq 2m \end{array} \right\} \mapsto h^{-}_{\epsilon}(r) = \\ &= \left(\frac{\sqrt{(2m-r)^2+\epsilon^2}}{r} \right)_{\epsilon} \in \mathbf{G}(\mathbb{R}^3, B^{-}(0,2m)), \end{aligned} \tag{B.4}$$

where $B^{-}(0,2m) = \{x \in \mathbb{R}^3 | 0 < \|x\| \leqslant 2m\}$

$$\left\{ \begin{array}{c} -\frac{r}{r-2m}, r < 2m \\ \infty, r = 2m \end{array} \right\} \mapsto (h^{-}_{\epsilon})^{-1}(r) =$$

$$= \left(\frac{r}{\sqrt{(r-2m)^2+\epsilon^2}} \right)_{\epsilon} \in \mathbf{G}(\mathbb{R}^3, B^{-}(0,2m))$$

Inserting (B.4) into (B.2) we obtain a generalized object modeling the singular Schwarzschild metric above (below) gorizon, i.e.,

$$\begin{aligned} (ds^{+2}_{\epsilon})_{\epsilon} &= (h^{+}_{\epsilon}(r)dt^2)_{\epsilon} - \left([h^{+}_{\epsilon}(r)]^{-1}dr^2\right)_{\epsilon} + r^2 d\Omega^2, \\ (ds^{-2}_{\epsilon})_{\epsilon} &= (h^{-}_{\epsilon}(r)dt^2)_{\epsilon} - \left([h^{-}_{\epsilon}(r)]^{-1}dr^2\right)_{\epsilon} + r^2 d\Omega^2 \end{aligned} \tag{B.5}$$

The generalized Ricci tensor above horizon $[\mathbf{R}^{+}]^{\beta}_{\alpha}$ may now be calculated componentwise using the classical formulae

$$\begin{aligned}\left([\mathbf{R}_\epsilon^+]_0^0\right)_\epsilon &= \left([\mathbf{R}_\epsilon^+]_1^1\right)_\epsilon = \frac{1}{2}\left((h_\epsilon^{+\prime\prime})_\epsilon + \frac{2}{r}(h_\epsilon^{+\prime})_\epsilon\right)\\ \left([\mathbf{R}_\epsilon^+]_2^2\right)_\epsilon &= \left([\mathbf{R}_\epsilon^+]_3^3\right)_\epsilon = \frac{(h_\epsilon^{+\prime})_\epsilon}{r} + \frac{1+(h_\epsilon^+)_\epsilon}{r^2}\,.\end{aligned} \tag{B.6}$$

From (B.4) by differentiation we obtain

$$h_{\epsilon}^{+\prime}(r) = -\frac{r-2m}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^2},$$

$$r(h_{\epsilon}^{+\prime})_{\epsilon} + 1 + (h_{\epsilon}^{+})_{\epsilon} =$$

$$r\left\{-\frac{r-2m}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^2}\right\} + 1 - \frac{\sqrt{(r-2m)^2+\epsilon^2}}{r} =$$

$$-\frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r} + 1 - \frac{\sqrt{(r-2m)^2+\epsilon^2}}{r} =$$

$$-\frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + 1.$$

$$h_{\epsilon}^{\prime\prime}(r) = -\left(\frac{r-2m}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}}\right)' + \left(\frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^2}\right)' =$$

$$= -\frac{1}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{(r-2m)^2}{r\left[(r-2m)^2+\epsilon^2\right]^{3/2}} + \frac{r-2m}{r^2\left[(r-2m)^2+\epsilon^2\right]^{1/2}} +$$

$$+\frac{r-2m}{r^2\left[(r-2m)^2+\epsilon^2\right]^{1/2}} - \frac{2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^3}.$$

$$r^2(h_{\epsilon}^{+\prime\prime})_{\epsilon} + 2r(h_{\epsilon}^{+\prime})_{\epsilon} = \qquad (B.7)$$

$$r^2\left\{-\frac{1}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{(r-2m)^2}{r\left[(r-2m)^2+\epsilon^2\right]^{3/2}} + \frac{r-2m}{r^2\left[(r-2m)^2+\epsilon^2\right]^{1/2}} +\right.$$

$$\left.+\frac{r-2m}{r^2\left[(r-2m)^2+\epsilon^2\right]^{1/2}} - \frac{2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^3}\right\} +$$

$$+2r\left\{-\frac{r-2m}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^2}\right\} =$$

$$-\frac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}} + \frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} +$$

$$+\frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} - \frac{2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r} +$$

$$-\frac{2(r-2m)}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r} =$$

$$-\frac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}}.$$

Investigating the weak limit of the angular components of the Ricci tensor (using the abbreviation

$$\tilde{\Phi}(r) = \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \Phi(\vec{x}) \qquad (B.8)$$

and let $\Phi(\vec{x})$ be the function $\Phi(\vec{x}) \in S_{2m}(\mathbb{R}^3, B^+(2m, R_0))$, where by $S_{2m}(\mathbb{R}^3, B^+(2m, R_0))$ we denote the class of the functions $\Phi(\vec{x})$ with compact support such that:

(i) **supp**$\left(\Phi(\vec{x})\right) \subset B^+(2m, R_0) \subset \left\{\vec{x} | R_0 \geq \|\vec{x}\| \geq 2m\right\}$ (ii) $\tilde{\Phi}(r) \in C^\infty(\mathbb{R})$.

Then for any function $\Phi(\vec{x}) \in S_{2m}(\mathbb{R}^3, B^+(2m, R_0))$ we get:

$$\begin{aligned} &\int_K \left([\mathbf{R}_\epsilon^+]_2^2\right)_\epsilon \Phi(\vec{x}) d^3x = \int_K \left([\mathbf{R}_\epsilon^+]_3^3\right)_\epsilon \Phi(\vec{x}) d^3x = \\ &= \int_{2m}^{R} (r(h_\epsilon^{+\prime})_\epsilon + 1 + (h_\epsilon^+)_\epsilon) \tilde{\Phi}(r) dr = \\ &= \int_{2m}^{R} \left\{ -\frac{r-2m}{\left[(r-2m)^2 + \epsilon^2\right]^{1/2}} \right\} \tilde{\Phi}(r) dr + \int_{2m}^{R} \tilde{\Phi}(r) dr. \end{aligned} \qquad (B.9)$$

By replacement $r - 2m = u$, from (B.9) we obtain

$$\begin{aligned} &\int_K \left([\mathbf{R}_\epsilon^+]_2^2\right)_\epsilon \Phi(x) d^3x = \int_K \left([\mathbf{R}_\epsilon^+]_3^3\right)_\epsilon \Phi(x) d^3x = \\ &= -\int_0^{R-2m} \frac{u\tilde{\Phi}(u+2m)du}{(u^2+\epsilon^2)^{1/2}} + \int_0^{R-2m} \tilde{\Phi}(u+2m) du. \end{aligned} \qquad (B.10)$$

By replacement $u = \epsilon\eta$, from (B.10) we obtain the expression

$$\begin{aligned} &\mathbf{I}_3^+(\epsilon) = \int_K \left([\mathbf{R}_\epsilon^+]_3^3\right)_\epsilon \Phi(x) d^3x = \mathbf{I}_2^+(\epsilon) = \int_K \left([\mathbf{R}_\epsilon^+]_2^2\right)_\epsilon \Phi(\vec{x}) d^3x = \\ &= -\epsilon \times \left(\int_0^{\frac{R-2m}{\epsilon}} \frac{\eta\tilde{\Phi}(\epsilon\eta + 2)d\eta}{(\eta^2+1)^{1/2}} - \int_0^{\frac{R-2m}{\epsilon}} \tilde{\Phi}(\epsilon\eta + 2m) d\eta \right). \end{aligned} \qquad (B.11)$$

From Eq.(B.11) we get

$$\mathbf{I}_3^+(\epsilon) = \mathbf{I}_2^+(\epsilon) = -\epsilon\frac{\tilde{\Phi}(2m)}{0!}\int_0^{\frac{R-2m}{\epsilon}}\left[\frac{\eta}{(\eta^2+1)^{1/2}} - 1\right]d\eta +$$
$$-\frac{\epsilon^2}{1!}\int_0^{\frac{R-2m}{\epsilon}}\left[\frac{\eta}{(\eta^2+1)^{1/2}} - 1\right]\tilde{\Phi}^{(1)}(\xi)\eta d\eta = \qquad (B.12)$$
$$-\epsilon\tilde{\Phi}(2m)\left[\sqrt{\left(\frac{R-2m}{\epsilon}\right)^2+1} - 1 - \frac{R-2m}{\epsilon}\right] +$$
$$-\frac{\epsilon^2}{1}\int_0^{\frac{R-2m}{\epsilon}}\left[\frac{\eta}{(\eta^2+1)^{1/2}} - 1\right]\tilde{\Phi}^{(1)}(\xi)\eta d\eta,$$

where we have expressed the function $\widetilde{\Phi}(\epsilon\eta + 2m)$ as

$$\widetilde{\Phi}(\epsilon\eta + 2m) = \sum_{l=0}^{n-1}\frac{\Phi^{(l)}(2m)}{l!}(\epsilon\eta)^l + \frac{1}{n!}(\epsilon\eta)^n\Phi^{(n)}(\xi)\ , \qquad (B.13)$$
$$\xi \triangleq \theta\epsilon\eta + 2m\ ,\quad 1 > \theta > 0\ ,\quad n = 1$$

with $\tilde{\Phi}^{(l)}(\xi) \triangleq d^l\tilde{\Phi}/d\xi^l$. Equations (B.12)-(3.13) gives

$$\lim_{\epsilon\to 0}\mathbf{I}_3^+(\epsilon) = \lim_{\epsilon\to 0}\mathbf{I}_2^+(\epsilon) =$$
$$\lim_{\epsilon\to 0}\left\{-\epsilon\tilde{\Phi}(2m)\left[\sqrt{\left(\frac{R-2m}{\epsilon}\right)^2+1} - 1 - \frac{R-2m}{\epsilon}\right]\right\} + \qquad (B.14)$$
$$+\lim_{\epsilon\to 0}\left\{-\frac{\epsilon^2}{1}\int_0^{\frac{R-2m}{\epsilon}}\left[\frac{\eta}{(\eta^2+1)^{1/2}} - 1\right]\tilde{\Phi}^{(1)}(\xi)\eta d\eta\right\} = 0.$$

Since $S'_{2m}(B^+(2m,R)) \subset D'(\mathbb{R}^3)$,where $B^+(2m,R) = \{x \in \mathbb{R}^3 | 2m \leqslant \|x\| \leqslant R\}$ from

$$w\text{-}\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^+]_3^3 = \lim_{\epsilon\to 0}\mathbf{I}_3^+(\epsilon) = 0,$$
$$w\text{-}\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^+]_2^2 = \lim_{\epsilon\to 0}\mathbf{I}_2^+(\epsilon) = 0. \qquad (B.15)$$

For $\left([\mathbf{R}_\epsilon^+]_1^1\right)_\epsilon, \left([\mathbf{R}_\epsilon^+]_0^0\right)_\epsilon$ we get:

$$2\int_K \left([\mathbf{R}^+_\epsilon]^1_1\right)_\epsilon \Phi(x)d^3x = 2\int_K \left([\mathbf{R}^+_\epsilon]^0_0\right)_\epsilon \Phi(x)d^3x =$$

$$\int\limits_{2m}^{R} (r^2(h^{+\prime\prime}_\epsilon)_\epsilon + 2r(h^{+\prime}_\epsilon)_\epsilon)\tilde{\Phi}(r)dr = \qquad (B.16)$$

$$= \int\limits_{2m}^{R} \left\{ -\frac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}} \right\} \tilde{\Phi}(r)dr.$$

where use is made of the relation

$$\lim_{s\to\infty} \left[\int\limits_0^s \frac{\eta^2 d\eta}{(\eta^2+1)^{3/2}} - \int\limits_0^s \frac{d\eta}{(u^2+1)^{1/2}} \right] = -1 \qquad (B.17)$$

Finally we obtain

$$- \lim_{\epsilon\to0} [\mathbf{R}^+_\epsilon]^1_1 = \lim_{\epsilon\to0} [\mathbf{R}^+_\epsilon]^0_0 = -m\tilde{\Phi}(2m). \qquad (B.18)$$

The Colombeau generalized Ricci tensor below horizon $[\mathbf{R}^-_\epsilon]^\beta_\alpha = [\mathbf{R}^-_\epsilon]^\beta_\alpha$ may now be calculated componentwise using the classical formulae

$$\begin{cases} \left([\mathbf{R}^-_\epsilon]^0_0\right)_\epsilon = \left([\mathbf{R}^-_\epsilon]^1_1\right)_\epsilon = \frac{1}{2}\left((h^{-\prime\prime}_\epsilon)_\epsilon + \frac{2}{r}(h^{-\prime}_\epsilon)_\epsilon\right), \\ \left([\mathbf{R}^-_\epsilon]^2_2\right)_\epsilon = \left([\mathbf{R}^-_\epsilon]^3_3\right)_\epsilon = \frac{(h^{-\prime}_\epsilon)_\epsilon}{r} + \frac{1+(h^-_\epsilon)_\epsilon}{r^2}. \end{cases} \qquad (B.19)$$

From (B.4) we obtain

$$h^-_\epsilon(r) = -\frac{r-2m}{r} \mapsto h^-_\epsilon(r) = \left(\frac{\sqrt{(2m-r)^2+\epsilon^2}}{r} \right) = -h^+_\epsilon(r), r < 2m.$$

$$h^{-\prime}_\epsilon(r) = -h^{+\prime}_\epsilon(r) = \frac{r-2m}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} - \frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^2},$$

$$r(h^{-\prime}_\epsilon)_\epsilon + 1 + (h^-_\epsilon)_\epsilon = -r(h^{+\prime}_\epsilon)_\epsilon + 1 - (h^+_\epsilon)_\epsilon =$$

$$\frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + 1. \qquad (B.20)$$

$$h^{-\prime\prime}_\epsilon(r) = -h^{+\prime\prime}_\epsilon(r) =$$

$$-\frac{r-2m}{r^2\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^3}.$$

$$r^2(h^{-\prime\prime}_\epsilon)_\epsilon + 2r(h^{-\prime}_\epsilon)_\epsilon = -r^2(h^{+\prime\prime}_\epsilon)_\epsilon - 2r(h^{+\prime}_\epsilon)_\epsilon =$$

$$\frac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} - \frac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}}.$$

Investigating the weak limit of the angular components of the Ricci tensor (using the abbreviation $\tilde{\Phi}(r) = \int\limits_0^{\pi} \sin\theta d\theta \int\limits_0^{2\pi} d\phi \Phi(\vec{x})$ and let $\Phi(\vec{x})$ be the function $\Phi(\vec{x}) \in S_{2m}(\mathbb{R}^3, B^-(0,2m))$, the class of the functions $\Phi(\vec{x})$ with compact support $K \subset B^-(0,2m), B^-(0,2m) = \left\{\vec{x} | 0 \leq \|\vec{x}\| \leq 2m\right\}$ such that:
(i) **supp**$(\Phi(\vec{x})) \subset \left\{\vec{x} | 0 \leq \|\vec{x}\| \leq 2m\right\}$ (ii) $\tilde{\Phi}(r) \in C^{\infty}(\mathbb{R})$.

Then for any function $\Phi(\vec{x}) \in S^-_{2m}(\mathbb{R}^3, B^-(0,2m))$ we get

$$\int_K \left([\mathbf{R}^-_\epsilon]^2_2\right)_\epsilon \Phi(\vec{x}) d^3x = \int_K \left([\mathbf{R}^-_\epsilon]^3_3\right)_\epsilon \Phi(\vec{x}) d^3x = \int\limits_0^{2m} \left(r(h^{-\prime}_\epsilon)_\epsilon + 1 + (h^-_\epsilon)_\epsilon\right)\tilde{\Phi}(r) dr = \int\limits_0^{2m} \left\{ \frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} \right\} \tilde{\Phi}(r) dr + \int\limits_0^{2m} \tilde{\Phi}(r) dr. \quad (B.21)$$

By replacement $r - 2m = u$, from Eq.(B.21) we obtain

$$\int_K \left([\mathbf{R}^-_\epsilon]^2_2\right)_\epsilon \Phi(x) d^3x = \int_K \left([\mathbf{R}^-_\epsilon]^3_3\right)_\epsilon \Phi(x) d^3x = \int\limits_{-2m}^{0} \frac{u\tilde{\Phi}(u+2m)du}{(u^2+\epsilon^2)^{1/2}} + \int\limits_{-2m}^{0} \tilde{\Phi}(u+2m)du. \quad (B.22)$$

By replacement $u = \epsilon\eta$, from (B.22) we obtain

$$\mathbf{I}^-_3(\epsilon) = \int_K \left([\mathbf{R}^-_\epsilon]^3_3\right)_\epsilon \Phi(x) d^3x = \mathbf{I}^-_2(\epsilon) = \int_K \left([\mathbf{R}^-_\epsilon]^2_2\right)_\epsilon \Phi(\vec{x}) d^3x = \epsilon \times \left(\int\limits_{-\frac{2m}{\epsilon}}^{0} \frac{\eta\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{1/2}} + \int\limits_{-\frac{2m}{\epsilon}}^{0} \tilde{\Phi}(\epsilon\eta+2m)d\eta \right), \quad (B.23)$$

which is calculated to give

$$\begin{aligned}
\mathbf{I}_3^-(\epsilon) = \mathbf{I}_2^-(\epsilon) = \epsilon\frac{\tilde{\Phi}(2m)}{0!}\int_{-\frac{2m}{\epsilon}}^{0}\left[\frac{\eta}{(\eta^2+1)^{1/2}}+1\right]d\eta + \\
+\frac{\epsilon^2}{1!}\int_{-\frac{2m}{\epsilon}}^{0}\left[\frac{\eta}{(\eta^2+1)^{1/2}}+1\right]\tilde{\Phi}^{(1)}(\xi)\eta d\eta = \\
\epsilon\tilde{\Phi}(2m)\left[1-\sqrt{\left(\frac{2m}{\epsilon}\right)^2+1}+\frac{2m}{\epsilon}\right] + \\
+\frac{\epsilon^2}{1}\int_{-\frac{2m}{\epsilon}}^{0}\left[\frac{\eta}{(\eta^2+1)^{1/2}}+1\right]\tilde{\Phi}^{(1)}(\xi)\eta d\eta,
\end{aligned} \qquad (B.24)$$

where we have expressed the function $\widetilde{\Phi}(\epsilon\eta+2m)$ as

$$\begin{cases} \widetilde{\Phi}(\epsilon\eta+2m) = \sum_{l=0}^{n-1}\frac{\Phi^{(l)}(2m)}{l!}(\epsilon\eta)^l + \frac{1}{n!}(\epsilon\eta)^n\Phi^{(n)}(\xi)\ , \\ \xi \triangleq \theta\epsilon\eta+2m\ ,\quad 1>\theta>0\ ,\quad n=1 \end{cases} \qquad (B.25)$$

with $\tilde{\Phi}^{(l)} \triangleq d^l\tilde{\Phi}/dr^l$. Equation (B.25) gives

$$\begin{cases} \lim_{\epsilon\to 0}\mathbf{I}_3^-(\epsilon) = \lim_{\epsilon\to 0}\mathbf{I}_2^-(\epsilon) = \\ \lim_{\epsilon\to 0}\left\{\epsilon\tilde{\Phi}(2m)\left[1-\sqrt{\left(\frac{2m}{\epsilon}\right)^2+1}+\frac{2m}{\epsilon}\right]\right\} + \\ +\lim_{\epsilon\to 0}\left\{\frac{\epsilon^2}{2}\int_{-\frac{2m}{\epsilon}}^{0}\left[\frac{\eta}{(\eta^2+1)^{1/2}}+1\right]\tilde{\Phi}^{(1)}(\xi)\eta d\eta\right\} = 0. \end{cases} \qquad (B.26)$$

Since $S'_{2m}(B^-(0,2m)) \subset D'(\mathbb{R}^3)$, where $B^-(0,2m) = \{x\in\mathbb{R}^3 | 0 \leqslant \|x\| \leqslant 2m\}$ from

$$\begin{cases} w-\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^-]_3^3 = \lim_{\epsilon\to 0}\mathbf{I}_3^-(\epsilon) = 0. \\ w-\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^-]_2^2 = \lim_{\epsilon\to 0}\mathbf{I}_2^-(\epsilon) = 0. \end{cases} \qquad (B.27)$$

For $\left([\mathbf{R}_\epsilon^-]_1^1\right)_\epsilon, \left([\mathbf{R}_\epsilon^-]_0^0\right)_\epsilon$ we get:

$$2\int_K\left([\mathbf{R}_\epsilon^-]_1^1\right)_\epsilon\Phi(x)d^3x = 2\int_K\left([\mathbf{R}_\epsilon^-]_0^0\right)_\epsilon\Phi(x)d^3x =$$

$$\int_0^{2m}(r^2(h_\epsilon^{-\prime\prime})_\epsilon + 2r(h_\epsilon^{-\prime})_\epsilon)\tilde{\Phi}(r)dr = \tag{B.28}$$

$$= \int_0^{2m}\left\{\frac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} - \frac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}}\right\}\tilde{\Phi}(r)dr.$$

By replacement $r - 2m = u$, from (B.28) we obtain

$$I_1^+(\epsilon) = 2\int\left([\mathbf{R}_\epsilon^-]_1^1\right)_\epsilon\Phi(x)d^3x = I_2^+(\epsilon) = 2\int\left([\mathbf{R}_\epsilon^-]_0^0\right)_\epsilon\Phi(x)d^3x$$

$$= \int_0^{2m}(r^2(h_\epsilon^{-\prime\prime})_\epsilon + 2r(h_\epsilon^{-\prime})_\epsilon)\tilde{\Phi}(r)dr = \tag{B.29}$$

$$= \int_{-2m}^{0}\left\{\frac{u+2m}{(u^2+\epsilon^2)^{1/2}} - \frac{u^2(u+2m)}{(u^2+\epsilon^2)^{3/2}}\right\}\tilde{\Phi}(u+2m)du.$$

By replacement $u = \epsilon\eta$, from (B.29) we obtain

$$2\int_K\left([\mathbf{R}_\epsilon^-]_1^1\right)_\epsilon\Phi(x)d^3x = 2\int_K\left([\mathbf{R}_\epsilon^-]_0^0\right)_\epsilon\Phi(x)d^3x =$$

$$\int_{-\frac{2m}{\epsilon}}^{0}(r^2(h_\epsilon^{-\prime\prime})_\epsilon + 2r(h_\epsilon^{-\prime})_\epsilon)\tilde{\Phi}(r)dr =$$

$$= \epsilon\int_{-\frac{2m}{\epsilon}}^{0}\left\{\frac{\epsilon\eta+2m}{(\epsilon^2\eta^2+\epsilon^2)^{1/2}} - \frac{\epsilon^2\eta^2(\epsilon\eta+2m)}{(\epsilon^2\eta^2+\epsilon^2)^{3/2}}\right\}\tilde{\Phi}(\epsilon\eta+2m)d\eta =$$

$$\int_{-\frac{2m}{\epsilon}}^{0}\frac{\epsilon^2\eta\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{1/2}} + 2m\int_{-\frac{2m}{\epsilon}}^{0}\frac{\epsilon\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{1/2}} - \tag{B.30}$$

$$-\int_{-\frac{2m}{\epsilon}}^{0}\frac{\epsilon^4\eta^3\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{3/2}} - 2m\int_{-\frac{2m}{\epsilon}}^{0}\frac{\epsilon^3\eta^2\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{3/2}} =$$

$$\epsilon\int_{-\frac{2m}{\epsilon}}^{0}\frac{\eta\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{1/2}} - \int_{-\frac{2m}{\epsilon}}^{0}\frac{\eta^3\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{3/2}} +$$

$$+2m\left[\int_{-\frac{2m}{\epsilon}}^{0}\frac{\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{1/2}} - \int_{-\frac{2m}{\epsilon}}^{0}\frac{\eta^2\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{3/2}}\right].$$

which is calculated to give

$$\mathbf{I}_0^-(\epsilon) = \mathbf{I}_1^-(\epsilon) = 2m\frac{\tilde{\Phi}(2m)}{0!}\epsilon^l \int\limits_{-\frac{2m}{\epsilon}}^{0}\left[\frac{1}{(\eta^2+1)^{1/2}} - \frac{\eta^2}{(\eta^2+1)^{3/2}}\right]d\eta + \qquad (B.31)$$

$$+\frac{\epsilon}{1!}\int\limits_{0}^{\frac{2m}{\epsilon}}\tilde{\Phi}^{(1)}(\xi)\left[\frac{1}{(\eta^2+1)^{1/2}} - \frac{\eta^2}{(\eta^2+1)^{3/2}}\right]\eta d\eta + O(\epsilon^2).$$

where we have expressed the function $\widetilde{\Phi}(\epsilon\eta + 2m)$ as

$$\widetilde{\Phi}(\epsilon\eta + 2m) = \sum_{l=0}^{n-1}\frac{\Phi^{\alpha\beta(l)}(2m)}{l!}(\epsilon\eta)^l + \frac{1}{n!}(\epsilon\eta)^n\Phi^{\alpha\beta(n)}(\xi)\ , \qquad (B.32)$$

$$\xi \triangleq \theta\epsilon\eta + 2m\ ,\quad 1 > \theta > 0\ ,\quad n = 1$$

with $\tilde{\Phi}^{(l)}(\xi) \triangleq d^l\tilde{\Phi}/d\xi^l$.Equation (B.32) gives

$$\lim_{\epsilon\to 0}\mathbf{I}_0^-(\epsilon) = \lim_{\epsilon\to 0}\mathbf{I}_1^-(\epsilon) =$$

$$2m\lim_{\epsilon\to 0}\left\{\frac{\tilde{\Phi}(2m)}{0!}\int\limits_{-\frac{2m}{\epsilon}}^{0}\left[\frac{1}{(\eta^2+1)^{1/2}} - \frac{\eta^2}{(\eta^2+1)^{3/2}}\right]d\eta\right\} = \qquad (B.33)$$

$$2m\tilde{\Phi}(2m)\lim_{s\to 0}\left[\int_{-s}^{0}\frac{d\eta}{(\eta^2+1)^{1/2}} - \int_{-s}^{0}\frac{\eta^2 d\eta}{(\eta^2+1)^{3/2}}\right] =$$

$$= 2m\tilde{\Phi}(2m).$$

where use is made of the relation

$$\lim_{s\to\infty}\left[\int\limits_{-s}^{0}\frac{d\eta}{(u^2+1)^{1/2}} - \int\limits_{-s}^{0}\frac{\eta^2 d\eta}{(\eta^2+1)^{3/2}}\right] = 1. \qquad (B.34)$$

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