The Dynamics of the Gravity Field

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Abstract

We derive the canonical momentum π_I of the gravity field e^I. Then we use it to derive the path integral of the gravity field. The canonical momentum π_I is represented in Lorentz group. We derive it from the holonomy $U(\gamma, A)$ of the connection A_a^i of Lorentz group. We derive the path integral of the gravity field as known in quantum fields theory and discuss the situation of free gravity field (like the electromagnetic field). We find that situation is only in the background spacetime, weak gravity, the situation of low matter density. We search for a theory in which the gravity field is dynamical at any energy in arbitrary curved spacetime $\{x^\mu\}$. For that, we suggest the duality $e^I \leftrightarrow \Sigma^{JK}$, where the field $\Sigma^{IJ} = e^I \wedge e^J$ is the Area field. That duality lets to the possibility to study both fields e^I and Σ^{IJ} in arbitrary curved spacetime. We find $e^I \rightarrow \Sigma^{JK}$ in spacelike and $\Sigma^{JK} \rightarrow e^I$ in timelike. We find that the tensor product of the gravity and area fields, in selfdual representation, satisfies reality condition. We apply that to derive the static potential of exchanging gravitons in scalar and spinor fields, the Newtonian gravitational potential.

1 The canonical conjugate field π^I and the path integral

We search for conditions to have a dynamical gravity field. The problem of the dynamics in general relativity is that the spacetime is itself a dynamical thing. It interacts with the matter, it is an operator $d\hat{x}^\mu$. Therefore we have to treat it as a quantum field like the other fields. But where they exist, this problem is solved by considering fields exist over fields not over the spacetime[1]. In background spacetime it is substantially different, as we will see, the gravity field becomes as usual fields.
As usual in quantum field theory we have to find the canonical conjugate field π^I(represented in Lorentz group) acts canonically on Lorentz vectors over 3d closed surface δM immersed in arbitrary curved spacetime x^μ of manifold M. That closed surface δM is parameterized by three parameters X^1, X^2, X^3. In a certain gauge, we consider them as a spatial part of Lorentz coordinate $X^I = X^0, X^1, X^2, X^3$ with the flat metric $(- + ++)$.
Therefore, the exterior derivative operator lets to the change along the norm of that surface, so it lets to the change in time X^0 direction. That lets the 3d surface extends and have four Lorentz spacetime $\left\{X^I\right\}$ parameterize the four dimensions x^μ of curved spacetime in the manifold M. That lets to propagation of the gravity field from surface to another.
For that, we suggest canonical states $\left|\tilde{e}^I\right\rangle$ and $\left|\pi^I\right\rangle$ represented in Lorentz group, we use them in deriving the path integral of the gravity field. We find that there is no propagation over the dynamical spacetime x^μ. But in background spacetime we find that the gravity field propagates freely like the electromagnetic fields.

Although the dynamics of the gravity field is built using Lorentz group elements, the measurements effect and depend on the dynamical spacetime x^μ. Because x^μ is itself a dynamical, it interacts with all fields. Therefore our need to the Lorentz representation is to have canonical dynamical laws, processes .... So we have to distinguish between the dynamics of the general relativity and its measurements.

The holonomy of the connection A in quantum gravity is[4]

$$U(\gamma, A) = TrPe^{i\oint_\gamma A} \tag{1.1}$$

The path ordered P is defined in:

$$Pe^{i\oint_\gamma A} = \sum_{n=0}^{\infty} \int\limits_0^1 ds_1 \int\limits_0^{s_1} ds_2 ... \int\limits_0^{s_{n-1}} ds_n iA\left(\gamma\left(s_n\right)\right)...iA\left(\gamma\left(s_1\right)\right) : \dot{\gamma}^\mu\left(s\right) = \frac{dx^\mu}{ds}$$

γ is a closed path in arbitrary curved spacetime x^μ. In irreducible representation in selfdual of Lorentz group we write $A = A^i\tau^i$, where τ^i are Pauli matrices. The element $U(\gamma, A)$ is invariant under local Lorentz transformation $V^I \to L^I_J(x)V^J$ and under arbitrary changing of the coordinates $dx^\mu \to \Lambda^\mu{}_\nu(x)dx^\nu$. Therefore the quantum gravity is studied using it[1].

The connection A is selfdual of Lorentz spin connection ω[1]:

$$A^i_\mu\left(x\right) = \left(P^i\right)_{IJ}\omega^{IJ}_\mu\left(x\right)$$

P^i are the selfdual projectors. We can write the holonomy using the spin connection $\omega^{IJ}_\mu dx^\mu$ of Lorentz group. We have

$$U(\gamma, \omega) = TrPe^{i\oint_\gamma \omega^I{}_J}$$

We expect that it has the same properties of $U(\gamma, A)$; satisfies the symmetries of GR. For free gravity field, we impose the relation:

$$\left(\omega_\mu\right)^{IJ} = \pi_K{}^{IJ}e^K_\mu$$

The conjugate field $\pi_K{}^{IJ}\left(x\right)$ is represented in Lorentz group and acts on its vectors. Therefore we consider it as a dynamical operator. The holonomy becomes

$$U(\gamma, \pi, e) = TrP\exp i\oint_\gamma \left(\pi_K{}^I{}_J\right)e^K_\mu dx^\mu$$

For free gravity field, we expect that the momentum π^{IJK} is antisymmetry. So we can write

$$\pi^{IJK} = \pi_L\varepsilon^{LIJK}$$

This is our starting point in studying the dynamics of the gravity. The holonomy becomes

$$U\left(\gamma, \pi, e\right) = TrP\exp i\oint_\gamma \left(\pi^{KI}{}_J\right)e_{K\mu}dx^\mu = TrP\exp i\oint_\gamma \left(\varepsilon^{LKI}{}_J\right)\pi_L e_{K\mu}dx^\mu$$

It becomes

$$\sum_{n=0}^{\infty}\int\limits_{0}^{1}ds_1\int\limits_{0}^{s_1}ds_2...\int\limits_{0}^{s_{n-1}}ds_n\left(i\varepsilon^{LKI}{}_J\pi_L e_{K\mu}\dot{\gamma}^{\mu}\right)(s_n)\left(i\varepsilon^{L_1K_1J}{}_{J_1}\pi_{L_1}e_{K_1\mu_1}\dot{\gamma}^{\mu_1}\right)(s_{n-1})$$
$$...\left(i\varepsilon^{L_{n-1}K_{n-1}J_{n-2}}{}_I\pi_{L_{n-1}}e_{K_{n-1}\mu_{n-1}}\dot{\gamma}^{\mu_{n-1}}\right)(s_1)$$

where $\left(i\varepsilon^{LKI}{}_J\pi_L e_{K\mu}\dot{\gamma}^{\mu}\right)(s_n)=i\varepsilon^{LKI}{}_J\pi_L(s_n)\,e_{K\mu}(s_n)\,\dot{\gamma}^{\mu}(s_n)$ with the tangent $\dot{\gamma}^{\mu}(s)=\frac{dx^{\mu}}{ds}$ on the closed path γ in the manifold M.

The integrals become over terms like

$$...\pi_I(s_j)e^I_{\mu}(s_i)\dot{\gamma}^{\mu}(s_i)ds_i...\pi_J(s_i)e^J_{\nu}(s_k)\dot{\gamma}^{\nu}(s_k)ds_k...$$

The holonomy $U(\gamma,\pi,e)$ satisfies the general relativity symmetries, invariance under local Lorentz transformation $V^I\to L^I{}_J(x)V^J$ and under arbitrary changing of the coordinates $dx^{\mu}\to\Lambda^{\mu}{}_{\nu}(x)dx^{\nu}$. Therefore we can use it in quantum gravity.

We expect $\pi_K e^K_{\mu}dx^{\mu}$ satisfies the same conditions if it is integrated over closed surface instead of the path γ. That is because

$$ed^4x=\frac{1}{4}d^3x_{\mu}\wedge dx^{\mu}=\frac{1}{4}e\varepsilon_{\mu\nu\rho\sigma}dx^{\nu}\wedge dx^{\rho}\wedge dx^{\sigma}\wedge dx^{\mu}/3!$$

is invariant element. Therefore we can replace $\pi_K e^K_{\mu}dx^{\mu}$ with

$$\pi_K e^{K\mu}d^3x_{\mu}=\pi_K e^{K\mu}e\varepsilon_{\mu\nu\rho\sigma}dx^{\nu}\wedge dx^{\rho}\wedge dx^{\sigma}/3!$$

With integrating it over three dimensions closed surface δM, it becomes invariant under GR transformations because in free gravity there are no sources for the gravity field. Therefore the flux of the vectors is invariant.
e is the determinant of the gravity field e^I_{μ}:

$$g_{\mu\nu}(x)=\eta_{IJ}e^I_{\mu}e^J_{\nu}\to\sqrt{-g}=e$$

In arbitrary transformation, we have the invariant element

$$\sqrt{g}\varepsilon_{i_1...i_n}=\sqrt{g'}\varepsilon'_{i_1...i_n}$$

Therefore

$$e\varepsilon_{\mu\nu\rho\sigma}dx^{\nu}\wedge dx^{\rho}\wedge dx^{\sigma}/3!=d^3x_{\mu}$$

Is a co-vector, as ∂_μ. By that, the integral

$$U(\delta M, \pi, e) = \exp i \oint_{\delta M} \pi_I e^{I\mu} e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3! = \exp i \oint_{\delta M} \pi_I e^{I\mu} d^3 x_\mu$$

Satisfies the same conditions of the holonomy $U(\gamma, A)$, invariant under local Lorentz transformation $V^I \to L^I_J(x) V^J$ and under arbitrary changing of the coordinates $dx^\mu \to \Lambda^\mu{}_\nu(x) dx^\nu$. That relates to physical reality, it is, the integral of free vector fields over a closed surface δM in a manifold M is invariant if there are no sources for those fields. It is the conversation. The spin connection ω^μ and so $\pi_K e^{K\mu}$, as vectors, satisfy that reality in free gravity.

The equation of motion of the gravity field e^I is

$$De^I = de^I + \omega^I{}_J \wedge e^J = 0$$

With our imposing $(\omega_\mu)^{IJ} = \pi_K{}^{IJ} e^K_\mu$, we get

$$de^I = -\pi_N{}^I{}_J e^N \wedge e^J$$

As we know, the tensor

$$e^N \wedge e^J = e^N_\mu e^J_\nu dx^\mu \wedge dx^\nu = \frac{1}{2}\left(e^N_\mu e^J_\nu - e^N_\nu e^J_\mu\right) dx^\mu \wedge dx^\nu$$

Measures the area in the manifold M. Therefore the changes of the gravity field around a closed path (rotation) relate to the flux of the momentum π throw the area which is determined by the closed path. It is like the magnetic field, generated by straight electric current. Therefore

$$e^N \wedge e^J \to Area$$

$$de^I = -\pi_N{}^I{}_J e^N \wedge e^J \to flux\ throw\ this\ Area$$

For that reason we suggested π^{IJK} is antisymmetry. We see that the flux depends on the momentum π.

In the integral

$$\exp i \oint_{\delta M} \pi_I e^{I\mu} e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!$$

We define canonical gravity field $\tilde{e}^I$:

$$\tilde{e}^I d^3X = \tilde{e}^I dX^1 dX^2 dX^3 \equiv e^{I\mu} e\varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!$$

We get

$$\tilde{U}_{\delta M}(\delta M, \pi, \tilde{e}) = \exp i \oint_{\delta M} \pi_I \tilde{e}^I d^3X$$

Where $X^I : I = 1,2,3$ parameterize the closed surface δM in the manifold M. In certain gauge, we consider $X^I : I = 1,2,3$ as a spatial part of Lorentz spacetime $X^I : I = 0,1,2,3$. Therefore the exterior derivative is along the time X^0. The time X^0 is the direction of the norm on the surface $\delta M(X^1, X^2, X^3)$. We will see that the result of the path integral is independent on this gauge.

The integral $\exp i \oint_{\delta M} \pi_I \tilde{e}^I d^3X$ satisfies the same conditions of the holonomy $U(\gamma, A)$, invariant under local Lorentz transformation and under arbitrary changing of the coordinates. Therefore we consider it as a canonical dynamical element.
Comparing it with

$$\langle \phi \mid \pi \rangle = \exp i \int d^3X \phi(X) \pi(X)/\hbar$$

the dynamical relation of scalar field ϕ. For $\hbar = 1$, we suggest canonical states $\left|\tilde{e}^I\right\rangle$ and $\left|\pi^I\right\rangle$ with

$$\left\langle \tilde{e}^I \mid \pi_I \right\rangle_{\delta M} = \exp i \int_{\delta M} \tilde{e}^I(X) \pi_I(X) d^3X$$

π_I is canonical conjugate field of $\tilde{e}^I$. We can write it over the surface δM like

$$\left\langle \tilde{e}^I \mid \pi_I \right\rangle_{\delta M} = \prod_{n,I} \left\langle \tilde{e}^I (x_n + dx_n) \mid \pi_I (x_n) \right\rangle_{\delta M}$$

With

$$\left\langle \tilde{e}^I (x_n + dx_n) \mid \pi_I (x_n) \right\rangle_{\delta M} = \exp i \tilde{e}^I (x_n + dx_n) \pi_I(x_n) d^3X \to \exp i \tilde{e}^I(x_n) \pi_I(x_n) d^3X$$

This relation is over the surface δM. In general, for two points in adjacent surfaces δM_1 and δM_2, we have

$$\left\langle \tilde{e}^I (x_n + dx_n) \mid \pi_I (x_n) \right\rangle = \exp i \tilde{e}^I (x_n + dx_n) \pi_I(x_n) d^3X \tag{1.2}$$

Here the variation

$$\tilde{e}^I(x_n + dx_n) - \tilde{e}^I(x_n)$$

Is exterior derivative along the time dX^0 in the direction of the norm on the surface δM_1, it lets to the propagation. That lets to extend the surface: $\delta M(X^1, X^2, X^3) \to M(X^0, X^1, X^2, X^3)$.

We need to make $\hat{e}d^4\hat{x}$ commutes with $\hat{\tilde{e}}^I d^3X$. For that we write

$$\begin{aligned}-\hat{e}d^4\hat{x} &= \hat{e}d\hat{x}^\mu \wedge \varepsilon_{\mu\nu\rho\sigma} d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma/4! \\ &= \hat{e}d\hat{x}^\mu \wedge \frac{\varepsilon_{\mu\nu\rho\sigma}}{4!}\frac{\partial\hat{x}^\nu}{\partial X^i}\frac{\partial\hat{x}^\rho}{\partial X^j}\frac{\partial\hat{x}^\sigma}{\partial X^k}\frac{\varepsilon^{ijk}}{3!}d^3X = \frac{1}{4}\hat{e}d\hat{x}^\mu\hat{n}_\mu d^3X\end{aligned}$$

The indexes ijk are Lorentz indexes for $I = 1, 2, 3$. As we assumed before, $X^I : I = 1, 2, 3$ parameterize the closed surface δM in the manifold M.
We can rewrite it(in certain gauge) like

$$-ed^4x = \frac{1}{4}edx^\mu n_\mu d^3X = \frac{1}{4}e\frac{\partial x^\mu}{\partial X^0}n_\mu d^3X dX^0 = \frac{1}{4}ee_0^\mu n_\mu d^3X dX^0$$

compare it with the term

$$\tilde{e}^I d^3X = e^{I\mu} e\varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma/3! = ee^{I\mu} n_\mu d^3X$$

We find it commutes with it

$$\left[\hat{e}\hat{e}^{I\mu}\hat{n}_\mu d^3X, \hat{e}\hat{e}_0^\mu\hat{n}_\mu d^3X dX^0\right] = 0 \to \left[\hat{\tilde{e}}^I d^3X, \hat{e}d^4\hat{x}\right] = 0$$

Where $\left[\hat{e}_\mu^I, \hat{e}_\nu^J\right] = 0$. Therefore the operator $\hat{e}d^4\hat{x}$ takes eigenvalues when it acts on the states $\left|\tilde{e}^I\right\rangle$.

The action of free gravity field is[1]

$$S(e,\omega) = \frac{1}{16\pi G}\int \varepsilon_{IJKL}\left(e^I \wedge e^J \wedge R^{KL}(\omega) + \lambda e^I \wedge e^J \wedge e^K \wedge e^L\right)$$

We consider only the first term

$$S(e,\omega) = c\int \varepsilon_{IJKL} e^I \wedge e^J \wedge R^{KL}(\omega)$$

C is constant. The Riemann curvature is

$$R^{KL}(\omega) = d\omega^{KL} + \omega^{K}{}_{M} \wedge \omega^{ML}$$

Using the relation we imposed before:

$$(\omega_\mu)^{IJ} = \pi_K{}^{IJ} e^K_\mu$$

the action becomes

$$S(e,\pi) = c\int \left[\varepsilon_{IJKL} e^I \wedge e^J \wedge d\left(\pi_M{}^{KL} e^M\right) + \varepsilon_{IJKL} e^I \wedge e^J \wedge \left(\pi_{K_1}{}^{K}{}_{M}\right) e^{K_1} \wedge \left(\pi_{K_2}{}^{ML}\right) e^{K_2}\right]$$

or

$$S(e,\pi) = c\int \left[\varepsilon_{IJKL} e^I \wedge e^J \wedge d\left(\pi_M{}^{KL} e^M\right) + \varepsilon_{IJKL} \left(\pi_{K_1}{}^{K}{}_{M}\right) \left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}\right] \tag{1.3}$$

We find the term $d\left(\pi_M{}^{KL} e^M\right)$ from

$$\varepsilon_{IJKL} d\left(e^I \wedge e^J \wedge \pi_M{}^{KL} e^M\right)$$

But we assume its integral is zero at infinity. We have

$$\begin{aligned}\varepsilon_{IJKL} d\left(e^I \wedge e^J \wedge \pi_M{}^{KL} e^M\right) =& \varepsilon_{IJKL}\left(de^I\right) \wedge e^J \wedge \pi_M{}^{KL} e^M - \varepsilon_{IJKL} e^I \wedge \left(de^J\right) \wedge \pi_M{}^{KL} e^M \\ &+ \varepsilon_{IJKL} e^I \wedge e^J \wedge d\left(\pi_M{}^{KL} e^M\right)\end{aligned}$$

Rewriting

$$\begin{aligned}-\varepsilon_{IJKL} e^I \wedge \left(de^J\right) \wedge \left(\pi_M{}^{KL} e^M\right) &= -\varepsilon_{IJKL}\left(de^J\right) \wedge e^I \wedge \left(\pi_M{}^{KL} e^M\right) \\ &= \varepsilon_{JIKL}\left(de^J\right) \wedge e^I \wedge \pi_M{}^{KL} e^M\end{aligned}$$

Therefore

$$\varepsilon_{IJKL} d\left(e^I \wedge e^J \wedge \pi_M{}^{KL} e^M\right) = 2\varepsilon_{IJKL}\left(de^I\right) \wedge e^J \wedge \pi_M{}^{KL} e^M + \varepsilon_{IJKL} e^I \wedge e^J \wedge d\left(\pi_M{}^{KL} e^M\right)$$

By that we write the action as

$$S(e,\pi) = c\int \left[-2\varepsilon_{IJKL}\left(de^I\right) \wedge e^J \wedge \left(\pi_M{}^{KL} e^M\right) + \varepsilon_{IJKL}\left(\pi_{K_1}{}^{K}{}_{M}\right)\left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}\right]$$

Using the equation of motion of the gravity field

$$0 = De^I = de^I + \omega^I{}_J \wedge e^J = de^I + \pi_N{}^I{}_J e^N \wedge e^J$$

We get

$$de^I = -\pi_N{}^I{}_J e^N \wedge e^J$$

Inserting it in the action, it becomes

$$S(e,\pi) = c\int 2\varepsilon_{IJKL}(\pi_N{}^I{}_B)e^N \wedge e^B \wedge e^J \wedge \left(\pi_M{}^{KL}e^M\right) \\ + \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}$$

Or

$$S(e,\pi) = c\int 2\varepsilon_{IJKL}\left(\pi_N{}^I{}_B\right)\left(\pi_M{}^{KL}\right) e^N \wedge e^B \wedge e^J \wedge e^M \\ + \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}$$

Rewriting it like

$$S(e,\pi) = c\int 2\varepsilon_{IJKL}\left(\pi_N{}^I{}_B\right)\left(\pi_M{}^{KL}\right) e^B \wedge e^J \wedge e^N \wedge e^M \\ + \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}$$

Replacing $B \rightleftarrows I, N \to K_1$ and $M \to K_2$ in the first term, we get

$$S(e,\pi) = c\int 2\varepsilon_{BJKL}\left(\pi_{K_1}{}^B{}_I\right)\left(\pi_{K_2}{}^{KL}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2} \\ + \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}$$

We replace

$$e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2} \to \varepsilon^{IJK_1K_2} e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

We get

$$S(e,\pi) = c\int \left[2\varepsilon_{BJKL}\left(\pi_{K_1}{}^B{}_I\right)\left(\pi_{K_2}{}^{KL}\right)\varepsilon^{IJK_1K_2} + \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right)\varepsilon^{IJK_1K_2}\right] \\ \times e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

Using the relation $\pi^{IJL} = \pi_K \varepsilon^{KIJL}$ we imposed before. The action:

$$S(e,\pi) = c\int \left[2\varepsilon^{B}{}_{JKL}\left(\pi_{K_1BI}\right)\left(\pi_{K_2}{}^{KL}\right)\varepsilon^{IJK_1K_2} - 2\left(\pi_K{}^K{}_M\right)\left(\pi_L{}^{ML}\right) + 2\left(\pi_L{}^K{}_M\right)\left(\pi_K{}^{ML}\right)\right]$$
$$\times e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

becomes:

$$S(e,\pi) = c\int \left[2\varepsilon^{B}{}_{JKL}\pi^N\varepsilon_{NK_1BI}\left(\pi_{K_2}{}^{KL}\right)\varepsilon^{IJK_1K_2} + 2\left(\pi_{LKM}\right)\left(\pi^{KML}\right)\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

Using $\varepsilon_{NK_1BI} = -\varepsilon_{IK_1BN} = \varepsilon_{IK_1NB}$, $\varepsilon_{ILKM} = -\varepsilon_{ILMK}$ and $\varepsilon^{JKML} = -\varepsilon^{JLMK}$, we get

$$S(e,\pi) = c\int \left[2\varepsilon^{B}{}_{JKL}\pi^N\varepsilon_{IK_1NB}\left(\pi_{K_2}{}^{KL}\right)\left(-\varepsilon^{IK_1JK_2}\right) + 2\pi^I\varepsilon_{ILMK}\pi_J\varepsilon^{JLMK}\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

Using the property

$$\varepsilon_{IK_1NB}\varepsilon^{IK_1JK_2} = -2\left(\delta^J_N\delta^{K_2}_B - \delta^J_B\delta^{K_2}_N\right) \text{ and } \varepsilon_{ILMK}\varepsilon^{JLMK} = -6\delta^J_I$$

The action becomes

$$S(e,\pi) = c\int \left[4\varepsilon^{K_2}{}_{JKL}\pi^J\left(\pi_{K_2}{}^{KL}\right) - 12\pi_I\pi^I\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

Or

$$S(e,\pi) = c\int \left[4\varepsilon_{K_2JKL}\pi^J\left(\pi^{K_2KL}\right) - 12\pi^2\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

Then

$$S(e,\pi) = c\int \left[-4\varepsilon_{JK_2KL}\pi^J\pi_I\varepsilon^{IK_2KL} - 12\pi^2\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

The action becomes

$$S_0(e,\pi) = c\int \left[24\pi^2 - 12\pi^2\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3 = c\int 12\pi^2 e^0 \wedge e^1 \wedge e^2 \wedge e^3$$
$$= c\int 12\pi^2 e d^4x$$

In the background spacetime, we have $e \to 1 + \delta e$, therefore

$$S_0(\delta e,\pi) \to \int 12c\pi^2 d^4x + ...$$

To find its meaning we compare it with scalar field Lagrange in background spacetime, for $\hbar = 1$:

$$Ld^4x = (\pi\partial_0\phi - H(\phi,\pi))\, d^4x \text{ with } H(\phi,\pi)d^4x = \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right) d^4x$$

We conclude that the term

$$\int 12c\pi^2 d^4x \succ 0$$

Is the energy of the gravity field in background spacetime. As we will find in result of the path integral, in background spacetime limit, we have to replace $c \to -c$ when we compare with the electromagnetic field, therefore, in the background spacetime, we replace

$$S(e,\pi) \to -\int 12c\pi^2 d^4x = -\int Hd^4x$$

That is not surprise, because the general relativity equation (Einstein field equation) is derived to satisfy the energy conservation over curved spacetime:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

It satisfies the energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$. But, as we know, in quantum field theory in background spacetime limit, we have to write the canonical law of the conservation like

$$\partial_\mu \left(T^{\mu\nu}_{matter} + T^{\mu\nu}_{gravity}\right) = 0$$

Therefore we write

$$T_{\mu\nu} + \frac{-1}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = T_{\mu\nu}\,(matter) + T_{\mu\nu}\,(gravity) = constant$$

By that we conclude

$$T_{\mu\nu}\,(gravity) = -\frac{1}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)$$

Therefore we have to replace $c \to -c$, we see that when we compare it with the electromagnetic field in background spacetime.

Now we derive the path integral as usual. As we saw before, the operator $\hat{e}d^4\hat{x}$ takes eigenvalues when it acts on the states $\left|\tilde{e}^I\right\rangle$, by using (1.2) we have the amplitude

$$\left\langle \tilde{e}^I\left(x+dx\right)\right| e^{iS}\left|\pi_I\left(x\right)\right\rangle \to \left\langle \tilde{e}^I\left(x+dx\right)\right| e^{i12c\hat{\pi}^2\hat{e}d^4\hat{x}}\left|\pi_I\left(x\right)\right\rangle$$

$$= \exp\left(i12c\pi^2\left(x\right)e\left(x+dx\right)d^4x+i\tilde{e}^I\left(x+dx\right)\pi_I(x)d^3X\right)$$

$$\to \exp\left(i12c\pi^2\left(x\right)e\left(x\right)d^4x+i\tilde{e}^I\left(x+dx\right)\pi_I(x)d^3X\right)$$

We let the momentum π_I acts on the left. The amplitude of the propagation between two points x and $x+dx$ in different adjacent surfaces $\delta M_1 \to \delta M_2$ is

$$\left\langle \tilde{e}_I\left(x+dx\right)\right| e^{ic12\hat{\pi}^2\hat{e}d^4\hat{x}}\left|\tilde{e}^I\left(x\right)\right\rangle_{\delta M_1\to\delta M_2}$$

$$= \int\prod_I d\pi^I \left\langle \tilde{e}_I\left(x+dx\right)\right| e^{ic12\hat{\pi}^2\hat{e}d^4\hat{x}}\left|\pi^I\left(x\right)\right\rangle_{\delta M_1\to\delta M_2}\left\langle \pi_I\left(x\right)\,\middle|\,\tilde{e}^I\left(x\right)\right\rangle_{\delta M_1}$$

$$= \int\prod_I d\pi^I \exp\left[i12c\pi^2\left(x\right)e\left(x+dx\right)d^4x+i\tilde{e}^I(x+dx)\pi_I(x)d^3X\right]\exp\left(-i\tilde{e}^I(x)\pi_I(x)d^3X\right)$$

$$\to \int\prod_I d\pi^I \exp\left[i12c\pi^2\left(x\right)e\left(x\right)d^4x+i\left(\tilde{e}^I(x+dx)-\tilde{e}^I(x)\right)\pi_I(x)d^3X\right]$$

The exterior derivative

$$\left(\tilde{e}^I(x+dx)-\tilde{e}^I(x)\right)d^3X = \frac{\partial\tilde{e}^I(x)}{\partial X^0}d^3X dX^0 = d\tilde{e}^I(x)d^3X$$

Is along the time dX^0 in the direction of the norm of the surface $\delta M(X^1,X^2,X^3)$, therefore it lets to propagate from surface to another.

We write the amplitude like

$$\left\langle \tilde{e}_I\left(x+dx\right)\right| e^{ic12\hat{\pi}^2\hat{e}d^4\hat{x}}\left|\tilde{e}^I\left(x\right)\right\rangle_{\delta M_1\to\delta M_2} = \int\prod_I d\pi^I \exp\left[i12c\pi^2\left(x\right)e\left(x\right)d^4x+i\pi_I(x)d\tilde{e}^I(x)d^3X\right]$$

The path integral is the integral of ordered product of those amplitudes over all spacetime points(over all ordered 3d surfaces).

$$\begin{aligned} W_{ST} &= \int\prod_I D\tilde{e}^I D\pi_I \exp i\int\left(12c\pi^2 e d^4x+\pi_I d\tilde{e}^I d^3X\right) \\ &= \int\prod_I D\tilde{e}^I D\pi_I \exp i\int\left(12c\pi^2 e^0\wedge e^1\wedge e^2\wedge e^3+\pi_I d\tilde{e}^I d^3X\right) \end{aligned}$$

For selfdual representation, we consider that propagation in the direction of expanding of the surface(positive direction).

There is no problem with Lorentz non-invariance in $\frac{\partial \tilde{e}^I(x)}{\partial X^0} d^3X dX^0$ because the equation of motion, we find in the result of the path integral, is

$$\frac{\partial \tilde{e}^I(x)}{\partial X^0} \propto -\pi^I$$

Therefore

$$\frac{\partial \tilde{e}^I(x)}{\partial X^0} \pi_I d^3X dX^0 \propto -\pi_I \pi^I d^3X dX^0$$

This is Lorentz invariant. This is like the equation of motion of the scalar field $\pi = \partial_0 \phi$ which solves the same problem.

In our gauge we have

$$\begin{aligned}\pi_I \pi^I d^3X dX^0 \to \pi^2 dX^0 \wedge dX^1 \wedge dX^2 \wedge dX^3 &= \pi^2 e^0_\mu e^1_\nu e^2_\rho e^3_\sigma dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \\ &= \pi^2 e^0_\mu e^1_\nu e^2_\rho e^3_\sigma \varepsilon^{\mu\nu\rho\sigma} d^4x = \pi^2 e d^4x\end{aligned}$$

It is invariant element; we find it in the path integral.
The path integral:

$$W_{ST} = \int \prod_I D\tilde{e}^I D\pi_I \exp i \int \left(12c\pi^2 e^0 \wedge e^1 \wedge e^2 \wedge e^3 + \pi_I d\tilde{e}^I d^3X\right)$$

Vanishes unless

$$\frac{\delta}{\delta \pi_I} \left(12c\pi^2 e^0 \wedge e^1 \wedge e^2 \wedge e^3 + \pi_I d\tilde{e}^I d^3X\right) = 24c\pi^I e^0 \wedge e^1 \wedge e^2 \wedge e^3 + d\tilde{e}^I d^3X = 0$$

Therefore we have the path(equation of motion)

$$\hat{\pi}^I = \frac{-1}{24c} \left(\hat{e}^0 \wedge \hat{e}^1 \wedge \hat{e}^2 \wedge \hat{e}^3\right)^{-1} d\hat{\tilde{e}}^I d^3X \tag{1.4}$$

Or

$$\pi^I \pi^J = \frac{1}{(24c)^2} \left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-2} d\tilde{e}^I d^3X d\tilde{e}^J d^3X \tag{1.5}$$

Therefore

$$12c\pi^2 e^0 \wedge e^1 \wedge e^2 \wedge e^3 + \pi_I d\tilde{e}^I d^3X = \frac{1}{48c}\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)$$
$$-\frac{1}{24c}\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)$$

The path integral becomes

$$W_{ST} = \int \prod_I D\tilde{e}^I Exp \frac{-i}{48c} \int \left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)$$

The canonical field $\tilde{e}^I$ is defined in

$$\tilde{e}^K d^3X = e^{K\mu} e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!$$

Therefore

$$\left(d\hat{\tilde{e}}^K\right) d^3X = \left(\hat{D}_{\mu_1} \hat{e}^{K\mu}\right) \hat{e} \varepsilon_{\mu\nu\rho\sigma} d\hat{x}^{\mu_1} \wedge d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma / 3!$$

Where D is the co-variant derivative defined in

$$DV^I = dV^I + \omega^I{}_J \wedge V^J$$

We have

$$\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right) = \frac{\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)}{e^0 \wedge e^1 \wedge e^2 \wedge e^3}$$

It becomes

$$\frac{\left(\hat{D}_{\mu_1} \hat{e}_I^\mu\right) \hat{e} \varepsilon_{\mu\nu\rho\sigma} d\hat{x}^{\mu_1} \wedge d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma \left(\hat{D}_{\mu_2} \hat{e}^{I\mu'}\right) \hat{e} \varepsilon_{\mu'\nu'\rho'\sigma'} d\hat{x}^{\mu_2} \wedge d\hat{x}^{\nu'} \wedge d\hat{x}^{\rho'} \wedge d\hat{x}^{\sigma'}}{3!3! \hat{e}^0_{\mu_3} \hat{e}^1_{\nu_3} \hat{e}^2_{\rho_3} \hat{e}^3_{\sigma_3} d\hat{x}^{\mu_3} \wedge d\hat{x}^{\nu_3} \wedge d\hat{x}^{\rho_3} \wedge d\hat{x}^{\sigma_3}}$$

Define the inverse:

$$\left(e^0_\mu e^1_\nu e^2_\rho e^3_\sigma dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma\right)^{-1} = E_0^{\mu'} E_1^{\nu'} E_2^{\rho'} E_3^{\sigma'} \frac{\partial}{\partial x^{\sigma'}} \wedge \frac{\partial}{\partial x^{\rho'}} \wedge \frac{\partial}{\partial x^{\nu'}} \wedge \frac{\partial}{\partial x^{\mu'}}$$

We write it in the form

$$e^0_\mu e^1_\nu e^2_\rho e^3_\sigma dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \frac{1}{4} e d^3x_\mu \wedge dx^\mu$$

Actually, we have to write the tensors $\varepsilon^{\mu\nu\rho\sigma}$ and $\varepsilon_{\mu\nu\rho\sigma}$ like $e^{-1}\varepsilon^{\mu\nu\rho\sigma}$ and $e\varepsilon_{\mu\nu\rho\sigma}$ but here we neglect that, because it gives the same results.

Therefore we write

$$E_0^{\mu'}E_1^{\nu'}E_2^{\rho'}E_3^{\sigma'}\partial_{\sigma'}\wedge\partial_{\rho'}\wedge\partial_{\nu'}\wedge\partial_{\mu'}=E\partial_\nu\wedge\partial^{3\nu}$$

With inner product like

$$\left(E\partial_\nu\wedge\partial^{3\nu}\right)\left(\frac{1}{4}ed^3x_\mu\wedge dx^\mu\right)=\frac{1}{4}Ee\partial_\nu\wedge\partial^{3\nu}d^3x_\mu\wedge dx^\mu=\frac{1}{4}Ee\left(\delta_\mu^\nu\right)\partial_\nu dx^\mu=Ee=1$$

also we can write it like

$$\left(E\partial_\nu\wedge\partial^{3\nu}\right)\left(ed^3x_{\mu'}\wedge dx^\mu\right)=Ee\partial_\nu\wedge\partial^{3\nu}d^3x_{\mu'}\wedge dx^\mu=Ee\delta_{\mu'}^\nu\partial_\nu dx^\mu=\delta_{\mu'}^\mu$$

We can write

$$\left(D_{\mu_1}e_I^\mu\right)e\varepsilon_{\mu\nu\rho\sigma}dx^{\mu_1}\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma/3!\to\left(D_{\mu_1}e_I^\mu\right)edx^{\mu_1}\wedge d^3x_\mu=-\left(D_{\mu_1}e_I^\mu\right)ed^3x_\mu\wedge dx^{\mu_1}$$

Also

$$\left(D_{\mu_2}e^{I\mu'}\right)e\varepsilon_{\mu'\nu'\rho'\sigma'}dx^{\mu_2}\wedge dx^{\nu'}\wedge dx^{\rho'}\wedge dx^{\sigma'}/3!\to-\left(D_{\mu_2}e^{I\mu'}\right)ed^3x_{\mu'}\wedge dx^{\mu_2}$$

We conclude

$$d^3x_\mu\wedge dx^\mu=-dx_\mu\wedge d^3x^\mu\to d^3x_\mu\wedge dx^{\mu_1}=-dx_\mu\wedge d^3x^{\mu_1}$$

Therefore

$$-\left(D_{\mu_1}e_I^\mu\right)ed^3x_\mu\wedge dx^{\mu_1}\to\left(D_{\mu_1}e_I^\mu\right)edx_\mu\wedge d^3x^{\mu_1}$$

By that the term

$$\frac{\left(\hat{D}_{\mu_1}\hat{e}_I^\mu\right)\hat{e}\varepsilon_{\mu\nu\rho\sigma}d\hat{x}^{\mu_1}\wedge d\hat{x}^\nu\wedge d\hat{x}^\rho\wedge d\hat{x}^\sigma\left(\hat{D}_{\mu_2}\hat{e}^{I\mu'}\right)\hat{e}\varepsilon_{\mu'\nu'\rho'\sigma'}d\hat{x}^{\mu_2}\wedge d\hat{x}^{\nu'}\wedge d\hat{x}^{\rho'}\wedge d\hat{x}^{\sigma'}}{3!3!\hat{e}_{\mu_3}^0\hat{e}_{\nu_3}^1\hat{e}_{\rho_3}^2\hat{e}_{\sigma_3}^3d\hat{x}^{\mu_3}\wedge d\hat{x}^{\nu_3}\wedge d\hat{x}^{\rho_3}\wedge d\hat{x}^{\sigma_3}}$$

becomes

$$-\left(E\partial_\nu\wedge\partial^{3\nu}\right)\left(\left(D_{\mu_1}e_I^\mu\right)edx_\mu\wedge d^3x^{\mu_1}\right)\left(\left(D_{\mu_2}e^{I\mu'}\right)ed^3x_{\mu'}\wedge dx^{\mu_2}\right)$$

$$=\left(D^{\mu_1}e_{I\mu}\right)\left(D_{\mu_2}e^{I\mu'}\right)e\left(\partial_\nu\wedge\partial^{3\nu}\right)\left(d^3x_{\mu_1}\wedge dx^\mu\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right)$$

We used

$$-dx_\mu \wedge d^3x^{\mu_1} = d^3x^{\mu_1} \wedge dx_\mu = d^3x_{\mu_1} \wedge dx^\mu$$

therefore we can write

$$\frac{\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)}{e^0 \wedge e^1 \wedge e^2 \wedge e^3} \to \left(D^{\mu_1} e_{I\mu}\right)\left(D_{\mu_2} e^{I\mu'}\right) e\left(\partial_\nu \wedge \partial^{3\nu}\right)\left(d^3x_{\mu_1} \wedge dx^\mu\right)\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right)$$

We can choose the contraction:

$$\left(\partial_\nu \wedge \partial^{3\nu}\right)\left(d^3x_{\mu_1} \wedge dx^\mu\right)\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right) = \left(\partial_\nu \wedge \partial^{3\nu} d^3x_{\mu_1} \wedge dx^\mu\right)\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right)$$

$$= \delta^\nu_{\mu_1}\left(\partial_\nu \wedge dx^\mu\right)\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right) = \delta^\nu_{\mu_1}\left(-dx^\mu \wedge \partial_\nu\right)\left(-dx^{\mu_2} \wedge d^3x_{\mu'}\right)$$

$$= \delta^\nu_{\mu_1} dx^\mu \wedge \partial_\nu dx^{\mu_2} \wedge d^3x_{\mu'} = \delta^\nu_{\mu_1}\delta^{\mu_2}_\nu dx^\mu \wedge d^3x_{\mu'}$$

Therefore we can write

$$\frac{\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)}{e^0 \wedge e^1 \wedge e^2 \wedge e^3} \to \left(D^{\mu_1} e_{I\mu}\right)\left(D_{\mu_2} e^{I\mu'}\right) e\delta^\nu_{\mu_1}\delta^{\mu_2}_\nu dx^\mu \wedge d^3x_{\mu'}$$

$$= \left(D_\nu e_{I\mu}\right)\left(D^\nu e^{I\mu'}\right) e dx^\mu \wedge d^3x_{\mu'} = -\left(D_\nu e_{I\mu}\right)\left(D^\nu e^{I\mu'}\right) e d^3x_{\mu'} \wedge dx^\mu$$

$$= -\left(D_\nu e_{I\mu}\right)\left(D^\nu e^{I\mu'}\right) e\delta^\mu_{\mu'} d^4x = -\left(D_\nu e_{I\mu}\right)\left(D^\nu e^{I\mu}\right) e d^4x$$

We can also choose another contraction:

$$\left(D^{\mu_1} e_{I\mu}\right)\left(D_{\mu_2} e^{I\mu'}\right) e\left(\partial_\nu \wedge \partial^{3\nu}\right)\left(d^3x_{\mu_1} \wedge dx^\mu\right)\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right) \to$$

$$\left(D^{\mu_1} e_{I\mu}\right)\left(D_{\mu_2} e^{I\mu'}\right) e\left(\partial_\nu \wedge \partial^{3\nu} d^3x_{\mu_1} \wedge dx^\mu\right)\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right)$$

$$= \left(D^{\mu_1} e_{I\mu}\right)\left(D_{\mu_2} e^{I\mu'}\right) e\left(\delta^\nu_{\mu_1}\partial_\nu \wedge dx^\mu\right)\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right)$$

$$= \delta^\nu_{\mu_1}\delta^\mu_\nu\left(D^{\mu_1} e_{I\mu}\right)\left(D_{\mu_2} e^{I\mu'}\right) e\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right)$$

It becomes

$$\frac{\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)}{e^0 \wedge e^1 \wedge e^2 \wedge e^3} \to \left(D^\mu e_{I\mu}\right)\left(D_{\mu'} e^{I\mu'}\right) e d^4x$$

By the two possible contractions, we can write the final result as

$$\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right) = \frac{-1}{2}\left(D_\mu e^\nu_I D^\mu e^I_\nu - D_\mu e^\nu_I D_\nu e^{I\mu}\right) e d^4x$$

This Lagrange is like the Lagrange of the electromagnetic field. Also it is independent on the gauge we chose for the surface δM. it is invariant under local Lorentz transformation $V^I \to L^I{}_J(x)V^J$ and any coordinate transformation $V^\mu \to \frac{\partial x^\mu}{\partial x'^\nu}V'^\nu$.

The path integral of the gravity field becomes, after replacing $c \to -c$.

$$W_{ST} = \int \prod_I De^I \exp \frac{i}{48c}\frac{1}{2}\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right) ed^4x$$

With the free gravity field Lagrange

$$Ld^4x = \frac{1}{48c}\frac{1}{2}\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right) ed^4x \qquad (1.6)$$

We determine the constant c in the Newtonian gravitational potential $c \succ 0$.
In background spacetime, weak gravity, $D_\mu \to \partial_\mu$ and $e \to 1+\delta e$, we have

$$L \to \frac{1}{48c}\frac{1}{2}\left(-\partial_\mu e_I^\nu \partial^\mu e_\nu^I + \partial_\mu e_I^\nu \partial_\nu e^{I\mu}\right)$$

Or

$$L_0 = \frac{1}{48c}\frac{1}{2}\eta_{IJ}e_\mu^I\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)e_\nu^J$$

Without background spacetime approximation, in strong gravity field, we have problem with the determinant e, it is

$$W_{ST} = \int \prod_I De^I exp \frac{i}{48c}\int \frac{1}{2}\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right) e_{\mu_1}^0 e_{\nu_1}^1 e_\rho^2 e_\sigma^3 \varepsilon^{\mu_1\nu_1\rho\sigma} d^4x$$

with $\eta_{0123} = -1$ we rewrite

$$W_{ST} = \int \prod_I De^I exp \frac{i}{48c}\int \frac{1}{2}\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right)\left(-\eta_{I_1JKL}\right) e_{\mu_1}^{I_1} e_{\nu_1}^J e_\rho^K e_\sigma^L \varepsilon^{\mu_1\nu_1\rho\sigma} d^4x/4!$$

Always there is a field e_ρ^K which is different from e_μ^I and e_ν^I therefore the integral over it gives delta Dirac:

$$\int \prod_I De^I exp \frac{i}{48c}\int \frac{1}{2}\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right)\left(-\eta_{I_1JKL}\right) e_{\mu_1}^{I_1} e_{\nu_1}^J e_\rho^K e_\sigma^L \varepsilon^{\mu_1\nu_1\rho\sigma} d^4x/4!$$

$$\to \delta\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right)$$
$$\to -D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu} = 0$$

it gives

$$\pi^2 = 0 \ \to \ S(\pi, e) = c\int 12\pi^2 e d^4x = 0 \ \to \ H(\pi, e) = 0$$

This path integral is trivial, there is no propagation because there is no gravity energy $H(\pi, e) = 0$. Like Wheeler-DeWitt equation $\hat{H}\psi = 0$. The reason is that because the gravity field e_μ^I has the entity of the spacetime, it is impossible for the spacetime to be a dynamical over itself, to propagate over itself.

But if we write $e_\mu^I(x) \to \delta_\mu^I + h_\mu^I(x)$ the path integral exists, the propagation is possible. Therefore the dynamics of the gravity is being only over background spacetime. This is the situation of weak gravity (low energy densities). In this situation the gravity field becomes like the other fields.

Latter we will search for conditions to make the gravity field propagate over x^μ, for that we impose the duality; Gravity-Area.

The path integral of weak gravity field in background spacetime is

$$w = \int \prod_I De^I \operatorname{exp} i \int \frac{1}{48c}\frac{1}{2} e_\mu^I \left(\eta_{IJ} g^{\mu\nu}\partial^2 - \eta_{IJ}\partial^\mu\partial^\nu\right) e_\nu^J d^4x \tag{1.7}$$

The gravity field propagator, $g = \eta$ and $k_\mu e^{\mu I} = 0$, is

$$\Delta_{IJ}^{\mu\nu}(x_2 - x_1) = 48c\int \frac{d^4k}{(2\pi)^4}\frac{\eta_{IJ} g^{\mu\nu} e^{ik(x_2 - x_1)}}{k^2 - i\varepsilon}$$

Or

$$\Delta_{\rho\sigma}^{\mu\nu}(x_2 - x_1) = 48c\int \frac{d^4k}{(2\pi)^4}\frac{g_{\rho\sigma} g^{\mu\nu} e^{ik(x_2 - x_1)}}{k^2 - i\varepsilon} \tag{1.8}$$

We will use this propagation in the gravity interaction with the scalar and spinor fields.

2 The need to the duality Gravity-Area

We search for conditions to have a dynamical gravity field in arbitrary curved spacetime without spacetime background approximation. We found that the spacetime path integral W_{ST} is trivial. There is no propagation without spacetime background. We can solve that problem by assuming that the fields exist over themselves not in the spacetime[1]. Therefore the spacetime is measured thing by its interactions with the matter.

According to general relativity, the length, the area and the volume are another form of the gravity. We can explain that by the duality *gravity* $\leftrightarrow$ *areas* and volumes. We try to find this duality using the trivial path integral W_{ST} by finding conditions allow the gravity field to propagate. That propagation is $e^I \leftrightarrow \Sigma^{JK}$ it means they propagate when they change to each other. Also we find that the tensor product of them $\left|e^I\right\rangle \otimes \left|\Sigma^{JK}\right\rangle$, in selfdual representation, satisfies the reality condition.

As we saw in the path integral of gravity field over curved spacetime we have problem in $e^0 \wedge e^1 \wedge e^2 \wedge e^3$. All of them must be different, the integral over one of them is delta Dirac. This is trivial path integral W_{ST}. Therefore there must be a new field, it is the area field $\Sigma^{KJ} = e^K \wedge e^J$ by that the path integral of the gravity field exists. It means that the gravity field is dynamical over the area field not over the spacetime.

Starting from the full Lagrange (1.6):

$$Ld^4x = \frac{1}{48c}\frac{1}{2}\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right) ed^4x$$

The covariant derivative is

$$De^I = de^I + \omega^I{}_J \wedge e^J$$

Using our assuming

$$\omega^{IJ} = \pi_K{}^{IJ} e^K$$

The covariant derivative becomes

$$De^I = de^I + \left(\pi_K{}^I{}_J\right) e^K \wedge e^J$$

The Area field is anti-symmetry field:

$$\Sigma_{\mu\nu}^{IJ} = \frac{1}{2}\left(e_\mu^I e_\nu^J - e_\nu^I e_\mu^J\right)$$

Inserting it in the covariant derivative, it becomes

$$De^I = de^I + \left(\pi_K{}^I{}_J\right)\Sigma^{KJ} = de^I + \pi^{KIJ}\Sigma_{KJ}$$

Using our assumption

$$\pi^{IJK} = \pi_L \varepsilon^{LIJK}$$

The derivative becomes

$$De^I = de^I + \pi^{KIJ}\Sigma_{KJ} = de^I + \pi_L \varepsilon^{LKIJ}\Sigma_{KJ} = de^I + \varepsilon^{ILKJ}\pi_L\Sigma_{KJ}$$

By that we have two fields e^I and Σ^{KJ} in the Lagrange. They interact, that lets to the duality $e^I \leftrightarrow \Sigma^{KJ}$.

The full Lagrange of the gravity field is

$$Ld^4x = \frac{1}{48c}\frac{1}{2}\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right) ed^4x$$

We have

$$-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu} = -D_\mu e_I^\nu \left(D^\mu e_\nu^I - D_\nu e^{I\mu}\right)$$

It becomes

$$-\left(\partial_\mu e_I^\nu + \varepsilon_{IJKL}\pi^J \Sigma_\mu^{KL\nu}\right)\left(\partial^\mu e_\nu^I + \varepsilon^{IJ_1K_1L_1}\pi_{J_1}\Sigma_{K_1L_1\nu}^\mu - \partial_\nu e^{I\mu} - \varepsilon^{IJ_1K_1L_1}\pi_{J_1}\Sigma_{K_1L_1\nu}{}^\mu\right)$$

Or

$$-\left(\partial_\mu e_I^\nu + \varepsilon_{IJKL}\pi^J \Sigma_\mu^{KL\nu}\right)\left(\partial^\mu e_\nu^I - \partial_\nu e^{I\mu} + 2\varepsilon^{IJ_1K_1L_1}\pi_{J_1}\Sigma_{K_1L_1\nu}^\mu\right)$$

It becomes

$$\begin{aligned} &-\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_\nu^I - \partial_\nu e^{I\mu}\right) - 2\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e_I^\nu\right)\pi_{J_1}\Sigma_{K_1L_1\nu}^\mu \\ &\quad - 2\varepsilon_{IJKL}\Sigma_\mu^{KL\nu}\pi^J\left(\frac{\partial^\mu e_\nu^I - \partial_\nu e^{I\mu}}{2}\right) - 2\varepsilon^{IJ_1K_1L_1}\varepsilon_{IJKL}\Sigma_\mu^{KL\nu}\pi^J\pi_{J_1}\Sigma_{K_1L_1\nu}^\mu \end{aligned}$$

To complete it, we need to replace the momentum π^I by its value, we had before (1.4) and (1.5):

$$\pi^I\pi^J = \frac{-1}{(48c)^2}\frac{1}{2}\left(D_\mu e^{I\nu} D^\mu e_\nu^J - D_\mu e^{I\nu} D_\nu e^{J\mu}\right)$$

We consider

$$\pi^I\pi^J = \frac{-1}{(48c)^2}\frac{1}{2}\left(\partial_\mu e^{I\nu}\partial^\mu e^J_\nu\right)$$

We expect the contraction

$$2\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e^\nu_I\right)\pi_{J_1}\Sigma^\mu_{K_1L_1\nu} \to \frac{-1}{(48c)^2}\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma^\rho_{K_1L_1\nu}$$

Therefore we rewrite

$$-\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e^I_\nu - \partial_\nu e^{I\mu}\right) + \frac{2}{(48c)^2}\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma^\rho_{K_1L_1\nu}$$

$$+\frac{1}{(48c)^2}\varepsilon^{IJ_1K_1L_1}\varepsilon_{IJKL}\Sigma^{KL\nu}_\mu\left(\partial_\sigma e^J_\rho\right)\left(\partial^\sigma e^\rho_{J_1}\right)\Sigma^\mu_{K_1L_1\nu}$$

The Lagrange

$$Ld^4x = \frac{1}{48c}\frac{1}{2}\left(-D_\mu e^\nu_I D^\mu e^I_\nu + D_\mu e^\nu_I D_\nu e^{I\mu}\right)ed^4x$$

Becomes

$$\begin{aligned} Ld^4x \to & \frac{1}{48c}\frac{-1}{2}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e^I_\nu\right)ed^4x + \frac{1}{(48c)^3}\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma^\rho_{K_1L_1\nu}ed^4x \\ & + \frac{1}{2*(48c)^3}\varepsilon^{IJ_1K_1L_1}\varepsilon_{IJKL}\left(\partial_\sigma e^J_\rho\right)\left(\partial^\sigma e^\rho_{J_1}\right)\Sigma^{KL}_{\mu\nu}\Sigma^{\mu\nu}_{K_1L_1}ed^4x \end{aligned}$$

We used the gauge $\partial_\mu e^{I\mu} = 0$.

Now we use the selfdual projection. For any real anti-symmetry tensor T^{IJ} we can write it in two unmixed representation, selfdual and anti-selfdual. In general relativity the selfdual is chosen, its projector is[1]

$$\left(P^i\right)_{jk} = \frac{1}{2}\varepsilon^i{}_{jk}\ ,\ \left(P^i\right)_{0j} = \frac{i}{2}\delta^i_j\ :\ i = I \text{ for } I = 1,2,3$$

We see that these projectors satisfy

$$2i\left(P^i\right)^{IJ}\left(P_i\right)^{KL} - 2i\left(\bar{P}^i\right)^{IJ}\left(\bar{P}_i\right)^{KL} \to \varepsilon^{IJKL}$$

It is a projection from $I \neq J$ and $K \neq L$ in the left to $I \neq J \neq K \neq L$ in the right.

The second term is for the anti-selfdual. Therefore we consider only the first term, we replace

$$\varepsilon^{IJKL} \to 2i \left(P^i\right)^{IJ} \left(P_i\right)^{KL}$$

We use it in the determinant e:

$$e = e^0_\mu e^1_\nu e^2_\rho e^3_\sigma \varepsilon^{\mu\nu\rho\sigma} \to -\varepsilon_{IJKL} e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma}/4! : \varepsilon_{0123} = -1$$

With selfdual projection, we have

$$e = -\varepsilon_{IJKL} e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma}/4! \to -2i \left(P^i\right)_{IJ} \left(P_i\right)_{KL} e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma}/4!$$

We can rewrite

$$e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma} = \frac{1}{2}\left(e^I_\mu e^J_\nu - e^I_\nu e^J_\mu\right) e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma} = \frac{1}{2}\left(e^I_\mu e^J_\nu - e^I_\nu e^J_\mu\right) \frac{1}{2}\left(e^K_\rho e^L_\sigma - e^K_\sigma e^L_\rho\right) \varepsilon^{\mu\nu\rho\sigma}$$

By that we can rewrite it using the area field Σ^{IJ}

$$e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma} = \Sigma^{IJ}_{\mu\nu} \Sigma^{KL}_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma}$$

Therefore the determinant e becomes

$$e = -\varepsilon_{IJKL} e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma}/4! \to -\frac{2i}{4!} \left(P^i\right)_{IJ} \left(P_i\right)_{KL} \Sigma^{IJ}_{\mu\nu} \Sigma^{KL}_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma} \tag{2.1}$$

Now we can write the area field as a vector $i = 1, 2, 3$ in the selfdual representation

$$\Sigma^i_{\mu\nu} = \left(P^i\right)_{IJ} \Sigma^{IJ}_{\mu\nu}$$

Therefore the determinant e becomes

$$e \to -\frac{2i}{4!} \left(\Sigma^i\right)_{\mu\nu} \left(\Sigma_i\right)_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma} \quad \text{or} \quad -\frac{2i}{4!} \Sigma^i_{\mu\nu} \Sigma_{i\rho\sigma} \varepsilon^{\mu\nu\rho\sigma} \tag{2.2}$$

We wrote it in this form to get rid of the gravity field in the path integral. As we saw it lets to delta Dirac, it cancels the propagation.

By that, the full Lagrange of the gravity field:

$$Ld^4x \to \frac{1}{48c}\frac{-1}{2}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e^I_\nu\right) ed^4x + \frac{1}{(48c)^3} \varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e_{\rho J_1}\right) \Sigma^\rho_{K_1L_1\nu} ed^4x$$
$$+ \frac{1}{2*(48c)^3} \varepsilon^{IJ_1K_1L_1} \varepsilon_{IJKL} \left(\partial_\sigma e^J_\rho\right)\left(\partial^\sigma e^\rho_{J_1}\right) \Sigma^{KL}_{\mu\nu} \Sigma^{\mu\nu}_{K_1L_1} ed^4x$$

becomes

$$\begin{aligned}Ld^4x \to & \frac{1}{48c}\frac{-1}{2}\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_\nu^I\right)\left(-\frac{2i}{4!}\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)d^4x \\ & + \frac{1}{(48c)^3}\left(2ip_i^{IJ_1}\right)\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma_\nu^{i\rho}\left(-\frac{2i}{4!}\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)d^4x \\ & - \frac{2}{(48c)^3}\left(p_i\right)^{IJ_1}\left(p_j\right)_{IJ}\left(\partial_\sigma e_\rho^J\right)\left(\partial^\sigma e_{J_1}^\rho\right)\Sigma_{\mu\nu}^j\Sigma^{i\mu\nu}\left(-\frac{2i}{4!}\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)d^4x\end{aligned}$$

Or

$$\begin{aligned}Ld^4x \to & \frac{2i}{48c}\frac{1}{2}\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_\nu^I\right)\left(\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x \\ & + \frac{4}{(48c)^3}\left(p_i\right)^{IJ_1}\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma_\nu^{i\rho}\left(\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x \\ & + \frac{4i}{(48c)^3}\left(p_i\right)^{IJ_1}\left(p_j\right)_{IJ}\left(\partial_\sigma e_\rho^J\right)\left(\partial^\sigma e_{J_1}^\rho\right)\Sigma_{\mu\nu}^j\Sigma^{i\mu\nu}\left(\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x\end{aligned}$$

It is quadratic in e^I therefore its integral is not trivial. Here we can consider the area field Σ^i as a background field that the gravity field propagate over it. Or suggest the duality $e^I \leftrightarrow \Sigma^i$, by that the amplitude of propagation of e^I between x and $x+dx$ is $\left\langle e^I(x+dx) \mid \Sigma^i(x)\right\rangle$.

If we considered the first term. To discover its behavior, we test one wave $\cos\left(k_\mu x^\mu\right)$. We have

$$\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_\nu^I\right) \to -e_I^\nu\partial_\mu\partial^\mu e_\nu^I \to -\partial_\mu\partial^\mu\cos\left(k_\mu x^\mu\right) = k_\mu k^\mu\cos\left(k_\mu x^\mu\right)$$

Therefore

$$\begin{aligned}e^{i\int Ld^4x} & \to \exp\int\frac{2}{48c}\frac{i^2}{2}\left(k_\mu k^\mu e_I^\nu e_\nu^I\right)\left(\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x + ... \\ & \to \exp\int\frac{2}{48c}\frac{1}{2}\left(-k_\mu k^\mu\right)\left(e_I^\nu e_\nu^I\right)\left(\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x + ...\end{aligned}$$

Or

$$e^{iS} \to \exp\int\frac{2}{48c}\frac{1}{2}\left(k_0^2 - \vec{k}^2\right)\left(e_I^\nu e_\nu^I\right)\left(\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x + ... \tag{2.3}$$

We consider the area field is in the positive direction $Re\left(\Sigma_{\mu\nu}^i dx^\mu \wedge dx^\nu\right) \succ 0$, the direction of the expanding, then $Re\left(\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right) \succ 0$.

We find, in time-like $k_0^2 - \vec{k}^2 \succ 0$ the gravity field is created. And in the space-like $k_0^2 - \vec{k}^2 \prec 0$ the gravity field is annihilated $e_\mu^I \to \Sigma_{\nu\rho}^i$, oppositely to the area field, as we will see. This is the duality $e_\mu^I \leftrightarrow \Sigma_{\nu\rho}^i$. It is like to say, in time-like we find the gravity field and in the space-like we find the area field.
The time-like phase is the phase of exchanging the energies (interactions). While the space-like is the phase of the static fields, the situation of located matter. Therefore the spacetime in which the matter is located is consisted of quanta of area and volume. The duality $e_\mu^I \leftrightarrow \Sigma_{\nu\rho}^i$, as we will see, satisfies the reality, it is like the right and left spinor fields.

3 The Lagrange of the Area field

We derive the Lagrange of the area field, we find that in the background spacetime it is like the electromagnetic field but with opposite sign in the Lagrange. We can get rid of that opposite sign by replacing $\partial_\mu \to i\partial_\mu$ it is equivalent to replace $k_\mu \to ik_\mu$ in the free solutions: $e^{ikx} \to e^{-kx}$ *or* e^{kx}. We find the behavior of the area field is opposite to the gravity behavior. For that reason we suggest the duality gravity-area, which satisfies the realty.

The area field is defined in

$$\Sigma^{IJ} = e^I \wedge e^J \quad \text{with} \quad \Sigma_{\mu\nu}^{IJ} = \frac{1}{2}\left(e_\mu^I e_\nu^I - e_\nu^I e_\mu^I\right)$$

Starting with the Lagrange (1.3)

$$S(e,\pi) = c\int \left[\varepsilon_{IJKL} e^I \wedge e^J \wedge d\left(\pi_M{}^{KL} e^M\right) + \varepsilon_{IJKL} e^I \wedge e^J \wedge \left(\pi_{K_1}{}^K{}_M\right) e^{K_1} \wedge \left(\pi_{K_2}{}^{ML}\right) e^{K_2}\right]$$

As before we assume the integral of

$$\varepsilon_{IJKL} d\left(e^I \wedge e^J \wedge \left(\pi_M{}^{KL} e^M\right)\right) = \varepsilon_{IJKL} d\left(\Sigma^{IJ} \wedge \left(\pi_M{}^{KL} e^M\right)\right)$$

is zero at infinity, it becomes

$$d\Sigma^{IJ} \wedge \left(\pi_M{}^{KL}\right) e^M + e^I \wedge e^J \wedge d\left(\pi_M{}^{KL} e^M\right) = -\left(\pi_M{}^{KL}\right) e^M \wedge d\Sigma^{IJ} + e^I \wedge e^J \wedge d\left(\pi_M{}^{KL} e^M\right)$$

The Action becomes

$$S(e,\pi) = c\int \left[\varepsilon_{IJKL}\left(\pi_M{}^{KL}\right) e^M \wedge d\Sigma^{IJ} + \varepsilon_{IJKL}\Sigma^{IJ} \wedge \left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right) e^{K_1} \wedge e^{K_2}\right]$$

Or

$$S(e,\pi) = c\int\left[\varepsilon_{IJKL}\left(\pi_M{}^{KL}\right)e^M\wedge d\Sigma^{IJ} + \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right)\Sigma^{IJ}\wedge\Sigma^{K_1K_2}\right]$$

Using our imposing

$$\pi^{IJK} = \pi_L\varepsilon^{LIJK}$$

We get

$$\varepsilon_{IJKL}\left(\pi_M{}^{KL}\right)e^M = \varepsilon_{IJKL}\pi^{MKL}e_M = \varepsilon_{IJKL}\pi_N\varepsilon^{NMKL}e_M = -2\left(\pi_I e_J - \pi_J e_I\right)$$

We write

$$\Sigma^{IJ}\wedge\Sigma^{K_1K_2}\to\varepsilon^{IJK_1K_2}\Sigma^{01}\wedge\Sigma^{23}$$

So we have

$$\begin{aligned}\varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right)\Sigma^{IJ}\wedge\Sigma^{K_1K_2} &= \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right)\varepsilon^{IJK_1K_2}\Sigma^{01}\wedge\Sigma^{23}\\ &= 2\left(\pi_L{}^K{}_M\right)\left(\pi_K{}^{ML}\right)\Sigma^{01}\wedge\Sigma^{23} = 2\left(\pi_{LKM}\right)\left(\pi^{KML}\right)\Sigma^{01}\wedge\Sigma^{23}\\ &= 2\left(\pi_{KML}\right)\left(\pi^{KML}\right)\Sigma^{01}\wedge\Sigma^{23} = 2\pi^I\varepsilon_{IKML}\pi_J\varepsilon^{JKML}\Sigma^{01}\wedge\Sigma^{23}\\ &= -12\pi^2\Sigma^{01}\wedge\Sigma^{23}\end{aligned}$$

The Action becomes

$$S(e,\pi,\Sigma) = c\int\left[-2\left(\pi_I e_J - \pi_J e_I\right)\wedge d\Sigma^{IJ} - 12\pi_I\pi^I\Sigma^{01}\wedge\Sigma^{23}\right]$$

Because the area field Σ^{IJ} is anti-symmetry, we write

$$S(e,\pi,\Sigma) = c\int\left[-4\pi_I e_J\wedge d\Sigma^{IJ} - 12\pi_I\pi^I\Sigma^{01}\wedge\Sigma^{23}\right]$$

Using $\varepsilon_{0123} = -1$ we can rewrite it like

$$S(e,\pi,\Sigma) = c\int\left[-4\pi_I e_J\wedge d\Sigma^{IJ} + 12\pi_I\pi^I\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}/4!\right]$$

Or

$$S(e,\pi,\Sigma) = c\int\left[-4\pi_I e_J\wedge d\Sigma^{IJ} + \frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}\right]$$

The path integral over momentum π^I vanishes unless (the equation of motion)

$$\frac{\delta}{\delta\pi_I}\int\left[-4\pi_I e_J\wedge d\Sigma^{IJ}+\frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}\right]=0$$

But it is not easy to separate Σ from e. It is like the gravity field, it is separable only in weak gravity(background spacetime). Therefore we solve it in background spacetime.

$$\int\left(-4\pi_I e_J\wedge d\Sigma^{IJ}+\frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}\right)$$

$$\to\int\left(-4\pi_I e_{\mu J}\partial_\nu\Sigma^{IJ}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}+\frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}_{\mu\nu}\Sigma^{KL}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)d^4x$$

The background spacetime is

$$e^I_\mu(x)\to\delta^I_\mu+h^I_\mu(x)\ ,\qquad e\to 1+\delta e$$

The area field becomes

$$\Sigma^{IJ}_{\mu\nu}=\frac{1}{2}\left(e^I_\mu e^J_\nu-e^I_\nu e^J_\mu\right)\to\frac{1}{2}\left(\delta^I_\mu\delta^J_\nu-\delta^I_\nu\delta^J_\mu\right)+\frac{1}{2}\left(h^I_\mu\delta^J_\nu-h^I_\nu\delta^J_\mu\right)+\frac{1}{2}\left(\delta^I_\mu h^J_\nu-\delta^I_\nu h^J_\mu\right)$$

inserting it in the action:

$$S\left(e,\Sigma\right)=c\int\left(-4\pi_I e_{\mu J}\partial_\nu\Sigma^{IJ}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}+\frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}_{\mu\nu}\Sigma^{KL}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)d^4x$$

it becomes

$$S\left(e,\Sigma\right)\to S\left(h,\delta\Sigma\right)=c\int\left(-4\pi_I\partial_\nu\Sigma^{IJ}_{\rho\sigma}\varepsilon_J{}^{\nu\rho\sigma}+\frac{1}{2}\pi^2\left(-24\right)+\dots\right)d^4x$$

Therefore the condition(equation of motion):

$$\frac{\delta}{\delta\pi_I}\int\left[-4\pi_I e_J\wedge d\Sigma^{IJ}+\frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}\right]=0$$

approximates to

$$\frac{\delta}{\delta\pi_I}\int\left(-4\pi_I\partial_\nu\Sigma^I{}_{J\rho\sigma}\varepsilon^{J\nu\rho\sigma}+\frac{1}{2}\pi^2\left(-24\right)\right)d^4x=0$$

Its solution is

$$\pi^I = -\frac{1}{6}\partial_\nu \Sigma^I_{J\rho\sigma}\varepsilon^{J\nu\rho\sigma} = -\frac{1}{6}\partial^\nu \Sigma^{IJ\rho\sigma}\varepsilon_{J\nu\rho\sigma}$$

By that, the action in background spacetime is approximated to

$$S(\Sigma) \to c\int \left[\frac{2}{3}\partial^{\nu_1}\Sigma^{IJ_1\rho_1\sigma_1}\varepsilon_{J_1\nu_1\rho_1\sigma_1}\partial_\nu \Sigma_{IJ\rho\sigma}\varepsilon^{J\nu\rho\sigma} + ...\right] d^4x$$

define inner product $\Sigma^{IJ_1\rho_1\sigma_1}\Sigma_{IJ\rho\sigma} = \Sigma^2 \delta^{J_1}_J \delta^{\rho_1}_\rho \delta^{\sigma_1}_\sigma$, we get

$$S(\Sigma) \to c\int \left(-4\partial_\mu \Sigma^{\nu\rho}_{IJ}\partial^\mu \Sigma^{IJ}_{\nu\rho} + ...\right) d^4x \quad \text{with} \quad \partial_\mu \Sigma^{\mu\rho}_{IJ} = 0$$

This is the action of the area field in weak gravity field (background spacetime). It is like the electromagnetic field.

$$L_0(\Sigma) \to -4c\left(\partial_\mu \Sigma^{\nu\rho}_{IJ}\right)\left(\partial^\mu \Sigma^{IJ}_{\nu\rho}\right) \quad \text{with} \quad \partial_\mu \Sigma^{\mu\rho}_{IJ} = 0$$

We rewrite it like

$$L_0(\Sigma)d^4x = -4c\left(\partial_\mu \Sigma^{\nu\rho}_{IJ}\right)\left(\partial^\mu \Sigma^{IJ}_{\nu\rho}\right) ed^4x + ...$$

As we did in deriving the gravity Lagrange we had to replace; $c \to -c$. This constant is determined in gravity potential $c \succ 0$. Therefore

$$L_0(\Sigma)d^4x \to 4c\left(\partial_\mu \Sigma^{\nu\rho}_{IJ}\right)\left(\partial^\mu \Sigma^{IJ}_{\nu\rho}\right) ed^4x + ... \tag{3.1}$$

To get rid of opposite sign, comparing with free electromagnetic Lagrange in background spacetime $e \to 1 + \delta e$, we can replace $\partial_\mu \to i\partial_\mu$ it is equivalent to replace $k_\mu \to ik_\mu$ in the free solutions: $e^{ikx} \to e^{-kx}$ *or* e^{kx} in the background spacetime. By that the area field becomes classical field, we can consider it as background field.

By using the selfdual projection (2.1) and (2.2):

$$e = -\varepsilon_{IJKL}e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma}/4! \to -\frac{2i}{4!}\left(P^i\right)_{IJ}\left(P_i\right)_{KL}\Sigma^{IJ}_{\mu\nu}\Sigma^{KL}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}$$

the Lagrange (3.1) becomes

$$L_0(\Sigma)ed^4x = -8ci\left(\partial_\mu \Sigma^{\nu\rho}_{IJ}\partial^\mu \Sigma^{IJ}_{\nu\rho}\right)\left(\Sigma^i_{\mu\nu}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right) d^4x + ...$$

To discover the area field behavior, we test one wave $\cos\left(k_\mu x^\mu\right)$. We have

$$L_0(\Sigma)ed^4x \to -8ci\left(k^\mu k_\mu \Sigma_{IJ}^{\nu\rho}\Sigma_{\nu\rho}^{IJ}\right)\left(\Sigma_{\mu\nu}^{i}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x$$

The action of that is

$$e^{iLed^4x} \to \exp 8c\left(k_\mu k^\mu \Sigma_{IJ}^{\nu\rho}\Sigma_{\nu\rho}^{IJ}\right)\left(\Sigma_{\mu\nu}^{i}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x$$

Then

$$e^{i\delta S} \to \exp 8c\left(-k_0^2+\vec{k}^2\right)\left(\Sigma_{IJ}^{\nu\rho}\Sigma_{\nu\rho}^{IJ}\right)\left(\Sigma_{\mu\nu}^{i}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x \tag{3.2}$$

It is opposite to the gravity field (2.3). In the time-like $-k_0^2+\vec{k}^2 \prec 0$ the area field is annihilated $\Sigma_{\nu\rho}^{JK} \to e_\mu^I$. And in the space-like $-k_0^2+\vec{k}^2 \succ 0$ the area field is created $e_\mu^I \to \Sigma_{\nu\rho}^{JK}$ this is the duality $e_\mu^I \leftrightarrow \Sigma_{\nu\rho}^{JK}$. It preserves the reality. It is like duality of the left and right spinor field under Lorentz transformation and party.

The opposite behavior is with the anti-selfdual representation, the hermitian of the selfdual

$$2i\left(P^i\right)^{IJ}\left(P_i\right)^{KL} - 2i\left(\bar{p}^i\right)^{IJ}\left(\bar{P}_i\right)^{KL} \to \varepsilon^{IJKL}$$

which is projection from $I \neq J$ and $K \neq L$ in the left to $I \neq J \neq K \neq L$ in the right.
The first term is for the selfdual, while the second is for the anti-selfdual. The tensor product of them satisfies the reality:

$$e^{i\Delta L(selfdual)d^4x}e^{i\Delta L(anti-selfdual)d^4x} = real$$

Instead of that we can satisfy the reality by gravity-area duality:

$$e^{i\Delta L(e)d^4x}e^{i\Delta L(\Sigma)d^4x} = real : invariant\ for\ selfdual$$

For one wave, it becomes

$$e^{\frac{2}{48c}\frac{1}{2}\left(k_0^2-\vec{k}^2\right)\left(e_I^\nu e_\nu^I\right)\left(\Sigma_{\mu\nu}^{i}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)}e^{8c'\left(-k_0^2+\vec{k}^2\right)\left(\Sigma_{IJ}^{\nu\rho}\Sigma_{\nu\rho}^{IJ}\right)\left(\Sigma_{\mu\nu}^{i}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)}$$

We wrote c' to distinguish it from c. For

$$\frac{2}{48c}\frac{1}{2}\left(e_I^\nu e_\nu^I\right) = 8c'\left(\Sigma_{IJ}^{\nu\rho}\Sigma_{\nu\rho}^{IJ}\right)$$

The product equals one, this satisfies the reality. By that we can determine c' like to choose $(48c)^{-1} = 16c'$. With

$$\left(e_I^{\nu}e_{\nu}^{I}\right) = \frac{1}{2}\left(\Sigma_{IJ}^{\nu\rho}\Sigma_{\nu\rho}^{IJ}\right) = \frac{1}{2}\left(\Sigma_i^{\nu\rho}\Sigma_{\nu\rho}^{i} + \bar{\Sigma}_i^{\nu\rho}\bar{\Sigma}_{\nu\rho}^{i}\right)$$

the hermitian conjugate $\bar{\Sigma}_i^{\nu\rho}\bar{\Sigma}_{\nu\rho}^{i}$ is represented in anti-selfdual: $\bar{\Sigma}^i = \bar{P}_{IJ}^{i}\Sigma^{IJ}$.

As done for left and right spinor fields; in left spinor field representation the right spinor field is zero. And in in right spinor field representation the left spinor field is zero[3]. Therefore in selfdual representation, we assume that the anti-selfdual is zero. Like that in anti-selfdual representation. By that we have in selfdual representation:

$$\bar{\Sigma}^i = \frac{1}{2}\varepsilon^{ijk}\Sigma_{jk} - i\Sigma^{0i} = 0 \rightarrow \frac{1}{2}\varepsilon^{ijk}\Sigma_{jk} = i\Sigma^{0i}$$

therefore the area field in selfdual representation becomes

$$\Sigma^i = \frac{1}{2}\varepsilon^{ijk}\Sigma_{jk} + i\Sigma^{0i} = \varepsilon^{ijk}\Sigma_{jk}$$

which is real as required for satisfying the reality. It is equivalent to replace $x^0 \rightarrow -ix^0$. Same result we get in anti-selfdual representation $\Sigma^i = 0 \rightarrow \bar{\Sigma}^i = \varepsilon^{ijk}\Sigma_{jk}$. It is equivalent to replace $x^0 \rightarrow ix^0$. That lets to the splitting $SO(3,1) \rightarrow SU(2)\otimes SU(2)$.

In the two representations, the condition $\left(e_I^{\nu}e_{\nu}^{I}\right) = \frac{1}{2}\left(\Sigma_{IJ}^{\nu\rho}\Sigma_{\nu\rho}^{IJ}\right) = \frac{1}{2}\left(\Sigma_i^{\nu\rho}\Sigma_{\nu\rho}^{i} + \bar{\Sigma}_i^{\nu\rho}\bar{\Sigma}_{\nu\rho}^{i}\right)$ becomes $\left(e_I^{\nu}e_{\nu}^{I}\right) = \frac{1}{2}\varepsilon_{ij'k'}\Sigma_{\nu\rho}^{j'k'}\varepsilon^{ijk}\Sigma_{jk}^{\nu\rho} = \Sigma_{jk}^{\nu\rho}\Sigma_{\nu\rho}^{jk}$.

The difference between the selfdual and anti-selfdual appeared in the opposite sign in the Lagrange:

$$L \rightarrow 8c'\left(-k_0^2 + \vec{k}^2\right)\left(\Sigma_i^{\nu\rho}\Sigma_{\nu\rho}^{i}\right)\left(\Sigma_{\mu\nu}^{i}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)\ for\ selfdual\ \bar{\Sigma}^i = 0$$

and

$$L \rightarrow -8c'\left(-k_0^2 + \vec{k}^2\right)\left(\bar{\Sigma}_i^{\nu\rho}\bar{\Sigma}_{\nu\rho}^{i}\right)\left(\bar{\Sigma}_{\mu\nu}^{i}\bar{\Sigma}_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)\ for\ anti\text{-}selfdual\ \Sigma^i = 0$$

The opposite sign comes from the projection (2.2):

$$e \rightarrow -\frac{2i}{4!}\Sigma_{\mu\nu}^{i}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4! + \frac{2i}{4!}\bar{\Sigma}_{\mu\nu}^{i}\bar{\Sigma}_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!$$

We chose the selfdual because the tensor product with the gravity field satisfies the reality. It is like the duality of the left and right spinor fields under Lorentz transformation and party: $\psi_L \leftrightarrow \psi_R$.

4 The static potential of weak gravity

We derive the static potential of the interactions of scalar and spinor fields with weak gravity field in static limit. We see it is the same in the both, the Newtonian gravitational potential. We see that potential relates to energy-energy interaction. By that we determine the constant $c \succ 0$.

The action of the scalar field in curved spacetime is[1]

$$S(e,\phi) = \int d^4xe\left(\eta^{IJ}e_I^\mu e_J^\nu D_\mu\phi^+ D_\nu\phi - V(\phi)\right)$$

In weak gravity, the background spacetime:

$$e_I^\mu(x) \to \delta_I^\mu + h_I^\mu(x) \quad , \quad e \to 1 + \delta e$$

the action is approximated to

$$S(e,\phi) = \int d^4x\left(\partial_\mu\phi^+\partial^\mu\phi + h^{\mu\nu}(x)\partial_\mu\phi^+\partial_\nu\phi + h^{\nu\mu}(x)\partial_\mu\phi^+\partial_\nu\phi - V(\phi) + ...\right)$$

The gravity field is symmetry, so

$$S(e,\phi) = \int d^4x\left(\partial_\mu\phi^+\partial^\mu\phi + 2h^{\mu\nu}(x)\partial_\mu\phi^+\partial_\nu\phi - V(\phi) + ...\right)$$

The energy-momentum tensor of the scalar field is[3]

$$T_{\mu\nu} = \partial_\mu\phi^+\partial_\nu\phi + g_{\mu\nu}L$$

Therefore

$$\partial_\mu\phi^+\partial_\nu\phi = T_{\mu\nu} - g_{\mu\nu}L$$

Using it in the Lagrange, it becomes

$$L = \partial_\mu\phi^+\partial^\mu\phi + 2h^{\mu\nu}(x)\left(T_{\mu\nu} - g_{\mu\nu}L\right) - V(\phi) + ...$$

By that we have

$$L = \partial_\mu\phi^+\partial^\mu\phi + 2h^{\mu\nu}T_{\mu\nu} - V(\phi) - 2h^{\mu\nu}g_{\mu\nu}L + ...$$

Therefore, in the interaction term, we have the replacement

$$\partial_\mu\phi^+\partial_\nu\phi \to T_{\mu\nu} \quad \text{and} \quad L \to L - 2h^{\mu\nu}g_{\mu\nu}L$$

Because the gravity field is weak (background spacetime), so $2h^{\mu\nu}g_{\mu\nu}L$ is neglected comparing with L .

We find the potential $V(r)$ of exchanged virtual gravitons by two particles k_1 and k_2 using $M\left(k_1+k_2 \to k_1'+k_2'\right)$ matrix element (like Born approximation to the scattering amplitude in non-relativistic quantum mechanics [7]).

For one diagram of Feynman diagrams, we have

$$iM\left(k_1+k_2 \to k_1'+k_2'\right) = i\left(-ik_2'\right)_\mu \left(ik_2\right)_\nu \frac{\bar{\Delta}^{\mu\nu\rho\sigma}\left(q\right)}{i} i\left(-ik_1'\right)_\rho \left(ik_1\right)_\sigma$$

with

$$q = k_1' - k_1 = k_2 - k_2'$$

The propagator $\Delta^{\mu\nu\rho\sigma}\left(x_2 - x_1\right)$ is the gravitons propagator (1.8), we find it in the Lagrange of the free gravity field (background spacetime) we had before

$$L_0 = \frac{1}{48c}\frac{1}{2}\eta_{IJ}e^I_\mu \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)e^J_\nu \to \frac{1}{48c}\frac{1}{2}\eta_{IJ}h^I_\mu \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)h^J_\nu$$

with the gauge $\partial^\mu e^I_\mu = 0$, we have

$$\Delta^{IJ}_{\mu\nu}\left(y-x\right) = \int \frac{d^4q}{\left(2\pi\right)^4}\bar{\Delta}^{IJ}_{\mu\nu}\left(q^2\right)e^{iq(y-x)} \ : \ \bar{\Delta}^{IJ}_{\mu\nu}\left(q^2\right) = 48c\frac{g_{\mu\nu}\eta^{IJ}}{q^2 - i\varepsilon}$$

The M matrix element becomes

$$iM\left(k_1+k_2 \to k_1'+k_2'\right) = i48c\left(-ik_2'\right)_\mu \left(ik_2\right)_\rho \frac{g^{\mu\nu}g^{\rho\sigma}}{q^2}\left(-ik_1'\right)_\sigma \left(ik_1\right)_\nu$$

where $\quad g = \eta$ and $q = k_1' - k_1 = k_2 - k_2'$

Comparing with[7]

$$iM\left(k_1+k_2 \to k_1'+k_2'\right) = -i\bar{V}\left(q\right)\delta^4\left(k_{out} - k_{in}\right)$$

We have

$$\bar{V}\left(q^2\right) = -48c\left(-ik_2'\right)_\mu \left(ik_2\right)_\rho \frac{g^{\mu\nu}g^{\rho\sigma}}{q^2}\left(-ik_1'\right)_\sigma \left(ik_1\right)_\nu$$

Comparing this relation with the replacement:

$$\partial_\mu \phi^+ \partial_\nu \phi \to T_{\mu\nu} \text{ and } \quad L \to L - 2h^{\mu\nu} g_{\mu\nu} L$$

and making the Fourier transformation, we get

$$V(y-x) = -48c T_{\mu\rho}(y)\, g^{\mu\nu} g^{\rho\sigma} T_{\nu\sigma}(x) \frac{1}{4\pi |y-x|} = -48c \frac{T_{\mu\nu}(y)\, T^{\mu\nu}(x)}{4\pi |y-x|}$$

With the transferred energy-momentum T , in the static limit, for one particle $T^{00} \to m$ the mass of the interacted particles.

Therefore we get the Newtonian gravitational potential

$$V(y-x) = -48c \frac{m^2}{4\pi |y-x|} = -G \frac{m^2}{|y-x|} \to 48c = 4\pi G$$

The weak gravity Lagrange becomes

$$L_0 = \frac{1}{4\pi G} \frac{1}{2} \eta_{IJ} e^I_\mu \left(g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu \right) e^J_\nu$$

We do the same thing for the gravity interaction with spinor fields. The action is[1]

$$S(e,\psi) = \int d^4x e \left(i e^\mu_I \bar{\psi} \gamma^I D_\mu \psi - m \bar{\psi} \psi \right)$$

The covariant derivative D_μ is

$$D_\mu = \partial_\mu + (\omega_\mu)^I_J L^J_I + A^a_\mu \mathrm{T}^a$$

In the background spacetime, we have

$$S(e,\psi) = \int d^4x \left(i\bar{\psi} \gamma^\mu D_\mu \psi + i h^\mu_I \bar{\psi} \gamma^I D_\mu \psi - m \bar{\psi} \psi + ... \right)$$

We consider only the terms

$$\int d^4x \left(i\bar{\psi} \gamma^\mu \partial_\mu \psi + i h^\mu_\nu \bar{\psi} \gamma^\nu \partial_\mu \psi - m \bar{\psi} \psi \right) : \quad g = \eta$$

The energy-momentum tensor is[3]

$$T^{\mu\nu} = -i\bar{\psi} \gamma^\mu \partial^\nu \psi + g^{\mu\nu} L$$

Therefore, as for the scalar field, in the interaction term, we have the replacement

$$i\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi \to -T_{\mu\nu} \text{ and } L \to L + h^{\mu\nu}g_{\mu\nu}L$$

The term $h^{\mu\nu}g_{\mu\nu}L$ is ignored comparing with the Lagrange L. We find the M element of exchanged virtual gravitons $p_1 + p_2 \to p_1' + p_2'$, for one diagram of Feynman diagrams[7]

$$iM\left(p_1 + p_2 \to p_1' + p_2'\right) = i48c\bar{u}\left(p_1'\right)\gamma^{\mu}\left(-ip_1\right)_{\nu}u\left(p_1\right)\frac{g_{\mu\sigma}g^{\nu\rho}}{q^2}\bar{u}\left(p_2'\right)\gamma^{\sigma}\left(-ip_2\right)_{\rho}u\left(p_2\right)$$

with

$$q = p_1' - p_1 = p_2 - p_2' \text{ and } \quad g = \eta$$

We have

$$\bar{V}\left(q^2\right) = -48c\bar{u}\left(p_1'\right)\gamma^{\mu}\left(-ip_1\right)_{\nu}u\left(p_1\right)\frac{g_{\mu\sigma}g^{\nu\rho}}{q^2}\bar{u}\left(p_2'\right)\gamma^{\sigma}\left(-ip_2\right)_{\rho}u\left(p_2\right)$$

Comparing this relation with the replacement

$$i\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi \to -T_{\mu\nu} \text{ and } \quad L \to L + h^{\mu\nu}g_{\mu\nu}L$$

And make the Fourier transformation, we get

$$V\left(y - x\right) = -48c\left(-T_{\mu\rho}\left(y\right)\right)g^{\mu\nu}g^{\rho\sigma}\left(-T_{\nu\sigma}\left(x\right)\right)\frac{1}{4\pi\left|y - x\right|} = -48c\frac{T_{\mu\nu}\left(y\right)T^{\mu\nu}\left(x\right)}{4\pi\left|y - x\right|}$$

With the transferred energy-momentum T , in the static limit, for one particle $T^{00} \to m$ is mass of the interacted particles(spinor).

Therefore we get the Newtonian gravitational potential.

$$V\left(y - x\right) = -48c\frac{m^2}{4\pi\left|y - x\right|} = -G\frac{m^2}{\left|y - x\right|} \to 48c = 4\pi G$$

It is the same potential as for the scalar particles.

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