The Dynamics of the Gravity Field

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Abstract

We derive the canonical momentum π_I of the gravity field e^I. Then we use it to derive the path integral of the gravity field. The canonical momentum π_I is represented in Lorentz group. We derive it from the holonomy $U(\gamma, A)$ of the connection A_a^i of the Lorentz group. We derive the path integral of the gravity field as known in the quantum fields theory and discuss the situation of the free gravity field (like the electromagnetic field). We find that situation is only in the background spacetime, the weak gravity situation. We search for a theory in which the gravity field is dynamical at any energy in arbitrary curved spacetime $\{x^\mu\}$. For that, we suggest the duality $e^I \leftrightarrow \Sigma^{JK}$, where the field $\Sigma^{IJ} = e^I \wedge e^J$ is the Area field. That duality lets to the possibility to study the both fields e^I and Σ^{IJ} in arbitrary curved spacetime. We find $e^I \to \Sigma^{JK}$ in spacelike and $\Sigma^{JK} \to e^I$ in timelike. We find that the tensor product of the gravity and area fields, in selfdual representation, satisfies the reality condition. We apply that to derive the static potential of exchanging gravitons in scalar and spinor fields, the Newtonian gravitational potential.

1 The canonical conjugate field π^I and the path integral

We search for conditions to have a dynamical gravity field. The problem of the dynamics in general relativity is that the spacetime is itself a dynamical thing. It interacts with the matter, it is an operator $d\hat{x}^\mu$. Therefore we have to treat the spacetime as a quantum field like the other fields. But where they exist, this problem is solved by considering fields exist over fields not over the spacetime[1].

As usual in quantum field theory we have to find the canonical conjugate field π^I(represented in Lorentz group) acts canonically on Lorentz vectors over 3d closed surface δM immersed in arbitrary curved spacetime x^μ of manifold M. That closed surface δM is parameterized by three parameters X^1, X^2, X^3. In a certain gauge, we consider them as a spatial part of Lorentz coordinate $X^I = X^0, X^1, X^2, X^3$ with the flat metric (- + + +).

Therefore, the exterior derivative operator lets to the change along the norm of that surface, so it lets to the change in the time X^0 direction. That lets the 3d surface extends and have the four Lorentz spacetime $\left\{X^I\right\}$ parameterize the four dimensions x^μ of the curved spacetime in the manifold M.

We suggest canonical states $\left|\tilde{e}^I\right\rangle$ and $\left|\pi^I\right\rangle$ represented in Lorentz group, we use them in deriving the path integral of the gravity field. We find that there is no propagation over the dynamical spacetime x^μ. But in the background spacetime we find that the gravity field propagates freely like the electromagnetic fields.

Although the dynamics of the gravity field is built using the Lorentz group elements, the measurements relate and depend on the dynamical spacetime x^μ . Because x^μ is itself a dynamical, it interacts with all fields. Therefore our need to the Lorentz representation is to have canonical dynamical laws, processes . . . , so we have to distinguish between the dynamics of the general relativity and its measurements.

The holonomy of the connection A in the quantum gravity is[1]

$$U(\gamma, A) = TrPe^{i\oint_\gamma A} \tag{1.1}$$

It is ordered integral along closed path γ in arbitrary curved spacetime x^μ. This element is invariant under local Lorentz transformation $V^I \to L^I_J(x)V^J$ and under arbitrary changing of the coordinates $dx^\mu \to \Lambda^\mu{}_\nu(x)dx^\nu$. Therefore the quantum gravity is studied using it.
The connection A is the selfdual of Lorentz spin connection ω:

$$A^i_\mu(x) = \left(P^i\right)_{IJ}\omega^{IJ}_\mu(x)$$

P^i is the selfdual projector[1]. We can write the holonomy using the spin connection $\omega_\mu^{IJ} dx^\mu$ of Lorentz group. We have

$$U(\gamma, \omega) = TrPe^{i \oint_\gamma \omega^I{}_J}$$

We expect that it has the same properties of $U(\gamma, A)$; satisfies the symmetries of GR. For free gravity field, we impose the relation:

$$(\omega_\mu)^{IJ} = \pi_K{}^{IJ} e_\mu^K$$

The conjugate field $\pi_K{}^{IJ}(x)$ is represented in Lorentz group and acts on its vectors. Therefore we consider it as a dynamical operator. The holonomy becomes

$$U(\gamma, \pi, e) = TrP \exp i \oint_\gamma \left(\pi_K{}^I{}_J\right) e_\mu^K dx^\mu$$

For free gravity field, we expect that the momentum π^{IJK} is antisymmetry. So we can write

$$\pi^{IJK} = \pi_L \varepsilon^{LIJK}$$

This is our starting point in study the dynamics of the gravity. The holonomy becomes

$$U(\gamma, \pi, e) = TrP \exp i \oint_\gamma \left(\pi^{KI}{}_J\right) e_{K\mu} dx^\mu = TrP \exp i \oint_\gamma \left(\varepsilon^{LKI}{}_J\right) \pi_L e_{K\mu} dx^\mu$$

The result of that integral depends on

$$P \int_{\gamma_i} \pi_K e_\mu^K dx^\mu$$

γ_i are ordered paths divide the closed path γ and P is the permutation of them. The holonomy $U(\gamma, \pi, e)$ satisfies the general relativity symmetries, invariance under local Lorentz transformation $V^I \to L^I{}_J(x) V^J$ and under arbitrary changing of the coordinates $dx^\mu \to \Lambda^\mu{}_\nu(x) dx^\nu$. Therefore we can use it in quantum gravity.

We expect $\pi_K e_\mu^K dx^\mu$ satisfies the same conditions if it is integrated over a closed surface instead of the path γ . That is because

$$ed^4x = \frac{1}{4} d^3x_\mu \wedge dx^\mu = \frac{1}{4} e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge dx^\mu / 3!$$

is invariant element. Therefore we can replace $\pi_K e_\mu^K dx^\mu$ with

$$\pi_K e^{K\mu} d^3 x_\mu = \pi_K e^{K\mu} e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!$$

With integrating it over closed three dimensions surface δM.
e is the determinant of the gravity field e_μ^I :

$$g_{\mu\nu}(x) = \eta_{IJ} e_\mu^I e_\nu^J \rightarrow \sqrt{-g} = e$$

For arbitrary transformation, we have the invariant element

$$\sqrt{g}\varepsilon_{i_1 \dots i_n} = \sqrt{g'}\varepsilon'_{i_1 \dots i_n}$$

Therefore

$$e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3! = d^3 x_\mu$$

Is a co-vector, as ∂_μ. By that, the integral

$$U(\gamma, \pi, e) = \exp i \oint_{\delta M} \pi_I e^{I\mu} e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3! = \exp i \oint_{\delta M} \pi_I e^{I\mu} d^3 x_\mu$$

Satisfies the same conditions of the holonomy $U(\gamma, A)$, invariant under local Lorentz transformation $V^I \rightarrow L_J^I(x) V^J$ and under arbitrary changing of the coordinates $dx^\mu \rightarrow \Lambda^\mu{}_\nu(x) dx^\nu$. That relates to the physical reality, it is, the integral of free vector fields over a closed surface δM in a manifold M is invariant if there are no sources for those fields. It is the conversation. The spin connection ω^μ and so $\pi_K e^{K\mu}$, as vectors, satisfy that reality in free gravity.

The equation of motion of the gravity field e^I is

$$De^I = de^I + \omega^I{}_J \wedge e^J = 0$$

With our imposing $(\omega_\mu)^{IJ} = \pi_K{}^{IJ} e_\mu^K$, we get

$$de^I = -\pi_N{}^I{}_J e^N \wedge e^J$$

As we know, the tensor

$$e^N \wedge e^J = e_\mu^N e_\nu^J dx^\mu \wedge dx^\nu = \frac{1}{2}\left(e_\mu^N e_\nu^J - e_\nu^N e_\mu^J\right) dx^\mu \wedge dx^\nu$$

Measures the area in the manifold M. Therefore the changes of the gravity field around a closed path (rotation) relate to the flux of the momentum π throw the area which is determined by the closed path. It is like the magnetic field, generated by straight electric current. Therefore

$$e^N \wedge e^J \to Area$$

$$de^I = -\pi_N{}^I{}_J e^N \wedge e^J \to flux\ throw\ this\ Area$$

For that reason we suggested π^{IJK} is antisymmetry. We see that the flux depends on the momentum π.

Now, in the integral

$$\exp i \oint_{\delta M} \pi_I e^{I\mu} e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!$$

We define the canonical gravity field

$$\tilde{e}^I d^3X = \tilde{e}^I dX^1 dX^2 dX^3 \equiv e^{I\mu} e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!$$

We get

$$\tilde{U}_{\delta M}(\delta M, \pi, \tilde{e}) = \exp i \oint_{\delta M} \pi_I \tilde{e}^I d^3X$$

Where $X^I : I = 1, 2, 3$ parameterize the closed surface δM in the manifold M. In certain gauge, we consider $X^I : I = 1, 2, 3$ as a spatial part of Lorentz spacetime $X^I : I = 0, 1, 2, 3$. Therefore the exterior derivative is along the time X^0. The time X^0 is the direction of the norm on the surface $\delta M(X1, X2, X3)$. We will see that the result of the path integral is independent on this gauge.

The integral $\exp i \oint_{\delta M} \pi_I \tilde{e}^I d^3X$ satisfies the same conditions of the holonomy $U(\gamma, A)$, invariant under local Lorentz transformation and under arbitrary changing of the coordinates. Therefore we consider it as a canonical dynamical element.

Comparing it with

$$\langle \phi \mid \pi \rangle = \exp i \int d^3X \phi(X) \pi(X) / \hbar$$

the dynamical relation of scalar field ϕ. For $\hbar = 1$, we suggest canonical states $\left| \tilde{e}^I \right\rangle$ and $\left| \pi^I \right\rangle$ with

$$\left\langle \tilde{e}^I \mid \pi_I \right\rangle_{\delta M} = \exp i \int_{\delta M} \tilde{e}^I(X) \pi_I(X) d^3X$$

π_I is canonical conjugate field of $\tilde{e}^I$. We can write it over the surface δM like

$$\left\langle \tilde{e}^I \mid \pi_I \right\rangle_{\delta M} = \prod_{n,I} \left\langle \tilde{e}^I \left(x_n + dx_n\right) \mid \pi_I \left(x_n\right) \right\rangle_{\delta M}$$

With

$$\left\langle \tilde{e}^I \left(x_n + dx_n\right) \mid \pi_I \left(x_n\right) \right\rangle_{\delta M} = \exp i\tilde{e}^I(x_n + dx_n)\pi_I(x_n)d^3X \to \exp i\tilde{e}^I(x_n)\pi_I(x_n)d^3X$$

This relation is over the surface δM. In general, for two points in different surfaces δM_1 and δM_2, we have

$$\left\langle \tilde{e}^I \left(x_n + dx_n\right) \mid \pi_I \left(x_n\right) \right\rangle = \exp i\tilde{e}^I(x_n + dx_n)\pi_I(x_n)d^3X \tag{1.2}$$

Here the variation

$$i\tilde{e}^I(x_n + dx_n) - i\tilde{e}^I(x_n)$$

Is exterior derivative along the time dX^0 in the direction of the norm on the surface δM_1, it lets to the propagation. That lets to extend the surface: $\delta M(X^1, X^2, X^3) \to M(X^0, X^1, X^2, X^3)$.

We need to make $\hat{e}d^4\hat{x}$ commutes with $\hat{\tilde{e}}^I d^3X$. For that we write

$$\begin{aligned} -\hat{e}d^4\hat{x} &= \hat{e}d\hat{x}^\mu \wedge \varepsilon_{\mu\nu\rho\sigma} d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma/4! \\ &= \hat{e}d\hat{x}^\mu \wedge \frac{\varepsilon_{\mu\nu\rho\sigma}}{4!}\frac{\partial \hat{x}^\nu}{\partial X^i}\frac{\partial \hat{x}^\rho}{\partial X^j}\frac{\partial \hat{x}^\sigma}{\partial X^k}\frac{\varepsilon^{ijk}}{3!}d^3X = \frac{1}{4}\hat{e}d\hat{x}^\mu \hat{n}_\mu d^3X \end{aligned}$$

The indexes ijk are Lorentz indexes for $I = 1, 2, 3$. As we assumed before, $X^I : I = 1, 2, 3$ parameterize the closed surface δM in the manifold M.
We can rewrite it(in certain gauge) like

$$-ed^4x = \frac{1}{4}edx^\mu n_\mu d^3X = \frac{1}{4}e\frac{\partial x^\mu}{\partial X^0}n_\mu d^3X dX^0 = \frac{1}{4}ee_0^\mu n_\mu d^3X dX^0$$

compare it with the term

$$\tilde{e}^I d^3X = e^{I\mu} e\varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma/3! = ee^{I\mu} n_\mu d^3X$$

We find it commutes with it

$$\left[\hat{e}\hat{e}^{I\mu}\hat{n}_\mu d^3X, \hat{e}\hat{e}_0^\mu \hat{n}_\mu d^3X dX^0\right] = 0 \to \left[\hat{\tilde{e}}^I d^3X, \hat{e}d^4\hat{x}\right] = 0$$

Where $\left[\hat{e}_\mu^I, \hat{e}_\nu^J\right] = 0$. Therefore the operator $\hat{e} d^4 \hat{x}$ takes eigenvalues when it acts on the states $\left|\tilde{e}^I\right\rangle$.

The action of the free gravity field is[1]

$$S(e,\omega) = \frac{1}{16\pi G} \int \varepsilon_{IJKL} \left(e^I \wedge e^J \wedge R^{KL}(\omega) + \lambda e^I \wedge e^J \wedge e^K \wedge e^L\right)$$

We consider only the first term

$$S(e,\omega) = c \int \varepsilon_{IJKL} e^I \wedge e^J \wedge R^{KL}(\omega)$$

C is constant. The Riemann curvature is

$$R^{KL}(\omega) = d\omega^{KL} + \omega^K{}_M \wedge \omega^{ML}$$

Using the relation we imposed before:

$$(\omega_\mu)^{IJ} = \pi_K{}^{IJ} e_\mu^K$$

the action becomes

$$S(e,\pi) = c \int \left[\varepsilon_{IJKL} e^I \wedge e^J \wedge d\left(\pi_M{}^{KL} e^M\right) + \varepsilon_{IJKL} e^I \wedge e^J \wedge \left(\pi_{K_1}{}^K{}_M\right) e^{K_1} \wedge \left(\pi_{K_2}{}^{ML}\right) e^{K_2}\right]$$

or

$$S(e,\pi) = c \int \left[\varepsilon_{IJKL} e^I \wedge e^J \wedge d\left(\pi_M{}^{KL} e^M\right) + \varepsilon_{IJKL} \left(\pi_{K_1}{}^K{}_M\right) \left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}\right] \tag{1.3}$$

We find the term $d\left(\pi_M{}^{KL} e^M\right)$ from

$$\varepsilon_{IJKL} d\left(e^I \wedge e^J \wedge \pi_M{}^{KL} e^M\right)$$

But we assume its integral is zero at infinity. We have

$$\begin{aligned} \varepsilon_{IJKL} d\left(e^I \wedge e^J \wedge \pi_M{}^{KL} e^M\right) =& \varepsilon_{IJKL} \left(de^I\right) \wedge e^J \wedge \pi_M{}^{KL} e^M - \varepsilon_{IJKL} e^I \wedge \left(de^J\right) \wedge \pi_M{}^{KL} e^M \\ &+ \varepsilon_{IJKL} e^I \wedge e^J \wedge d\left(\pi_M{}^{KL} e^M\right) \end{aligned}$$

Rewriting

$$-\varepsilon_{IJKL} e^I \wedge \left(de^J\right) \wedge \left(\pi_M{}^{KL} e^M\right) = -\varepsilon_{IJKL} \left(de^J\right) \wedge e^I \wedge \left(\pi_M{}^{KL} e^M\right)$$
$$= \varepsilon_{JIKL} \left(de^J\right) \wedge e^I \wedge \pi_M{}^{KL} e^M$$

Therefore

$$\varepsilon_{IJKL} d\left(e^I \wedge e^J \wedge \pi_M{}^{KL} e^M\right) = 2\varepsilon_{IJKL} \left(de^I\right) \wedge e^J \wedge \pi_M{}^{KL} e^M + \varepsilon_{IJKL} e^I \wedge e^J \wedge d\left(\pi_M{}^{KL} e^M\right)$$

By that we write the action as

$$S(e,\pi) = c \int \left[-2\varepsilon_{IJKL} \left(de^I\right) \wedge e^J \wedge \left(\pi_M{}^{KL} e^M\right) + \varepsilon_{IJKL} \left(\pi_{K_1}{}^K{}_M\right) \left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}\right]$$

Using the equation of motion of the gravity field

$$0 = De^I = de^I + \omega^I{}_J \wedge e^J = de^I + \pi_N{}^I{}_J e^N \wedge e^J$$

We get

$$de^I = -\pi_N{}^I{}_J e^N \wedge e^J$$

Inserting it in the action, it becomes

$$S(e,\pi) = c \int 2\varepsilon_{IJKL} (\pi_N{}^I{}_B) e^N \wedge e^B \wedge e^J \wedge \left(\pi_M{}^{KL} e^M\right)$$
$$+ \varepsilon_{IJKL} \left(\pi_{K_1}{}^K{}_M\right) \left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}$$

Or

$$S(e,\pi) = c \int 2\varepsilon_{IJKL} \left(\pi_N{}^I{}_B\right) \left(\pi_M{}^{KL}\right) e^N \wedge e^B \wedge e^J \wedge e^M$$
$$+ \varepsilon_{IJKL} \left(\pi_{K_1}{}^K{}_M\right) \left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}$$

Rewriting it like

$$S(e,\pi) = c \int 2\varepsilon_{IJKL} \left(\pi_N{}^I{}_B\right) \left(\pi_M{}^{KL}\right) e^B \wedge e^J \wedge e^N \wedge e^M$$
$$+ \varepsilon_{IJKL} \left(\pi_{K_1}{}^K{}_M\right) \left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}$$

Replacing $B \rightleftarrows I, N \to K_1$ and $M \to K_2$ in the first term, we get

$$S(e,\pi) = c \int 2\varepsilon_{BJKL} \left(\pi_{K_1}{}^B{}_I\right) \left(\pi_{K_2}{}^{KL}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}$$
$$+ \varepsilon_{IJKL} \left(\pi_{K_1}{}^K{}_M\right) \left(\pi_{K_2}{}^{ML}\right) e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2}$$

We replace

$$e^I \wedge e^J \wedge e^{K_1} \wedge e^{K_2} \to \varepsilon^{IJK_1K_2} e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

We get

$$S(e,\pi) = c \int \left[2\varepsilon_{BJKL} \left(\pi_{K_1}{}^B{}_I\right) \left(\pi_{K_2}{}^{KL}\right) \varepsilon^{IJK_1K_2} + \varepsilon_{IJKL} \left(\pi_{K_1}{}^K{}_M\right) \left(\pi_{K_2}{}^{ML}\right) \varepsilon^{IJK_1K_2}\right] \times e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

Using the relation $\pi^{IJL} = \pi_K \varepsilon^{KIJL}$ we imposed before. The action:

$$S(e,\pi) = c \int \left[2\varepsilon^B{}_{JKL} \left(\pi_{K_1BI}\right) \left(\pi_{K_2}{}^{KL}\right) \varepsilon^{IJK_1K_2} - 2\left(\pi_K{}^K{}_M\right)\left(\pi_L{}^{ML}\right) + 2\left(\pi_L{}^K{}_M\right)\left(\pi_K{}^{ML}\right)\right] \times e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

becomes:

$$S(e,\pi) = c \int \left[2\varepsilon^B{}_{JKL} \pi^N \varepsilon_{NK_1BI} \left(\pi_{K_2}{}^{KL}\right) \varepsilon^{IJK_1K_2} + 2\left(\pi_{LKM}\right)\left(\pi^{KML}\right)\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

Or

$$S(e,\pi) = c \int \left[2\varepsilon^B{}_{JKL} \pi^N \varepsilon_{IK_1NB} \left(\pi_{K_2}{}^{KL}\right) \left(-\varepsilon^{IK_1JK_2}\right) + 2\pi^I \varepsilon_{ILMK} \pi_J \varepsilon^{JLMK}\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

Using the property

$$\varepsilon_{IK_1NB}\varepsilon^{IK_1JK_2} = -2\left(\delta^J_N \delta^{K_2}_B - \delta^J_B \delta^{K_2}_N\right) \text{ and } \varepsilon_{ILMK}\varepsilon^{JLMK} = -6\delta^J_I$$

The action becomes

$$S(e,\pi) = c \int \left[4\varepsilon^{K_2}{}_{JKL} \pi^J \left(\pi_{K_2}{}^{KL}\right) - 12\pi_I \pi^I\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

Or

$$S(e,\pi) = c \int \left[4\varepsilon_{K_2JKL} \pi^J \left(\pi^{K_2KL}\right) - 12\pi^2\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

Then

$$S(e,\pi) = c \int \left[4\varepsilon_{K_2JKL} \pi^J \pi_I \varepsilon^{IK_2KL} - 12\pi^2\right] e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

The action becomes

$$\begin{aligned} S_0(e,\pi) &= c\int \left[24\pi^2 - 12\pi^2\right] e^0\wedge e^1\wedge e^2\wedge e^3 = c\int 12\pi^2 e^0\wedge e^1\wedge e^2\wedge e^3 \\ &= c\int 12\pi^2 e d^4x \end{aligned}$$

In the background spacetime, we have $e = 1+\delta e$, therefore

$$S_0(\delta e,\pi) \rightarrow \int 12c\pi^2 d^4x + ...$$

To find its meaning we compare it with scalar field Lagrange in the background spacetime, for $\hbar = 1$:

$$Ld^4x = \left(\pi\partial_0\phi - H(\phi,\pi)\right)d^4x \text{ with } H(\phi,\pi)d^4x = \left(\frac{1}{2}\pi^2 + \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi^2\right)d^4x$$

We conclude that the term

$$\int 12c\pi^2 d^4x \succ 0$$

Is the energy of the gravity field in background spacetime. As we will find in result of the path integral, in background spacetime limit, we have to replace $c\rightarrow -c$ when we compare with the electromagnetic field, therefore, in the background spacetime, we replace

$$S(e,\pi) \rightarrow -\int 12c\pi^2 d^4x = -\int Hd^4x$$

That is not surprising, because the general relativity equation (Einstein field equation) is derived to satisfy the energy conservation over curved spacetime:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

It satisfies the energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$. But, as we know, in quantum field theory in background spacetime limit, we have to write the canonical law of the conservation like

$$\partial_\mu\left(T^{\mu\nu}_{matter} + T^{\mu\nu}_{gravity}\right) = 0$$

Therefore we write

$$T_{\mu\nu}+\frac{-1}{8\pi G}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)=T_{\mu\nu}\left(matter\right)+T_{\mu\nu}\left(gravity\right)=constant$$

By that we conclude

$$T_{\mu\nu}\left(gravity\right)=-\frac{1}{8\pi G}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)$$

Therefore we have to replace $c\rightarrow -c$, we see that when we compare it with the electromagnetic field in background spacetime.

Now we derive the path integral as usual. As we saw before, the operator $\hat{e}d^4\hat{x}$ takes eigenvalues when it acts on the states $\left|\tilde{e}^I\right\rangle$ therefore we have the amplitude

$$\left\langle \tilde{e}^I\left(x+dx\right)\right| e^{iS}\left|\pi_I\left(x\right)\right\rangle \rightarrow \left\langle \tilde{e}^I\left(x+dx\right)\right| e^{i12c\hat{\pi}^2\hat{e}d^4\hat{x}}\left|\pi_I\left(x\right)\right\rangle$$

$$=\exp\left(i12c\pi^2 e\left(x+dx\right)d^4x+i\tilde{e}^I\left(x+dx\right)\pi_I(x)d^3X\right)$$

$$\rightarrow\exp\left(i12c\pi^2 e\left(x\right)d^4x+i\tilde{e}^I\left(x+dx\right)\pi_I(x)d^3X\right)$$

We let the momentum π_I acts on the left. The amplitude of the propagation between two points x and $x+dx$ in different adjacent surfaces $\delta M_1\rightarrow\delta M_2$ is

$$\left\langle \tilde{e}_I\left(x+dx\right)\right| e^{ic12\hat{\pi}^2\hat{e}d^4\hat{x}}\left|\tilde{e}^I\left(x\right)\right\rangle_{\delta M_1\rightarrow\delta M_2}$$

$$=\int\prod_I d\pi^I\left\langle \tilde{e}_I\left(x+dx\right)\right| e^{ic12\hat{\pi}^2\hat{e}d^4\hat{x}}\left|\pi^I\left(x\right)\right\rangle_{\delta M_1\rightarrow\delta M_2}\left\langle \pi_I\left(x\right)\ \middle|\ \tilde{e}^I\left(x\right)\right\rangle_{\delta M_1}$$

$$=\int\prod_I d\pi^I\exp\left[i12c\pi^2\left(x\right)e\left(x+dx\right)d^4x+i\tilde{e}^I(x+dx)\pi_I(x)d^3X\right]\exp\left(-i\tilde{e}^I(x)\pi_I(x)d^3X\right)$$

$$\rightarrow\int\prod_I d\pi^I\exp\left[i12c\pi^2\left(x\right)e\left(x\right)d^4x+i\left(\tilde{e}^I(x+dx)-\tilde{e}^I(x)\right)\pi_I(x)d^3X\right]$$

The exterior derivative

$$\left(\tilde{e}^I(x+dx)-\tilde{e}^I(x)\right)d^3X=\frac{\partial\tilde{e}^I(x)}{\partial X^0}d^3XdX^0=d\tilde{e}^I(x)d^3X$$

Is along the time dX^0 in the direction of the norm of the surface $M(X^1, X^2, X^3)$, therefore it lets to propagate from surface to another.

We write the amplitude like

$$\left\langle \tilde{e}_I\left(x+dx\right)\right| e^{ic12\hat{\pi}^2\hat{e}d^4\hat{x}} \left|\tilde{e}^I\left(x\right)\right\rangle_{\delta M_1\to\delta M_2} = \int \prod_I d\pi^I \exp\left[i12c\pi^2\left(x\right)e\left(x\right)d^4x + i\pi_I(x)d\tilde{e}^I(x)d^3X\right]$$

The path integral is the integral of ordered product of those amplitudes over all spacetime points(over all ordered 3d surfaces).

$$\begin{aligned} W_{ST} &= \int \prod_I D\tilde{e}^I D\pi_I \exp i \int \left(12c\pi^2 e d^4x + \pi_I d\tilde{e}^I d^3X\right) \\ &= \int \prod_I D\tilde{e}^I D\pi_I \exp i \int \left(12c\pi^2 e^0\wedge e^1\wedge e^2\wedge e^3 + \pi_I d\tilde{e}^I d^3X\right) \end{aligned}$$

For selfdual representation, we consider that propagation in the direction of expanding of the surface(positive direction).

There is no problem with Lorentz non-invariance in $\frac{\partial\tilde{e}^I(x)}{\partial X^0}d^3XdX^0$ because the equation of motion, we find in the result of the path integral, is

$$\frac{\partial\tilde{e}^I(x)}{\partial X^0} \propto -\pi^I$$

Therefore

$$\frac{\partial\tilde{e}^I(x)}{\partial X^0}\pi_I d^3XdX^0 \propto -\pi_I\pi^I d^3XdX^0$$

This is Lorentz invariant. This is like the equation of motion of the scalar field $\pi = \partial_0\phi$ which solves the same problem.
In our gauge we have

$$\begin{aligned} \pi_I\pi^I d^3XdX^0 \to \pi^2 dX^0\wedge dX^1\wedge dX^2\wedge dX^3 &= \pi^2 e^0_\mu e^1_\nu e^2_\rho e^3_\sigma dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma \\ &= \pi^2 e^0_\mu e^1_\nu e^2_\rho e^3_\sigma \varepsilon^{\mu\nu\rho\sigma} d^4x = \pi^2 e d^4x \end{aligned}$$

It is invariant element; we find it in the path integral.
The path integral:

$$W_{ST} = \int \prod_I D\tilde{e}^I D\pi_I \exp i \int \left(12c\pi^2 e^0\wedge e^1\wedge e^2\wedge e^3 + \pi_I d\tilde{e}^I d^3X\right)$$

Vanishes unless

$$\frac{\delta}{\delta\pi_I}\left(12c\pi^2 e^0 \wedge e^1 \wedge e^2 \wedge e^3 + \pi_I d\tilde{e}^I d^3X\right) = 24c\pi^I e^0 \wedge e^1 \wedge e^2 \wedge e^3 + d\tilde{e}^I d^3X = 0$$

Therefore we have the path(equation of motion)

$$\hat{\pi}^I = \frac{-1}{24c}\left(\hat{e}^0 \wedge \hat{e}^1 \wedge \hat{e}^2 \wedge \hat{e}^3\right)^{-1} d\hat{\tilde{e}}^I d^3X \tag{1.4}$$

Or

$$\pi^I\pi^J = \frac{1}{(24c)^2}\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-2} d\tilde{e}^I d^3X d\tilde{e}^J d^3X \tag{1.5}$$

Therefore

$$12c\pi^2 e^0 \wedge e^1 \wedge e^2 \wedge e^3 + \pi_I d\tilde{e}^I d^3X = \frac{1}{48c}\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)$$
$$-\frac{1}{24c}\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)$$

The path integral becomes

$$W_{ST} = \int \prod_I D\tilde{e}^I Exp\frac{-i}{48c}\int\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)$$

The canonical field $\tilde{e}^I$ is defined in

$$\tilde{e}^K d^3X = e^{K\mu} e\varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!$$

Therefore

$$\left(d\hat{\tilde{e}}^K\right) d^3X = \left(\hat{D}_{\mu_1}\hat{e}^{K\mu}\right)\hat{e}\varepsilon_{\mu\nu\rho\sigma} d\hat{x}^{\mu_1} \wedge d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma / 3!$$

Where D is the co-variant derivative defined in

$$DV^I = dV^I + \omega^I{}_J \wedge V^J$$

We have

$$\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right) = \frac{\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)}{e^0 \wedge e^1 \wedge e^2 \wedge e^3}$$

It becomes

$$\frac{\left(\hat{D}_{\mu_1}\hat{e}_I^{\mu}\right)\hat{e}\varepsilon_{\mu\nu\rho\sigma}d\hat{x}^{\mu_1}\wedge d\hat{x}^{\nu}\wedge d\hat{x}^{\rho}\wedge d\hat{x}^{\sigma}\left(\hat{D}_{\mu_2}\hat{e}^{I\mu'}\right)\hat{e}\varepsilon_{\mu'\nu'\rho'\sigma'}d\hat{x}^{\mu_2}\wedge d\hat{x}^{\nu'}\wedge d\hat{x}^{\rho'}\wedge d\hat{x}^{\sigma'}}{3!3!\hat{e}^0_{\mu_3}\hat{e}^1_{\nu_3}\hat{e}^2_{\rho_3}\hat{e}^3_{\sigma_3}d\hat{x}^{\mu_3}\wedge d\hat{x}^{\nu_3}\wedge d\hat{x}^{\rho_3}\wedge d\hat{x}^{\sigma_3}}$$

We have, the inverse:

$$\left(e^0_\mu e^1_\nu e^2_\rho e^3_\sigma dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma\right)^{-1}=E_0^{\mu'}E_1^{\nu'}E_2^{\rho'}E_3^{\sigma'}\frac{\partial}{\partial x^{\sigma'}}\wedge\frac{\partial}{\partial x^{\rho'}}\wedge\frac{\partial}{\partial x^{\nu'}}\wedge\frac{\partial}{\partial x^{\mu'}}$$

We write it in the form

$$e^0_\mu e^1_\nu e^2_\rho e^3_\sigma dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma=\frac{1}{4}ed^3x_\mu\wedge dx^\mu$$

Therefore we write

$$E_0^{\mu'}E_1^{\nu'}E_2^{\rho'}E_3^{\sigma'}\partial_{\sigma'}\wedge\partial_{\rho'}\wedge\partial_{\nu'}\wedge\partial_{\mu'}=E\partial_\nu\wedge\partial^{3\nu}$$

With inner product like

$$\left(E\partial_\nu\wedge\partial^{3\nu}\right)\left(\frac{1}{4}ed^3x_\mu\wedge dx^\mu\right)=\frac{1}{4}Ee\partial_\nu\wedge\partial^{3\nu}d^3x_\mu\wedge dx^\mu=\frac{1}{4}Ee\left(\delta^\nu_\mu\right)\partial_\nu dx^\mu=Ee=1$$

Actually, we have to write the tensors $\varepsilon^{\mu\nu\rho\sigma}$ and $\varepsilon_{\mu\nu\rho\sigma}$ like $e^{-1}\varepsilon^{\mu\nu\rho\sigma}$ and $e\varepsilon_{\mu\nu\rho\sigma}$ but here we neglect that, because it gives the same results.

We can write

$$\left(D_{\mu_1}e_I^\mu\right)e\varepsilon_{\mu\nu\rho\sigma}dx^{\mu_1}\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma/3!\to\left(D_{\mu_1}e_I^\mu\right)edx^{\mu_1}\wedge d^3x_\mu=-\left(D_{\mu_1}e_I^\mu\right)ed^3x_\mu\wedge dx^{\mu_1}$$

Also

$$\left(D_{\mu_2}e^{I\mu'}\right)e\varepsilon_{\mu'\nu'\rho'\sigma'}dx^{\mu_2}\wedge dx^{\nu'}\wedge dx^{\rho'}\wedge dx^{\sigma'}/3!\to-\left(D_{\mu_2}e^{I\mu'}\right)ed^3x_{\mu'}\wedge dx^{\mu_2}$$

We conclude

$$d^3x_\mu\wedge dx^\mu=-dx_\mu\wedge d^3x^\mu\to d^3x_\mu\wedge dx^{\mu_1}=-dx_\mu\wedge d^3x^{\mu_1}$$

Therefore

$$-\left(D_{\mu_1}e_I^\mu\right)ed^3x_\mu\wedge dx^{\mu_1}\to\left(D_{\mu_1}e_I^\mu\right)edx_\mu\wedge d^3x^{\mu_1}$$

By that we get

$$\frac{(d\tilde{e}_I d^3X)\left(d\tilde{e}^I d^3X\right)}{e^0\wedge e^1\wedge e^2\wedge e^3} \to -\left(E\partial_\nu\wedge\partial^{3\nu}\right)\left((D_{\mu_1}e_I^\mu)\,edx_\mu\wedge d^3x^{\mu_1}\right)\left(\left(D_{\mu_2}e^{I\mu'}\right)ed^3x_{\mu'}\wedge dx^{\mu_2}\right)$$

It becomes

$$\frac{(d\tilde{e}_I d^3X)\left(d\tilde{e}^I d^3X\right)}{e^0\wedge e^1\wedge e^2\wedge e^3} \to (D^{\mu_1}e_{I\mu})\left(D_{\mu_2}e^{I\mu'}\right)e\left(\partial_\nu\wedge\partial^{3\nu}\right)\left(d^3x_{\mu_1}\wedge dx^\mu\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right)$$

We can write the contraction

$$\left(\partial_\nu\wedge\partial^{3\nu}\right)\left(d^3x_{\mu_1}\wedge dx^\mu\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right) = \left(\partial_\nu\wedge\partial^{3\nu}d^3x_{\mu_1}\wedge dx^\mu\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right)$$
$$= \delta^\nu_{\mu_1}\left(\partial_\nu\wedge dx^\mu\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right) = \delta^\nu_{\mu_1}\left(-dx^\mu\wedge\partial_\nu\right)\left(-dx^{\mu_2}\wedge d^3x_{\mu'}\right)$$
$$= \delta^\nu_{\mu_1}dx^\mu\wedge\partial_\nu dx^{\mu_2}\wedge d^3x_{\mu'} = \delta^\nu_{\mu_1}\delta^{\mu_2}_\nu dx^\mu\wedge d^3x_{\mu'}$$

Therefore we can write

$$\frac{(d\tilde{e}_I d^3X)\left(d\tilde{e}^I d^3X\right)}{e^0\wedge e^1\wedge e^2\wedge e^3} \to (D^{\mu_1}e_{I\mu})\left(D_{\mu_2}e^{I\mu'}\right)e\delta^\nu_{\mu_1}\delta^{\mu_2}_\nu dx^\mu\wedge d^3x_{\mu'}$$
$$= (D_\nu e_{I\mu})\left(D^\nu e^{I\mu'}\right)edx^\mu\wedge d^3x_{\mu'} = -(D_\nu e_{I\mu})\left(D^\nu e^{I\mu'}\right)ed^3x_{\mu'}\wedge dx^\mu$$
$$= -(D_\nu e_{I\mu})\left(D^\nu e^{I\mu'}\right)e\delta^\mu_{\mu'}d^4x = -(D_\nu e_{I\mu})\left(D^\nu e^{I\mu}\right)ed^4x$$

We can also write another contraction:

$$(D^{\mu_1}e_{I\mu})\left(D_{\mu_2}e^{I\mu'}\right)e\left(\partial_\nu\wedge\partial^{3\nu}\right)\left(d^3x_{\mu_1}\wedge dx^\mu\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right)\to$$

$$(D^{\mu_1}e_{I\mu})\left(D_{\mu_2}e^{I\mu'}\right)e\left(\partial_\nu\wedge\partial^{3\nu}d^3x_{\mu_1}\wedge dx^\mu\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right)$$
$$= (D^{\mu_1}e_{I\mu})\left(D_{\mu_2}e^{I\mu'}\right)e\left(\delta^\nu_{\mu_1}\partial_\nu\wedge dx^\mu\right)\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right)$$
$$= \delta^\nu_{\mu_1}\delta^\mu_\nu(D^{\mu_1}e_{I\mu})\left(D_{\mu_2}e^{I\mu'}\right)e\left(d^3x_{\mu'}\wedge dx^{\mu_2}\right)$$

It becomes

$$\frac{(d\tilde{e}_I d^3X)\left(d\tilde{e}^I d^3X\right)}{e^0\wedge e^1\wedge e^2\wedge e^3} \to (D^\mu e_{I\mu})\left(D_{\mu'}e^{I\mu'}\right)ed^4x$$

By the two possible contractions, we can write the final result as

$$\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1} \left(d\tilde{e}_I d^3X\right) \left(d\tilde{e}^I d^3X\right) = \frac{-1}{2} \left(D_\mu e_I^\nu D^\mu e_\nu^I - D_\mu e_I^\nu D_\nu e^{I\mu}\right) ed^4x$$

This Lagrange is like the Lagrange of the electromagnetic field. Also it is independent on the gauge we chose for the surface δM. it is invariant under local Lorentz transformation $V^I \to L^I{}_J(x)V^J$ and any coordinate transformation $V^\mu \to \frac{\partial x^\mu}{\partial x'^\nu} V'^\nu$.

The path integral of the gravity field becomes, after replacing $c \to -c$.

$$W_{ST} = \int \prod_I De^I \exp \frac{i}{48c} \frac{1}{2} \left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right) ed^4x$$

With the free gravity field Lagrange

$$Ld^4x = \frac{1}{48c} \frac{1}{2} \left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right) ed^4x \tag{1.6}$$

We determine the constant c in the Newtonian gravitational potential $c \succ 0$.
For free gravity $D_\mu \to \partial_\mu$ and $e \to 1 + \delta e$, therefore we have

$$L \to \frac{1}{48c} \frac{1}{2} \left(-\partial_\mu e_I^\nu \partial^\mu e_\nu^I + \partial_\mu e_I^\nu \partial_\nu e^{I\mu}\right)$$

Or

$$L_0 = \frac{1}{48c} \frac{1}{2} \eta_{IJ} e_\mu^I \left(g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu\right) e_\nu^J$$

Without background spacetime approximation, we have a problem with the determinant e, it is

$$W_{ST} = \int \prod_I De^I exp \frac{i}{48c} \int \frac{1}{2} \left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right) e_{\mu_1}^0 e_{\nu_1}^1 e_\rho^2 e_\sigma^3 \varepsilon^{\mu_1\nu_1\rho\sigma} d^4x$$

with $\eta_{0123} = -1$ we get

$$W_{ST} = \int \prod_I De^I exp \frac{i}{48c} \int \frac{1}{2} \left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right) \left(-\eta_{I_1JKL}\right) e_{\mu_1}^{I_1} e_{\nu_1}^J e_\rho^K e_\sigma^L \varepsilon^{\mu_1\nu_1\rho\sigma} d^4x/4!$$

Always there is a field e_{ρ}^{K} which is different from e_{μ}^{I} and e_{ν}^{I} therefore the integral over it gives delta Dirac:

$$\int \prod_I De^I exp \frac{i}{48c} \int \frac{1}{2} \left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right) \left(-\eta_{I_1 JKL}\right) e_{\mu_1}^{I_1} e_{\nu_1}^{J} e_{\rho}^{K} e_{\sigma}^{L} \varepsilon^{\mu_1 \nu_1 \rho\sigma} d^4x/4!$$

$$\to \delta \left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right)$$

$$\to -D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu} = 0$$

This path integral is trivial, there is no propagation. The gravity field e_{μ}^{I} has the entity of the spacetime. It is impossible for the spacetime to propagate over itself. But if we write $e_{\mu}^{I}(x) \to \delta_{\mu}^{I} + h_{\mu}^{I}(x)$ the path integral exists, the propagation is possible. Therefore the propagation of the gravity is possible only over a background spacetime, this is the situation of weak gravity (low energy densities).

Latter we will search for conditions to make the gravity field propagate over x^μ , for that we impose the duality; Gravity-Area.

The path integral of the weak gravity field in the background spacetime is

$$w = \int \prod_I De^I \, \mathrm{exp} i \int \frac{1}{48c} \frac{1}{2} e_\mu^I \left(\eta_{IJ} g^{\mu\nu} \partial^2 - \eta_{IJ} \partial^\mu \partial^\nu\right) e_\nu^J d^4x \tag{1.7}$$

The gravity field propagator, $g = \eta$ and $k_\mu e^{\mu I} = 0$, is

$$\Delta_{IJ}^{\mu\nu}(x_2 - x_1) = 48c \int \frac{d^4k}{(2\pi)^4} \frac{\eta_{IJ} g^{\mu\nu} e^{ik(x_2 - x_1)}}{k^2 - i\varepsilon}$$

Or

$$\Delta_{\rho\sigma}^{\mu\nu}(x_2 - x_1) = 48c \int \frac{d^4k}{(2\pi)^4} \frac{g_{\rho\sigma} g^{\mu\nu} e^{ik(x_2 - x_1)}}{k^2 - i\varepsilon} \tag{1.8}$$

We will use this propagation in the gravity interaction with the scalar and spinor fields.

2 The need to the duality Gravity-Area

We search for conditions to have a dynamical gravity field in arbitrary curved spacetime without spacetime background approximation.
We found that the spacetime path integral W_{ST} is trivial. There is no propagation without spacetime background. We can solve that problem by assuming that the fields exist over themselves not in the spacetime[1]. Therefore the spacetime is measured thing by its interactions with the matter.

According to general relativity, the length, the area and the volume are another form of the gravity. We can explain that by the duality *gravity* $\leftrightarrow$ *areas* and volumes. We try to find this duality using the trivial path integral W_{ST} by finding conditions allow the gravity field to propagate. That propagation is $e^I \leftrightarrow \Sigma^{JK}$ it means they propagate when they change to each other. Also we find that the tensor product of them $\left|e^I\right\rangle \otimes \left|\Sigma^{JK}\right\rangle$, in selfdual representation, satisfies the reality condition.

Starting from the full Lagrange (1.6):

$$Ld^4x = \frac{1}{48c}\frac{1}{2}\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right) ed^4x$$

The covariant derivative is

$$De^I = de^I + \omega^I{}_J \wedge e^J$$

Using our assuming

$$\omega^{IJ} = \pi_K{}^{IJ} e^K$$

The covariant derivative becomes

$$De^I = de^I + \left(\pi_K{}^I{}_J\right) e^K \wedge e^J$$

The second term have two multiple fields $e^K \wedge e^J$, they become four multiple fields in the Lagrange. All of them must be different as $(de_I)\left(\pi_K{}^I{}_J\right) e^K \wedge e^J$. Like what we saw with the determinant e, the path integral over one of them is delta Dirac. This is trivial path integral W_{ST}. Therefore the second term must be a new field, it is the area field $\Sigma^{KJ} = e^K \wedge e^J$ by that the path integral exists.

The Area field is anti-symmetry field:

$$\Sigma_{\mu\nu}^{IJ} = \frac{1}{2}\left(e_\mu^I e_\nu^J - e_\nu^I e_\mu^J\right)$$

The derivative becomes

$$De^I = de^I + \left(\pi_K{}^I{}_J\right)\Sigma^{KJ} = de^I + \pi^{KIJ}\Sigma_{KJ}$$

Using our assumption

$$\pi^{IJK} = \pi_L \varepsilon^{LIJK}$$

The derivative becomes

$$De^I = de^I + \pi^{KIJ}\Sigma_{KJ} = de^I + \pi_L\varepsilon^{LKIJ}\Sigma_{KJ} = de^I + \varepsilon^{ILKJ}\pi_L\Sigma_{KJ}$$

By that we have two fields e^I and Σ^{IJ} in the Lagrange. They interact, that lets to the duality.

The full Lagrange of the gravity field is

$$Ld^4x = \frac{1}{48c}\frac{1}{2}\left(-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu}\right) ed^4x$$

We have

$$-D_\mu e_I^\nu D^\mu e_\nu^I + D_\mu e_I^\nu D_\nu e^{I\mu} = -D_\mu e_I^\nu \left(D^\mu e_\nu^I - D_\nu e^{I\mu}\right)$$

It becomes

$$-\left(\partial_\mu e_I^\nu + \varepsilon_{IJKL}\pi^J\Sigma_\mu^{KL\nu}\right)\left(\partial^\mu e_\nu^I + \varepsilon^{IJ_1K_1L_1}\pi_{J_1}\Sigma^\mu_{K_1L_1\nu} - \partial_\nu e^{I\mu} - \varepsilon^{IJ_1K_1L_1}\pi_{J_1}\Sigma_{K_1L_1\nu}{}^\mu\right)$$

Or

$$-D_\mu e_I^\nu\left(D^\mu e_\nu^I - D_\nu e^{I\mu}\right) = -\left(\partial_\mu e_I^\nu + \varepsilon_{IJKL}\pi^J\Sigma_\mu^{KL\nu}\right)\left(\partial^\mu e_\nu^I - \partial_\nu e^{I\mu} + 2\varepsilon^{IJ_1K_1L_1}\pi_{J_1}\Sigma^\mu_{K_1L_1\nu}\right)$$

It becomes

$$\begin{aligned}&-\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_\nu^I - \partial_\nu e^{I\mu}\right) - 2\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e_I^\nu\right)\pi_{J_1}\Sigma^\mu_{K_1L_1\nu}\\ &\quad - 2\varepsilon_{IJKL}\Sigma_\mu^{KL\nu}\pi^J\left(\frac{\partial^\mu e_\nu^I - \partial_\nu e^{I\mu}}{2}\right) - 2\varepsilon^{IJ_1K_1L_1}\varepsilon_{IJKL}\Sigma_\mu^{KL\nu}\pi^J\pi_{J_1}\Sigma^\mu_{K_1L_1\nu}\end{aligned}$$

To complete it, we need to replace the momentum π^I by its value, we had before (1.4) and (1.5):

$$\pi^I\pi^J = \frac{-1}{(48c)^2}\frac{1}{2}\left(D_\mu e^{I\nu}D^\mu e_\nu^J - D_\mu e^{I\nu}D_\nu e^{J\mu}\right)$$

We approximate it to

$$\pi^I \pi^J = \frac{-1}{(48c)^2}\frac{1}{2}\left(\partial_\mu e^{I\nu}\partial^\mu e^J_\nu\right)$$

We expect the contraction

$$2\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e^\nu_I\right)\pi_{J_1}\Sigma^\mu_{K_1L_1\nu} \to \frac{-1}{(48c)^2}\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma^\rho_{K_1L_1\nu}$$

Therefore we rewrite

$$-\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e^I_\nu - \partial_\nu e^{I\mu}\right) + \frac{2}{(48c)^2}\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma^\rho_{K_1L_1\nu}$$

$$+\frac{1}{(48c)^2}\varepsilon^{IJ_1K_1L_1}\varepsilon_{IJKL}\Sigma^{KL\nu}_\mu\left(\partial_\sigma e^J_\nu\right)\left(\partial^\sigma e^\rho_{J_1}\right)\Sigma^\mu_{K_1L_1\rho}$$

The Lagrange

$$Ld^4x = \frac{1}{48c}\frac{1}{2}\left(-D_\mu e^\nu_I D^\mu e^I_\nu + D_\mu e^\nu_I D_\nu e^{I\mu}\right) ed^4x$$

Becomes

$$\begin{aligned} Ld^4x \to & \frac{1}{48c}\frac{-1}{2}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e^I_\nu\right) ed^4x + \frac{1}{(48c)^3}\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma^\rho_{K_1L_1\nu} ed^4x \\ & + \frac{1}{2*(48c)^3}\varepsilon^{IJ_1K_1L_1}\varepsilon_{IJKL}\Sigma^{KL\nu}_\mu\left(\partial_\sigma e^J_\nu\right)\left(\partial^\sigma e^\rho_{J_1}\right)\Sigma^\mu_{K_1L_1\rho} ed^4x \end{aligned}$$

We used the gauge $\partial_\mu e^{I\mu} = 0$.

Now we use the selfdual projection. For any anti-symmetry tensor T^{IJ} we can write it in two unmixed representation, selfdual and anti-selfdual. In general relativity the selfdual is chosen, its projector is[1]

$$\left(P^i\right)_{jk} = \frac{1}{2}\varepsilon^i{}_{jk}\ ,\ \left(P^i\right)_{0j} = \frac{i}{2}\delta^i_j\ :\ i = I \text{ for } I = 1,2,3$$

We see that these projectors satisfy

$$2i\left(P^i\right)^{IJ}\left(P_i\right)^{KL} - 2i\left(\bar{P}^i\right)^{IJ}\left(\bar{P}_i\right)^{KL} \to \varepsilon^{IJKL}$$

It is a projection from $I \neq J$ and $K \neq L$ to $I \neq J \neq K \neq L$.

The second term is for the anti-selfdual. Therefore we consider only the first term, we replace

$$\varepsilon^{IJKL} \to 2i \left(P^i\right)^{IJ} \left(P_i\right)^{KL}$$

We use it in the determinant e:

$$e = e^0_\mu e^1_\nu e^2_\rho e^3_\sigma \varepsilon^{\mu\nu\rho\sigma} \to -\varepsilon_{IJKL} e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma}/4! : \varepsilon_{0123} = -1$$

With selfdual projection, we have

$$e = -\varepsilon_{IJKL} e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma}/4! \to -2i \left(P^i\right)_{IJ} \left(P_i\right)_{KL} e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma}/4!$$

We can rewrite

$$e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma} = \frac{1}{2}\left(e^I_\mu e^J_\nu - e^I_\nu e^J_\mu\right) e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma} = \frac{1}{2}\left(e^I_\mu e^J_\nu - e^I_\nu e^J_\mu\right)\frac{1}{2}\left(e^K_\rho e^L_\sigma - e^K_\sigma e^L_\rho\right)\varepsilon^{\mu\nu\rho\sigma}$$

By that we can rewrite it using the area field Σ^{IJ}

$$e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma} = \Sigma^{IJ}_{\mu\nu}\Sigma^{KL}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}$$

Therefore the determinant e becomes

$$e = -\varepsilon_{IJKL} e^I_\mu e^J_\nu e^K_\rho e^L_\sigma \varepsilon^{\mu\nu\rho\sigma}/4! \to -\frac{2i}{4!}\left(P^i\right)_{IJ}\left(P_i\right)_{KL}\Sigma^{IJ}_{\mu\nu}\Sigma^{KL}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} \tag{2.1}$$

Now we can write the area field as a vector $i = 1, 2, 3$ in the selfdual representation

$$\Sigma^i_{\mu\nu} = \left(P^i\right)_{IJ}\Sigma^{IJ}_{\mu\nu}$$

Therefore the determinant e becomes

$$e \to -\frac{2i}{4!}\left(\Sigma^i\right)_{\mu\nu}\left(\Sigma_i\right)_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} \quad \text{or} \quad -\frac{2i}{4!}\Sigma^i_{\mu\nu}\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} \tag{2.2}$$

We wrote it in this form to get rid of it in the path integral. As we saw it lets to delta Dirac, it cancels the propagation.

By that, the full Lagrange of the gravity field:

$$\begin{aligned} Ld^4x \to \frac{1}{48c}\frac{-1}{2}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e^I_\nu\right) ed^4x + \frac{1}{(48c)^3}\varepsilon^{IJ_1K_1L_1}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma^\rho_{K_1L_1\nu} ed^4x \\ + \frac{1}{2*(48)^3}\varepsilon^{IJ_1K_1L_1}\varepsilon_{IJKL}\Sigma^{KL\nu}_\mu\left(\partial_\sigma e^J_\nu\right)\left(\partial^\sigma e^\rho_{J_1}\right)\Sigma^\mu_{K_1L_1\rho} ed^4x \end{aligned}$$

becomes

$$
\begin{aligned}
Ld^4x \to \frac{1}{48c}\frac{-1}{2}\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_\nu^I\right)\left(-\frac{2i}{4!}\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)d^4x \\
+\frac{1}{(48c)^3}\left(2ip_i^{IJ_1}\right)\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma_\nu^{i\rho}\left(-\frac{2i}{4!}\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)d^4x \\
-\frac{2}{(48c)^3}\left(p_i\right)^{IJ_1}\left(p_j\right)_{IJ}\Sigma_\mu^{j\nu}\left(\partial_\sigma e_\nu^J\right)\left(\partial^\sigma e_{J_1}^\rho\right)\Sigma_\rho^{i\mu}\left(-\frac{2i}{4!}\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)d^4x
\end{aligned}
$$

Or

$$
\begin{aligned}
Ld^4x \to \frac{2i}{48c}\frac{1}{2}\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_\nu^I\right)\left(\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x \\
+\frac{4}{(48c)^3}\left(p_i\right)^{IJ_1}\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_{\rho J_1}\right)\Sigma_\nu^{i\rho}\left(\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x \\
+\frac{4i}{(48c)^3}\left(p_i\right)^{IJ_1}\left(p_j\right)_{IJ}\Sigma_\mu^{j\nu}\left(\partial_\sigma e_\nu^J\right)\left(\partial^\sigma e_{J_1}^\rho\right)\Sigma_\rho^{i\mu}\left(\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x
\end{aligned}
$$

It is quadratic in e^I and Σ^i therefore its integral is not trivial.

If we considered the first term. To discover its behavior, we test one wave $\cos\left(k_\mu x^\mu\right)$. We have

$$
\left(\partial_\mu e_I^\nu\right)\left(\partial^\mu e_\nu^I\right) \to -e_I^\nu\partial_\mu\partial^\mu e_\nu^I \to -\partial_\mu\partial^\mu\cos\left(k_\mu x^\mu\right) = k_\mu k^\mu\cos\left(k_\mu x^\mu\right)
$$

Therefore

$$
\begin{aligned}
e^{i\int Ld^4x} &\to \exp\int\frac{2}{48c}\frac{i^2}{2}\left(k_\mu k^\mu e_I^\nu e_\nu^I\right)\left(\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x + ... \\
&\to \exp\int\frac{2}{48c}\frac{1}{2}\left(-k_\mu k^\mu\right)\left(e_I^\nu e_\nu^I\right)\left(\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x + ...
\end{aligned}
$$

Or

$$
e^{iS} \to \exp\int\frac{2}{48c}\frac{1}{2}\left(k_0^2-\vec{k}^2\right)\left(e_I^\nu e_\nu^I\right)\left(\Sigma_{\mu\nu}^i\Sigma_{i\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}/4!\right)d^4x + ... \tag{2.3}
$$

We find in the time-like $k_0^2-\vec{k}^2 \succ 0$ the gravity field is created. And in the space-like $k_0^2-\vec{k}^2 \prec 0$ the gravity field is annihilated $e_\mu^I \to \Sigma_{\nu\rho}^i$, oppositely to the area field, as we will see.
This is the duality $e_\mu^I \leftrightarrow \Sigma_{\nu\rho}^i$. It is like to say, in time-like we find the gravity field

and in the space-like we find the area field.
The time-like phase is the phase of exchanging the energies (interactions). While the space-like is the phase of the static fields, the situation of located matter. Therefore the spacetime in which the matter is located is consisted of quanta of area and volume. The duality $e^I_\mu \leftrightarrow \Sigma^i_{\nu\rho}$ satisfies the reality, it is like the right and left spinor fields.

3 The Lagrange of the Area field

We derive the Lagrange of the area field; we find it is like the electromagnetic field. We find the behavior of the area field is opposite to the gravity behavior. For that reason we suggest the duality gravity-area.

The area field is defined in

$$\Sigma^{IJ} = e^I \wedge e^J \quad \text{with} \quad \Sigma^{IJ}_{\mu\nu} = \frac{1}{2}\left(e^I_\mu e^I_\nu - e^I_\nu e^I_\mu\right)$$

Starting with the Lagrange (1.3)

$$S(e,\pi) = c\int \left[\varepsilon_{IJKL} e^I \wedge e^J \wedge d\left({\pi_M}^{KL} e^M\right) + \varepsilon_{IJKL} e^I \wedge e^J \wedge \left({\pi_{K_1}}^K{}_M\right) e^{K_1} \wedge \left({\pi_{K_2}}^{ML}\right) e^{K_2}\right]$$

As before we assume the integral of

$$\varepsilon_{IJKL} d\left(e^I \wedge e^J \wedge \left({\pi_M}^{KL} e^M\right)\right) = \varepsilon_{IJKL} d\left(\Sigma^{IJ} \wedge \left({\pi_M}^{KL} e^M\right)\right)$$

is zero at infinity, it becomes

$$d\Sigma^{IJ} \wedge \left({\pi_M}^{KL}\right) e^M + e^I \wedge e^J \wedge d\left({\pi_M}^{KL} e^M\right) = -\left({\pi_M}^{KL}\right) e^M \wedge d\Sigma^{IJ} + e^I \wedge e^J \wedge d\left({\pi_M}^{KL} e^M\right)$$

The Action becomes

$$S(e,\pi) = c\int \left[\varepsilon_{IJKL}\left({\pi_M}^{KL}\right) e^M \wedge d\Sigma^{IJ} + \varepsilon_{IJKL} \Sigma^{IJ} \wedge \left({\pi_{K_1}}^K{}_M\right)\left({\pi_{K_2}}^{ML}\right) e^{K_1} \wedge e^{K_2}\right]$$

Or

$$S(e,\pi) = c\int \left[\varepsilon_{IJKL}\left({\pi_M}^{KL}\right) e^M \wedge d\Sigma^{IJ} + \varepsilon_{IJKL}\left({\pi_{K_1}}^K{}_M\right)\left({\pi_{K_2}}^{ML}\right) \Sigma^{IJ} \wedge \Sigma^{K_1 K_2}\right]$$

Using our imposing

$$\pi^{IJK} = \pi_L \varepsilon^{LIJK}$$

We get

$$\varepsilon_{IJKL}\left(\pi_M{}^{KL}\right)e^M = \varepsilon_{IJKL}\pi^{MKL}e_M = \varepsilon_{IJKL}\pi_N\varepsilon^{NMKL}e_M = -2\left(\pi_I e_J - \pi_J e_I\right)$$

We write

$$\Sigma^{IJ}\wedge\Sigma^{K_1K_2} \to \varepsilon^{IJK_1K_2}\Sigma^{01}\wedge\Sigma^{23}$$

So we have

$$\begin{aligned}\varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right)\Sigma^{IJ}\wedge\Sigma^{K_1K_2} &= \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right)\varepsilon^{IJK_1K_2}\Sigma^{01}\wedge\Sigma^{23}\\ &= 2\left(\pi_L{}^K{}_M\right)\left(\pi_K{}^{ML}\right)\Sigma^{01}\wedge\Sigma^{23} = 2\left(\pi_{LKM}\right)\left(\pi^{KML}\right)\Sigma^{01}\wedge\Sigma^{23}\\ &= 2\left(\pi_{KML}\right)\left(\pi^{KML}\right)\Sigma^{01}\wedge\Sigma^{23} = 2\pi^I\varepsilon_{IKML}\pi_J\varepsilon^{JKML}\Sigma^{01}\wedge\Sigma^{23}\\ &= -12\pi^2\Sigma^{01}\wedge\Sigma^{23}\end{aligned}$$

The Action becomes

$$S(e,\pi,\Sigma) = c\int\left[-2\left(\pi_I e_J - \pi_J e_I\right)\wedge d\Sigma^{IJ} - 12\pi_I\pi^I\Sigma^{01}\wedge\Sigma^{23}\right]$$

Because the area field Σ^{IJ} is anti-symmetry, we write

$$S(e,\pi,\Sigma) = c\int\left[-4\pi_I e_J\wedge d\Sigma^{IJ} - 12\pi_I\pi^I\Sigma^{01}\wedge\Sigma^{23}\right]$$

Using $\varepsilon_{0123} = -1$ we can rewrite it like

$$S(e,\pi,\Sigma) = c\int\left[-4\pi_I e_J\wedge d\Sigma^{IJ} + 12\pi_I\pi^I\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}/4!\right]$$

Or

$$S(e,\pi,\Sigma) = c\int\left[-4\pi_I e_J\wedge d\Sigma^{IJ} + \frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}\right]$$

The path integral over momentum π^I vanishes unless (the equation of motion)

$$\frac{\delta}{\delta\pi_I}\int\left[-4\pi_I e_J\wedge d\Sigma^{IJ} + \frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}\right] = 0$$

But it is not easy to separate Σ from e. It is like the gravity field, it is separable only in weak gravity(background spacetime). Therefore we solve it in the background spacetime.

$$\int \left(-4\pi_I e_J \wedge d\Sigma^{IJ} + \frac{1}{2}\pi^2 \varepsilon_{IJKL}\Sigma^{IJ} \wedge \Sigma^{KL}\right)$$

$$\to \int \left(-4\pi_I e_{\mu J}\partial_\nu \Sigma^{IJ}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} + \frac{1}{2}\pi^2 \varepsilon_{IJKL}\Sigma^{IJ}_{\mu\nu}\Sigma^{KL}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right) d^4x$$

The background spacetime is

$$e^I_\mu(x) \to \delta^I_\mu + h^I_\mu(x) \ , \quad e \to 1 + \delta e$$

The area field becomes

$$\Sigma^{IJ}_{\mu\nu} = \frac{1}{2}\left(e^I_\mu e^J_\nu - e^I_\nu e^J_\mu\right) \to \frac{1}{2}\left(\delta^I_\mu \delta^J_\nu - \delta^I_\nu \delta^J_\mu\right) + \frac{1}{2}\left(h^I_\mu \delta^J_\nu - h^I_\nu \delta^J_\mu\right) + \frac{1}{2}\left(\delta^I_\mu h^J_\nu - \delta^I_\nu h^J_\mu\right)$$

inserting it in the action:

$$S\left(e, \Sigma\right) = \int \left(-4\pi_I e_{\mu J}\partial_\nu \Sigma^{IJ}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} + \frac{1}{2}\pi^2 \varepsilon_{IJKL}\Sigma^{IJ}_{\mu\nu}\Sigma^{KL}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right) d^4x$$

it becomes

$$S\left(e, \Sigma\right) \to S\left(h, \delta\Sigma\right) = \int \left(-4\pi_I \partial_\nu \Sigma^{IJ}_{\rho\sigma}\varepsilon_J{}^{\nu\rho\sigma} + \frac{1}{2}\pi^2\left(-24\right) + \ldots\right) d^4x$$

Therefore the condition(the equation of motion):

$$\frac{\delta}{\delta\pi_I}\int \left[-4\pi_I e_J \wedge d\Sigma^{IJ} + \frac{1}{2}\pi^2 \varepsilon_{IJKL}\Sigma^{IJ} \wedge \Sigma^{KL}\right] = 0$$

approximates to

$$\frac{\delta}{\delta\pi_I}\int \left(-4\pi_I \partial_\nu \Sigma^I{}_{J\rho\sigma}\varepsilon^{J\nu\rho\sigma} + \frac{1}{2}\pi^2\left(-24\right)\right) d^4x = 0$$

Its solution is

$$\pi^I = -\frac{1}{6}\partial_\nu \Sigma^I_{J\rho\sigma}\varepsilon^{J\nu\rho\sigma} = -\frac{1}{6}\partial^\nu \Sigma^{IJ\rho\sigma}\varepsilon_{J\nu\rho\sigma}$$

By that, the action is approximated to

$$S(\Sigma) \to c \int \left[\frac{2}{3} \partial^{\nu_1} \Sigma^{IJ_1\rho_1\sigma_1} \varepsilon_{J_1\nu_1\rho_1\sigma_1} \partial_\nu \Sigma_{IJ\rho\sigma} \varepsilon^{J\nu\rho\sigma} + ... \right] d^4x$$

define inner product $\Sigma^{IJ_1\rho_1\sigma_1} \Sigma_{IJ\rho\sigma} = \Sigma^2 \delta_J^{J_1} \delta_\rho^{\rho_1} \delta_\sigma^{\sigma_1}$, we get

$$S(\Sigma) \to c \int \left(-4\partial_\mu \Sigma_{IJ}^{\nu\rho} \partial^\mu \Sigma_{\nu\rho}^{IJ} + ... \right) d^4x \quad \text{with} \quad \partial_\mu \Sigma_{IJ}^{\mu\rho} = 0$$

This is the action of the area field in weak gravity field (background spacetime). It is like the electromagnetic field.

$$L_0(\Sigma) \to -4c \left(\partial_\mu \Sigma_{IJ}^{\nu\rho} \right) \left(\partial^\mu \Sigma_{\nu\rho}^{IJ} \right) \text{ with } \quad \partial_\mu \Sigma_{IJ}^{\mu\rho} = 0$$

We rewrite it like

$$L_0(\Sigma) e d^4x = -4c \left(\partial_\mu \Sigma_{IJ}^{\nu\rho} \right) \left(\partial^\mu \Sigma_{\nu\rho}^{IJ} \right) e d^4x + ...$$

By using the selfdual projection (2.1) and (2.2)

$$e = -\varepsilon_{IJKL} e_\mu^I e_\nu^J e_\rho^K e_\sigma^L \varepsilon^{\mu\nu\rho\sigma}/4! \to -\frac{2i}{4!} \left(P^i \right)_{IJ} \left(P_i \right)_{KL} \Sigma_{\mu\nu}^{IJ} \Sigma_{\rho\sigma}^{KL} \varepsilon^{\mu\nu\rho\sigma}$$

the Lagrange becomes

$$L_0(\Sigma) e d^4x = 8ci \left(\partial^\mu \Sigma_{IJ}^{\nu\rho} \partial_\mu \Sigma_{\nu\rho}^{IJ} \right) \left(\Sigma_{\mu\nu}^i \Sigma_{i\rho\sigma} \varepsilon^{\mu\nu\rho\sigma}/4! \right) d^4x + ...$$

As we did, in deriving the gravity Lagrange we had to replace; $c \to -c$. This constant is determined in gravity potential $c \succ 0$.

To discover the area field behavior, we test one wave $\cos\left(k_\mu x^\mu\right)$. We have

$$L_0(\Sigma) e d^4x \to -8ci \left(k^\mu k_\mu \Sigma_{IJ}^{\nu\rho} \Sigma_{\nu\rho}^{IJ} \right) \left(\Sigma_{\mu\nu}^i \Sigma_{i\rho\sigma} \varepsilon^{\mu\nu\rho\sigma}/4! \right) d^4x$$

The action of that is

$$e^{iLed^4x} \to \exp 8c \left(k_\mu k^\mu \Sigma_{IJ}^{\nu\rho} \Sigma_{\nu\rho}^{IJ} \right) \left(\Sigma_{\mu\nu}^i \Sigma_{i\rho\sigma} \varepsilon^{\mu\nu\rho\sigma}/4! \right) d^4x$$

Then

$$e^{i\delta S} \to \exp 8c \left(-k_0^2 + \vec{k}^2 \right) \left(\Sigma_{IJ}^{\nu\rho} \Sigma_{\nu\rho}^{IJ} \right) \left(\Sigma_{\mu\nu}^i \Sigma_{i\rho\sigma} \varepsilon^{\mu\nu\rho\sigma}/4! \right) d^4x \tag{3.1}$$

It is opposite to the gravity field (2.3). In the time-like $-k_0^2+\vec{k}^2 \prec 0$ the area field is annihilated $\Sigma^i_{\nu\rho} \to e^I_\mu$. And in the space-like $-k_0^2+\vec{k}^2 \succ 0$ the area field is created $e^I_\mu \to \Sigma^i_{\nu\rho}$ this is the duality $e^I_\mu \leftrightarrow \Sigma^i_{\nu\rho}$. It preserves the reality. It is like duality of the left and right spinor field under Lorentz transformation and the party.

The opposite behavior is with the anti-selfdual representation, the hermitian of the selfdual

$$2i\left(P^i\right)^{IJ}\left(P_i\right)^{KL} - 2i\left(\bar{p}^i\right)^{IJ}\left(\bar{P}_i\right)^{KL} \to \varepsilon^{IJKL}$$

which is projection from $I \neq J$ and $K \neq L$, in the left, to $I \neq J \neq K \neq L$ in the right.
The first term is for the selfdual, while the second is for the anti-selfdual. The tensor product of them gives the reality:

$$e^{\Delta L(selfdual)d^4x}e^{\Delta L(anti-selfdual)d^4x} = real$$

instead of that we can satisfy the reality by gravity-area duality:

$$e^{\Delta L(e)d^4x}e^{\Delta L(\Sigma)d^4x} = \text{real : invariant for selfdual}$$

Mathematically, there must be another term like it for the anti-selfdual. But that is not necessary because by that relation the energy is conserved, as we saw in (2.3) and (3.1), we conclude:

$$\delta\text{E}\left(\text{gravity}\right) + \delta\text{E}\left(\text{area}\right) = 0$$

It is like the duality of the left ψ_L and right ψ_R spinor fields under the Lorentz transformation and the party. $\psi_L^+\psi_L$ and $\psi_R^+\psi_R$ are not Lorentz invariant. While $\psi_L\psi_R$ is Lorentz invariant, but it is not hermitian. To satisfy the both we write $\psi_L\psi_R+\psi_L^+\psi_R^+$.

4 The static potential of the weak gravity

We derive the static potential of the interactions of the scalar and spinor fields with the weak gravity field in the static limit. We see it is the same in the both, the Newtonian gravitational potential. We see that potential relates to the energy-energy interaction. By that we determine the constant $c \succ 0$.

The action of the scalar field in the curved spacetime is[1]

$$S(e,\phi) = \int d^4xe\left(\eta^{IJ}e_I^{\mu}e_J^{\nu}D_{\mu}\phi^{+}D_{\nu}\phi - V(\phi)\right)$$

In weak gravity, the background spacetime:

$$e_I^{\mu}(x) \to \delta_I^{\mu} + h_I^{\mu}(x) \quad , \quad e \to 1 + \delta e$$

the action is approximated to

$$S(e,\phi) = \int d^4x\left(\partial_{\mu}\phi^{+}\partial^{\mu}\phi + h^{\mu\nu}(x)\partial_{\mu}\phi^{+}\partial_{\nu}\phi + h^{\nu\mu}(x)\partial_{\mu}\phi^{+}\partial_{\nu}\phi - V(\phi) + L_{\text{int}} + ...\right)$$

The gravity field is symmetry, so

$$S(e,\phi) = \int d^4x\left(\partial_{\mu}\phi^{+}\partial^{\mu}\phi + 2h^{\mu\nu}(x)\partial_{\mu}\phi^{+}\partial_{\nu}\phi - V(\phi) + ...\right)$$

The energy-momentum tensor of the scalar field is[3]

$$T_{\mu\nu} = \partial_{\mu}\phi^{+}\partial_{\nu}\phi + g_{\mu\nu}L$$

Therefore

$$\partial_{\mu}\phi^{+}\partial_{\nu}\phi = T_{\mu\nu} - g_{\mu\nu}L$$

Using it in the Lagrange, it becomes

$$L = \partial_{\mu}\phi^{+}\partial^{\mu}\phi + 2h^{\mu\nu}(x)\left(T_{\mu\nu} - g_{\mu\nu}L\right) - V(\phi) + ...$$

By that we have

$$L = \partial_{\mu}\phi^{+}\partial^{\mu}\phi + 2h^{\mu\nu}T_{\mu\nu} - V(\phi) - 2h^{\mu\nu}g_{\mu\nu}L + ...$$

Therefore we have the replacement

$$\partial_{\mu}\phi^{+}\partial_{\nu}\phi \to T_{\mu\nu} \quad \text{and} \quad L \to L - 2h^{\mu\nu}g_{\mu\nu}L$$

Because the gravity field is weak (in this study), so $2h^{\mu\nu}g_{\mu\nu}L$ is neglected comparing with L .

We find the potential $V(r)$ of exchanged virtual gravitons by two particles k_1 and k_2 using $M\left(k_1+k_2 \to k_1'+k_2'\right)$ matrix element (like Born approximation to the scattering amplitude in non-relativistic quantum mechanics [7]).

For one diagram of Feynman diagrams, we have

$$iM\left(k_1+k_2 \to k_1'+k_2'\right) = i\left(-ik_2'\right)_\mu \left(ik_2\right)_\nu \frac{\bar{\Delta}^{\mu\nu\rho\sigma}(q)}{i} i\left(-ik_1'\right)_\rho \left(ik_1\right)_\sigma$$

with

$$q = k_1' - k_1 = k_2 - k_2'$$

The propagator $\Delta^{\mu\nu\rho\sigma}\left(x_2-x_1\right)$ is the gravitons propagator (1.7), we find it in the Lagrange of the free gravity field (background spacetime) we had before

$$L_0 = \frac{1}{48c}\frac{1}{2}\eta_{IJ} e_\mu^I \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right) e_\nu^J \to \frac{1}{48c}\frac{1}{2}\eta_{IJ} h_\mu^I \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right) h_\nu^J$$

with the gauge $\partial^\mu e_\mu^I = 0$, we have

$$\Delta_{\mu\nu}^{IJ}(y-x) = \int \frac{d^4q}{(2\pi)^4} \bar{\Delta}_{\mu\nu}^{IJ}\left(q^2\right) e^{iq(y-x)} \;:\; \bar{\Delta}_{\mu\nu}^{IJ}\left(q^2\right) = 48c \frac{g_{\mu\nu}\eta^{IJ}}{q^2 - i\varepsilon}$$

The M matrix element becomes

$$iM\left(k_1+k_2 \to k_1'+k_2'\right) = i48c\left(-ik_2'\right)_\mu \left(ik_2\right)_\rho \frac{g^{\mu\nu}g^{\rho\sigma}}{q^2} \left(-ik_1'\right)_\sigma \left(ik_1\right)_\nu$$

where $\quad g = \eta$ and $q = k_1' - k_1 = k_2 - k_2'$

Comparing with[7]

$$iM\left(k_1+k_2 \to k_1'+k_2'\right) = -i\bar{V}(q)\,\delta^4\left(k_{out} - k_{in}\right)$$

We have

$$\bar{V}\left(q^2\right) = -48c\left(-ik_2'\right)_\mu \left(ik_2\right)_\rho \frac{g^{\mu\nu}g^{\rho\sigma}}{q^2} \left(-ik_1'\right)_\sigma \left(ik_1\right)_\nu$$

Comparing this relation with the replacement:

$$\partial_\mu\phi^+\partial_\nu\phi \to T_{\mu\nu} \text{ and } \quad L \to L - 2h^{\mu\nu}g_{\mu\nu}L$$

and making the Fourier transformation, we get

$$V\left(y-x\right)=-48cT_{\mu\rho}\left(y\right)g^{\mu\nu}g^{\rho\sigma}T_{\nu\sigma}\left(x\right)\frac{1}{4\pi\left|y-x\right|}=-48c\frac{T_{\mu\nu}\left(y\right)T^{\mu\nu}\left(x\right)}{4\pi\left|y-x\right|}$$

With the transferred energy-momentum T , in the static limit, for one particle $T^{00}\rightarrow m$ the mass of the interacted particles.

Therefore we get the Newtonian gravitational potential

$$V\left(y-x\right)=-48c\frac{m^2}{4\pi\left|y-x\right|}=-G\frac{m^2}{\left|y-x\right|}\rightarrow 48c=4\pi G$$

The weak gravity Lagrange becomes

$$L_0=\frac{1}{4\pi G}\frac{1}{2}\eta_{IJ}e_\mu^I\left(g^{\mu\nu}\partial^2-\partial^\mu\partial^\nu\right)e_\nu^J$$

We do the same thing for the gravity interaction with spinor field. The action is[1]

$$S(e,\psi)=\int d^4xe\left(ie_I^\mu\bar{\psi}\gamma^ID_\mu\psi-m\bar{\psi}\psi\right)$$

The covariant derivative D_μ is

$$D_\mu=\partial_\mu+\left(\omega_\mu\right)_J^I L_I^J+A_\mu^a\mathrm{T}^a$$

In the background spacetime, we have

$$S(e,\psi)=\int d^4x\left(i\bar{\psi}\gamma^\mu D_\mu\psi+ih_I^\mu\bar{\psi}\gamma^ID_\mu\psi-m\bar{\psi}\psi\right)+...$$

We consider only the terms

$$\int d^4x\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi+ih_\nu^\mu\bar{\psi}\gamma^\nu\partial_\mu\psi-m\bar{\psi}\psi\right):\quad g=\eta$$

The energy-momentum tensor is[3]

$$T^{\mu\nu}=-i\bar{\psi}\gamma^\mu\partial^\nu\psi+g^{\mu\nu}L$$

Therefore, as for the scalar field, we have the replacement

$$i\bar{\psi}\gamma^\mu\partial^\nu\psi\rightarrow -T_{\mu\nu}\ \text{ and }\ L\rightarrow L+h^{\mu\nu}g_{\mu\nu}L$$

The term $h^{\mu\nu}g_{\mu\nu}L$ is ignored comparing with the Lagrange L. We find the M element of exchanged virtual gravitons $p_1 + p_2 \rightarrow p_1' + p_2'$, for one diagram of Feynman diagrams[7]

$$iM\left(p_1 + p_2 \rightarrow p_1' + p_2'\right) = i48c\bar{u}\left(p_1'\right)\gamma^{\mu}\left(-ip_1\right)_{\nu}u\left(p_1\right)\frac{g_{\mu\sigma}g^{\nu\rho}}{q^2}\bar{u}\left(p_2'\right)\gamma^{\sigma}\left(-ip_2\right)_{\rho}u\left(p_2\right)$$

with

$$q = p_1' - p_1 = p_2 - p_2' \text{ and } \quad g = \eta$$

We have

$$\bar{V}\left(q^2\right) = -48c\bar{u}\left(p_1'\right)\gamma^{\mu}\left(-ip_1\right)_{\nu}u\left(p_1\right)\frac{g_{\mu\sigma}g^{\nu\rho}}{q^2}\bar{u}\left(p_2'\right)\gamma^{\sigma}\left(-ip_2\right)_{\rho}u\left(p_2\right)$$

Comparing this relation with the replacement

$$i\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi \rightarrow -T_{\mu\nu} \text{ and } \quad L \rightarrow L + h^{\mu\nu}g_{\mu\nu}L$$

And make the Fourier transformation, we get

$$V\left(y-x\right) = -48c\left(-T_{\mu\rho}\left(y\right)\right)g^{\mu\nu}g^{\rho\sigma}\left(-T_{\nu\sigma}\left(x\right)\right)\frac{1}{4\pi\left|y-x\right|} = -48c\frac{T_{\mu\nu}\left(y\right)T^{\mu\nu}\left(x\right)}{4\pi\left|y-x\right|}$$

With the transferred energy-momentum T , in the static limit, for one particle $T^{00} \rightarrow m$ is mass of the interacted particles(spinor).

Therefore we get the Newtonian gravitational potential.

$$V\left(y-x\right) = -48c\frac{m^2}{4\pi\left|y-x\right|} = -G\frac{m^2}{\left|y-x\right|} \rightarrow 48c = 4\pi G$$

It is the same potential as for the scalar particles.

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