An Identity for a q-Hypergeometric Series

By Edigles Guedes and Cícera Guedes

August 5, 2017

"For now we see through a glass, darkly; but then face to face: now I know in part; but then shall I know even as also I am known." - 1 Corinthians 13:12.

Abstract. In this paper, we construct a identity for a q-hypergeometric series.

1. Introduction

The q-binomial coefficient is given by [1]

$$\left[\begin{array}{c} n \\ m \end{array}\right]_q = \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}} = \prod_{\ell=0}^{m-1} \frac{1-q^{n-\ell}}{1-q^{\ell+1}}. \tag{1}$$

The q-binomial coefficient satisfies the recurrence equation

$$\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q = q^k \left[\begin{array}{c} n \\ k \end{array}\right]_q + \left[\begin{array}{c} n \\ k-1 \end{array}\right]_q, \tag{2}$$

for $n \geqslant 1$ and $1 \leqslant k \leqslant n$.

In this paper, we demonstrate that

$$1 - zq + G(z;q) = F(z;q) + G_5(z;q),$$

where

$$G(z;q) := \sum_{n=0}^{\infty} \frac{(-1)^n (q;q)_n (1/z;q)_n (zq;q)_{n+1} q^{n(n+1)/2}}{(q;q)_{2n+1}},$$

$$G_5(z;q) := \sum_{n=0}^{\infty} \frac{(-1)^n (q;q)_n (1/z;q)_n (zq;q)_{n+1} q^{n(n+5)/2}}{(q;q)_{2n+1}}$$

and

$$E(z;q) := \sum_{n=0}^{\infty} \frac{(-1)^n (1/z;q)_n (zq;q)_{n+1} q^{n(n+1)/2}}{(-q;q)_n (q;q^2)_n}.$$

2. Preliminary

Lemma 1. *If $n \geqslant 1$ and $1 \leqslant k \leqslant n$, then*

$$\frac{1}{(q;q)_n} = \frac{(q;q)_{n-k+1}}{(q;q)_{n-k}(q;q)_{n+1}} q^k + \frac{(q;q)_k}{(q;q)_{k-1}(q;q)_{n+1}}.$$

Proof. Using the recurrence equation, we obtain

$$q^k = \frac{\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q}{\left[\begin{array}{c} n \\ k \end{array}\right]_q} - \frac{\left[\begin{array}{c} n \\ k-1 \end{array}\right]_q}{\left[\begin{array}{c} n \\ k \end{array}\right]_q}. \tag{3}$$

Applying the definition for q-binomial coefficient, see equation (1), we get

$$\begin{aligned} q^k &= \frac{\frac{(q;q)_{n+1}}{(q;q)_k (q;q)_{n-k+1}}}{\frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}} - \frac{\frac{(q;q)_n}{(q;q)_{k-1} (q;q)_{n-k+1}}}{\frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}} \\ &= \frac{(q;q)_{n+1} (q;q)_{n-k}}{(q;q)_n (q;q)_{n-k+1}} - \frac{(q;q)_k (q;q)_{n-k}}{(q;q)_{k-1} (q;q)_{n-k+1}} \\ &= \left(\frac{(q;q)_{n+1}}{(q;q)_n} - \frac{(q;q)_k}{(q;q)_{k-1}} \right) \frac{(q;q)_{n-k}}{(q;q)_{n-k+1}} \\ &\Rightarrow \frac{(q;q)_{n-k+1}}{(q;q)_{n-k}} q^k = \frac{(q;q)_{n+1}}{(q;q)_n} - \frac{(q;q)_k}{(q;q)_{k-1}} \\ &\Leftrightarrow \frac{(q;q)_{n+1}}{(q;q)_n} = \frac{(q;q)_{n-k+1}}{(q;q)_{n-k}} q^k + \frac{(q;q)_k}{(q;q)_{k-1}} \\ &\Leftrightarrow \frac{1}{(q;q)_n} = \frac{(q;q)_{n-k+1}}{(q;q)_{n-k}(q;q)_{n+1}} q^k + \frac{(q;q)_k}{(q;q)_{k-1}(q;q)_{n+1}}, \end{aligned} \tag{4}$$

which is the desired result. □

Corollary 2. *If* $n \geqslant 1$, *then*

$$\frac{1}{(q;q)_n} = q\frac{(q;q)_n}{(q;q)_{n-1}(q;q)_{n+1}} + (1-q)\frac{1}{(q;q)_{n+1}}.$$

Proof. Set $k=1$ in previous Lemma. □

Corollary 3. *If* $n \geqslant 1$, *then*

$$\frac{1}{(q;q^2)_n} = (1-q^{2n})\frac{(-q;q)_n(q;q)_n}{(q;q)_{2n+1}}.$$

Proof. Setting $n \to 2n$ in Corollary 2, we obtain

$$\frac{1}{(q;q)_{2n}} = q\frac{(q;q)_{2n}}{(q;q)_{2n-1}(q;q)_{2n+1}} + (1-q)\frac{1}{(q;q)_{2n+1}}. \tag{5}$$

In [3, p. 242], we have

$$(a;q)_{2n} = (a;q^2)_n(aq;q^2)_n,$$

whence, we get $a \to q$

$$(q;q)_{2n} = (q;q^2)_n(q^2;q^2)_n. \tag{6}$$

Substituting the right hand side of (6) in the left hand side of (5), we obtain

$$\begin{aligned} \frac{1}{(q;q^2)_n(q^2;q^2)_n} &= q\frac{(q;q)_{2n}}{(q;q)_{2n-1}(q;q)_{2n+1}} + (1-q)\frac{1}{(q;q)_{2n+1}} \\ \Leftrightarrow \frac{1}{(q;q^2)_n} &= q\frac{(q;q)_{2n}(q^2;q^2)_n}{(q;q)_{2n-1}(q;q)_{2n+1}} + (1-q)\frac{(q^2;q^2)_n}{(q;q)_{2n+1}} \\ \Leftrightarrow \frac{1}{(q;q^2)_n} &= q\frac{(-q;q)_n(q;q)_n(q;q)_{2n}}{(q;q)_{2n-1}(q;q)_{2n+1}} + (1-q)\frac{(-q;q)_n(q;q)_n}{(q;q)_{2n+1}}. \end{aligned} \tag{7}$$

Using the definition of q-series [see 4]:

$$(a;q)_k = \prod_{m=0}^{k-1}(1-aq^m),$$

we find

$$(q;q)_{2n-1} = \prod_{m=0}^{2n-2}(1-q^{m+1}), \tag{8}$$

$$(q;q)_{2n} = \prod_{m=0}^{2n-1}(1-q^{m+1}), \tag{9}$$

thereby, dividing (9) by (8), we have

$$\frac{(q;q)_{2n}}{(q;q)_{2n-1}} = \frac{\prod_{m=0}^{2n-1}(1-q^{m+1})}{\prod_{m=0}^{2n-2}(1-q^{m+1})} = 1-q^{2n}. \tag{10}$$

From (7) and (10), it follows that

$$\begin{aligned} \frac{1}{(q;q^2)_n} &= q(1-q^{2n})\frac{(-q;q)_n(q;q)_n}{(q;q)_{2n+1}} + (1-q)\frac{(-q;q)_n(q;q)_n}{(q;q)_{2n+1}} \\ \Rightarrow \frac{1}{(q;q^2)_n} &= (q-q^{2n}+1-q)\frac{(-q;q)_n(q;q)_n}{(q;q)_{2n+1}} = (1-q^{2n})\frac{(-q;q)_n(q;q)_n}{(q;q)_{2n+1}}, \end{aligned}$$

which is the desired result. □

Theorem 4. *We have*

$$1 - zq + G(z;q) = E(z;q) + G_5(z;q).$$

Proof. In Corollary 3, we have

$$\frac{1}{(-q;q)_n(q;q^2)_n} = (1-q^{2n})\frac{(q;q)_n}{(q;q)_{2n+1}}. \tag{11}$$

Multiply both members of (11) by $(-1)^n(1/z;q)_n(zq;q)_{n+1}q^{n(n+1)/2}$ and sum from 1 at infinity

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n(1/z;q)_n(zq;q)_{n+1}q^{n(n+1)/2}}{(-q;q)_n(q;q^2)_n} &= \sum_{n=1}^{\infty}\frac{(-1)^n(q;q)_n(1/z;q)_n(zq;q)_{n+1}q^{n(n+1)/2}}{(q;q)_{2n+1}}\\ &-\sum_{n=1}^{\infty}\frac{(-1)^n(q;q)_n(1/z;q)_n(zq;q)_{n+1}q^{n(n+5)/2}}{(q;q)_{2n+1}}\\ &=\frac{(zq;q)_1}{(q;q)_1}+\sum_{n=1}^{\infty}\frac{(-1)^n(q;q)_n(1/z;q)_n(zq;q)_{n+1}q^{n(n+1)/2}}{(q;q)_{2n+1}}\\ &-\frac{(zq;q)_1}{(q;q)_1}-\sum_{n=1}^{\infty}\frac{(-1)^n(q;q)_n(1/z;q)_n(zq;q)_{n+1}q^{n(n+5)/2}}{(q;q)_{2n+1}}\\ &=\sum_{n=0}^{\infty}\frac{(-1)^n(q;q)_n(1/z;q)_n(zq;q)_{n+1}q^{n(n+1)/2}}{(q;q)_{2n+1}}\\ &-\sum_{n=0}^{\infty}\frac{(-1)^n(q;q)_n(1/z;q)_n(zq;q)_{n+1}q^{n(n+5)/2}}{(q;q)_{2n+1}}\end{aligned} \tag{12}$$

On the other hand, we knew that [5,]

$$G(z;q):=\sum_{n=0}^{\infty}\frac{(-1)^n(q;q)_n(1/z;q)_n(zq;q)_{n+1}q^{n(n+1)/2}}{(q;q)_{2n+1}}. \tag{13}$$

From (12) and (13), it follows that

$$\begin{aligned}G(z;q) &= \sum_{n=1}^{\infty}\frac{(-1)^n(1/z;q)_n(zq;q)_{n+1}q^{n(n+1)/2}}{(-q;q)_n(q;q^2)_n}\\ &+\sum_{n=0}^{\infty}\frac{(-1)^n(q;q)_n(1/z;q)_n(zq;q)_{n+1}q^{n(n+5)/2}}{(q;q)_{2n+1}}\\ &=(zq;q)_1-(zq;q)_1+\sum_{n=1}^{\infty}\frac{(-1)^n(1/z;q)_n(zq;q)_{n+1}q^{n(n+1)/2}}{(-q;q)_n(q;q^2)_n}\\ &+\sum_{n=0}^{\infty}\frac{(-1)^n(q;q)_n(1/z;q)_n(zq;q)_{n+1}q^{n(n+5)/2}}{(q;q)_{2n+1}}\\ &=\sum_{n=0}^{\infty}\frac{(-1)^n(1/z;q)_n(zq;q)_{n+1}q^{n(n+1)/2}}{(-q;q)_n(q;q^2)_n}-(1-zq)\\ &+\sum_{n=0}^{\infty}\frac{(-1)^n(q;q)_n(1/z;q)_n(zq;q)_{n+1}q^{n(n+5)/2}}{(q;q)_{2n+1}}\\ &\Rightarrow 1-zq+G(z;q)\\ &=\sum_{n=0}^{\infty}\frac{(-1)^n(1/z;q)_n(zq;q)_{n+1}q^{n(n+1)/2}}{(-q;q)_n(q;q^2)_n}\\ &+\sum_{n=0}^{\infty}\frac{(-1)^n(q;q)_n(1/z;q)_n(zq;q)_{n+1}q^{n(n+5)/2}}{(q;q)_{2n+1}}.\end{aligned} \tag{14}$$

We define the functions

$$G_5(z;q):=\sum_{n=0}^{\infty}\frac{(-1)^n(q;q)_n(1/z;q)_n(zq;q)_{n+1}q^{n(n+5)/2}}{(q;q)_{2n+1}} \tag{15}$$

and

$$E(z;q):=\sum_{n=0}^{\infty}\frac{(-1)^n(1/z;q)_n(zq;q)_{n+1}q^{n(n+1)/2}}{(-q;q)_n(q;q^2)_n} \tag{16}$$

From (14), (15) and (16), it follows that

$$1-zq+G(z;q)=E(z;q)+G_5(z;q),$$

which is the desired result. □

References

[1] Weisstein, Eric W. "q-Binomial Coefficient." From *MathWorld*–A Wolfram Web Resource. http://mathworld.wolfram.com/q-BinomialCoefficient.html.

[2] Andrews, George and Berndt, Bruce, *Ramanujan's Lost Notebook*, Part II, Springer-Verlag, 2008.

[3] Slater, Lucy Joan, *Generalized Hypergeometric Functions*, Cambridge University Press, 1966.

[4] Weisstein, Eric W. “q-Series.” From *MathWorld*–A Wolfram Web Resource. http://mathworld.wolfram.com/q-Series.html.

[5] Lovejoy, Jeremy and Ono, Ken, *Hypergeometric generating functions for values of Dirichlet and other L functions*, PNAS, June 10, 2003, vol. 100, nº 12, www.pnas.org/cgi/doi/10.1073/pnas.1131697100.