On the Pythagorean triples $(12, y, z)$

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August 10, 2017

Abstract:
I found a pattern in the Pythagorean triples formed of the natural number 12; $\{(12,5,13),\ (12,9,15),\ (12,16,20),\ (12,35,37)\}$. The pattern is the decreasing value of the difference $z-y$ for the triples such that the differences form a sequence of the even numbers $\{8,6,4,2\}$ in that order. The existence of such sequence for *other* natural numbers transforms the Pythagorean equation into a linear equation in y.

MSC numbers: 11D04, 11D09, 11D45[2]
Keywords: Number theory, Diophantine equations, Linear equation, Quadratic and bilinear equations, counting solutions of Diophantine equations

Introduction:
It was known that a triangle whose sides were in the ratio $3:4:5$ would have a right angle as one of its angles. This was used in construction and later in early geometry. This had already been discovered to be just one example of a general rule that any triangle where the length of two sides, each squared and then added together $3^2+4^2=9+16=25$, equaled the length of the third side squared $5^2=25$, would also be a right angle triangle.
This is now known as the Pythagorean Theorem, and a triple of numbers that meets this condition is called a Pythagorean triple – both are named after the ancient Greek Pythagoras. Examples include $(3,4,5)$ and $(5,12,13)$.
There are infinitely many such triples, and methods for generating such triples have been studied in many cultures, beginning with the Babylonians and later ancient Greek, Chinese, and Indian mathematicians. Mathematically, the definition of a Pythagorean triple is a set of three integers (a,b,c) that satisfy the equation: $a^2+b^2=c^2$

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[2] https://cran.r-project.org/web/classifications/MSC.html
http://www.ams.org/mathscinet/msc/msc2010.html

Diophantine equations

$x^n + y^n = z^n$, where n is a positive integer is an example of a Diophantine equation, named for the third century Alexandrian mathematician, Diophantus, who studied them and developed methods for the solution of some kinds of Diophantine equations. A typical Diophantine problem is to find two integers *x* and *y* such that their sum, and the sum of their squares, equal two given numbers *A* and *B*, respectively:

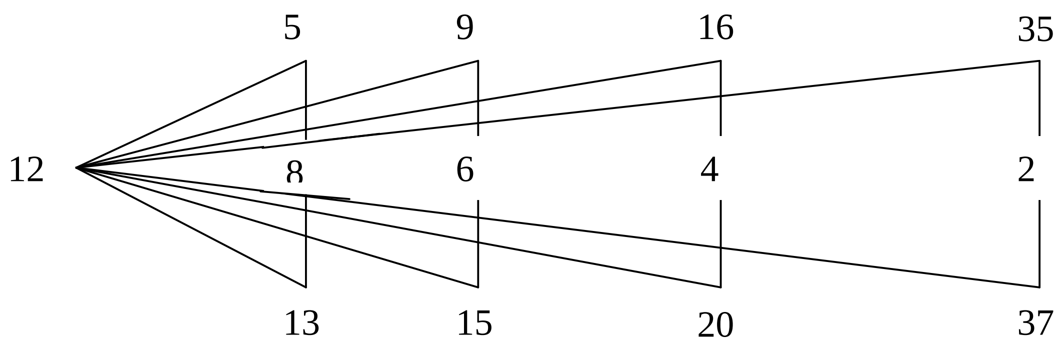
$$A = x + y$$

$$B = x^2 + y^2$$

Diophantine equations have been studied for thousands of years. For example, the solutions to the quadratic Diophantine equation $x^2 + y^2 = z^2$ are given by the Pythagorean triples, originally solved by the Babylonians (c. 1800 BC). Solutions to linear Diophantine equations, such as $(26\ x + 65\ y = 13)$, may be found using the Euclidean algorithm (c. 5th century BC). Many Diophantine equations have no *cross terms* mixing two letters, without sharing its particular properties. For example, it is known that there are infinitely many positive integers x, y and z such that $x^2 + y^2 = z^m$ where n and m are relatively prime natural numbers.

Discussion:

The set of the Pythagorean triples $\{(12,5,13),(12,9,15),(12,16,20),(12,35,37)\}$ for the natural number 12 derived from the Pythagorean equation $12^2 + y^2 = z^2$ is drawn in a diagram below.



By studying the pattern of the difference $z - y$ in the diagram above it is obvious that $z - y$ wouldn't be greater than 8 and wouldn't be less than 2 for *natural numbers* solution of the Pythagorean equation $12^2 + y^2 = z^2$. *I concluded* that $\{(12,5,13),(12,9,15),(12,16,20),(12,35,37)\}$ is the only set of the Pythagorean triples for the natural number 12.

The conclusion made above stems from substituting

$$z = y + k$$

where k is a natural number, into the Pythagorean equation

$$12^2 + y^2 = z^2$$

so it becomes

$$12^2 + y^2 = (y + k)^2$$

and, further becomes

$$12^2 + y^2 = y^2 + 2y\,k + k^2$$

and finally becomes

$$12^2 = 2y\,k + k^2$$

which is *a linear equation* in y.

Now, substituting different natural number values for k starting from 1 wouldn't produce natural numbers solution to the Pythagorean equation $12^2 + y^2 = z^2$ except those 4 Pythagorean triples for the natural number 12 given above.

Conclusion:

The Pythagorean equation has been transformed into a linear equation which can be applied to *any* natural number to determine its Pythagorean triples. The number of Pythagorean triples for *any* natural number is finite, so that there are infinite but countable Pythagorean triples solution to the Pythagorean equation.

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