Euler-Lagrange equations of the Einstein-Hilbert action

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Abstract:

I applied the method of the second variations of the Calculus of Variations to the Einstein-Hilbert action. Since the Einstein-Hilbert's Lagrangian for a gravitational field is proportional to the Ricci curvature scalar, construction of the Euler-Lagrange equations requires dealing with the tensor quantities from which the Ricci curvature scalar is composed rather than the Ricci curvature scalar itself. As the result of applying this method two Euler-Lagrange equations emerged; both equations yielded the same Einstein's field equations in absence of energy-momentum fields.

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Introduction:

It is of interest to derive equations of motion of various fields (gravitational, electromagnetic, etc) from action because it has several advantages. First of all, it allows for easy unification of different fields theories e.g. gravitational fields with other classical field theories such as electromagnetic fields theory, which are also formulated in terms of an action. In the process the derivation of equation of motion of the gravitational fields from an action identifies a natural candidate for the source term coupling the metric tensor to energy fields. Moreover, the action allows for the easy identification of conserved quantities through Noether's theorem by studying symmetries of the action.

In the general theory of relativity, the action is usually assumed to be a functional of the metric tensor (and energy-momentum tensor fields). In this work we consider only the Euler-Lagrange equations in absence of energy-momentum fields.

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The Einstein-Hilbert action

The Einstein-Hilbert action for the gravitational field may be written as

$$I_{E-H} = \int (\frac{1}{2\kappa})\ R\ (\sqrt{-g}\ d^4x) \qquad (1.1)$$

Where κ, R, g, and $\sqrt{-g}\ d^4x$ are Einstein constant ($\kappa = 8\pi G/c^4$), Ricci curvature scalar, determinant of the metric tensor and the invariant hyper-volume element of a 4-dimenasional spacetime, respectively.

The Ricci curvature scalar R is a function of the metric tensor, its first and second derivatives with respect to the spacetime coordinates $x^i \equiv (x^0, x^1, x^2, x^3, x^4) = (ct, x^1, x^2, x^3, x^4)$; i.e. ($R = R(g_{ab}, g_{ab,c}, g_{ab,cd})$).

The action principle then tells us that the variation of the action in equation (1.1) with respect to the arguments of the integrand is zero, yielding

$$0 = \delta\ I_{E-H} = \delta\ [\int (\frac{1}{2\kappa})\ R\ (\sqrt{-g}\ d^4x)] = \int (\frac{1}{2\kappa})\ \delta\ (g^{ab}\ R_{ab}\ \sqrt{-g})\ d^4x \qquad (1.2)$$

we have made use of the following relation

$$R = g^{ab}\ R_{ab}$$

where g^{ab} is the contravariant metric tensor and R_{ab} is the Ricci curvature tensor.

We may write the integral in equation (1.2) as a sum of two integrals, namely,

$$0 = [\int (\frac{1}{2\kappa})\ (R_{ab})\ \delta\ (g^{ab}\ \sqrt{-g})\ d^4x] + [\int (\frac{1}{2\kappa})\ (g^{ab}\ \sqrt{-g})\ (\delta\ R_{ab})\ d^4x] \qquad (1.3)$$

Second Variations of the Calculus of Variations of scalar functions

It is well known that the Euler-Lagrange equation resulting from applying the second variations of the Calculus of Variations of a Lagrangian functional $L(x, y(x), \dot{y}(x), \ddot{y}(x))$ of a single independent variable $y(x)$, its first and second derivatives of the following action

$$I[y(x)] = \int L(x, y(x), \dot{y}(x), \ddot{y}(x))\ dx \qquad (1.4)$$

When varied with respect to the arguments of integrand and the variation are set to zero, i.e.

$$0 = \delta\ I[y(x)] = \delta \int L(x, y(x), \dot{y}(x), \ddot{y}(x))\ dx \qquad (1.5)$$

is given by

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial \dot{y}} + \frac{d^2}{dx^2}\frac{\partial L}{\partial \ddot{y}} = 0 \qquad (1.6)$$

provided that the variations δy and $\delta \dot{y}$ vanish at the end points of the integration.

Second Variations of the Calculus of Variations of tensor functions

We assume that equation (1.6) to be valid for tensor functions whose arguments are the metric tensor, its first derivative and its second derivative with respect to the spacetime coordinates, provided that the variations δg_{ab} and $\delta g_{ab,c}$ vanish at the end points of the integration.

As an example, for a tensor function $K = K(g_{ab}, g_{ab,c}, g_{ab,cd})$, the Euler-Lagrange equations may written as

$$\frac{\partial\ K_{ab}}{\partial g_{mp}} - \frac{\partial}{\partial x^s}\ \frac{\partial\ K_{ab}}{\partial g_{mp,s}} + \frac{\partial^2}{\partial x^s\ \partial x^r}\ \frac{\partial\ K_{ab}}{\partial g_{mp,sr}} = 0 \qquad (1.7)$$

provided that the variations δg_{mp} and $\delta g_{mp,s}$ vanish at the end points of the integration.

Euler-Lagrange equations of the Einstein's field equations from the first Einstein-Hilbert integral

The Euler-Lagrange equation corresponding to the first integral in equation (1.3) may be written as

$$(\frac{1}{2\kappa})\ R_{ab}\ [\frac{\partial\ (\sqrt{-g}\ \ g^{ab})}{\partial g_{mp}} - \frac{\partial}{\partial x^s}\ \frac{\partial\ (\sqrt{-g}\ \ g^{ab})}{\partial g_{mp,s}} + \frac{\partial^2}{\partial x^s\ \partial x^r}\ \frac{\partial\ (\sqrt{-g}\ \ g^{ab})}{\partial g_{mp,sr}}] = 0 \qquad (1.8)$$

Since $\sqrt{-g}$ and g^{ab} don't depend on the first and the second derivatives of the metric tensor with respect the spacetime coordinates, their partial derivatives with respect to them give zero. So equation (1.8) becomes

$$R_{ab}\ [\frac{\partial\ (\sqrt{-g}\ \ g^{ab})}{\partial g_{mp}}] = 0 \qquad (1.9)$$

This is the Euler-Lagrange equation that produces Einstein's field equations.

It should be noted that in equation (1.9) the Ricci tensor shouldn't be dropped out by diving both sides by it, for it would lead to an absurd result.

Derivation of Einstein's field equations from the Euler-Lagrange equation of the first Einstein-Hilbert integral

To proof that equation (1.9) produces Einstein's field equations, we perform the partial differentiation and use the following useful equations

$$\frac{\partial\ \sqrt{-g}}{\partial\ g_{mp}} = (1/2)\ \sqrt{-g}\ \ g^{mp}$$

and,

$$\frac{\partial\ g^{ab}}{\partial\ g_{mp}} = -\ g^{am}\ g^{bp}$$

Then, equation (1.9) yields,

$$\begin{aligned}
0 &= R_{ab}\ [\frac{\partial\ (\sqrt{-g}\ g^{ab})}{\partial g_{mp}}] \\
&= R_{ab}\ [\sqrt{-g}\ \ \frac{\partial\ (g^{ab})}{\partial g_{mp}} + g^{ab}\ \ \frac{\partial\ (\sqrt{-g})}{\partial g_{mp}}] \\
&= R_{ab}\ [\sqrt{-g}\ \ (-\ g^{am}\ g^{bp}) + g^{ab}\ \ (1/2)\ \sqrt{-g}\ \ g^{mp}] \\
&= R_{ab}\ [\sqrt{-g}\ \ (-\ g^{am}\ g^{bp} + (1/2)\ g^{ab}\ g^{mp})] \\
&= \sqrt{-g}\ \ [-\ g^{am}\ g^{bp}\ R_{ab} + (1/2)\ g^{ab}\ g^{mp}\ R_{ab}] \\
&= \sqrt{-g}\ \ [-\ R^{mp} + (1/2)\ g^{mp}\ R]
\end{aligned} \qquad (1.10)$$

Now, dividing both sides of the equation (1.10) by $\sqrt{-g}$ or either say $\sqrt{-g}$ is not zero, we arrive at the Einstein's field equations in absence of energy-momentum fields.

Euler-Lagrange equations of the Einstein's field equations from the second Einstein-Hilbert integral

The Euler-Lagrange equation corresponding to the second integral in equation (1.3) may be written as

$$(\frac{1}{2\kappa})\ (\sqrt{-g}\ \ g^{ab})\ [\frac{\partial\ R_{ab}}{\partial g_{mp}} - \frac{\partial}{\partial x^s}\ \frac{\partial\ R_{ab}}{\partial g_{mp,s}} + \frac{\partial^2}{\partial x^s\ \partial x^r}\ \frac{\partial\ R_{ab}}{\partial g_{mp,sr}}] = 0 \qquad (1.11)$$

Differentiation of the Ricci curvature tensor with respect to its arguments

To calculate the differentiation of the Ricci curvature tensor with respect to the metric tensor, its first derivative and second derivative we write first Riemann curvature tensor and Ricci tensor as explicit functions of the metric tensor, its first derivative and second derivative. So, the Riemann curvature tensor is defined as,

$$R^d{}_{abc} = \Gamma_{ac}{}^d{}_{,b} - \Gamma_{ab}{}^d{}_{,c} + \Gamma_{be}{}^d\ \Gamma_{ac}{}^e - \Gamma_{ce}{}^d\ \Gamma_{ab}{}^e$$

using the Christoffel's symbols of the first and the second kinds defined respectively as,

$$\Gamma_{ijk} = \frac{1}{2}\ (g_{ik,j} + g_{jk,i} - g_{ij,k})$$

and

$$\Gamma_{ij}^{\ l} \equiv g^{kl}\ \Gamma_{ijk} = \frac{1}{2}\ g^{kl}\ (g_{ik,j} + g_{jk,i} - g_{ij,k})$$

The Riemann curvature tensor may be put in the following form,

$$R^{d}{}_{abc} = g^{df}\ (\Gamma_{acf,b} - \Gamma_{abf,c}) - g^{de}\ g^{fh}\ (\Gamma_{acf}\ \Gamma_{beh} - \Gamma_{abf}\ \Gamma_{ceh}) \qquad (1.12)$$

We may now obtain the Ricci curvature tensor simply by contracting two indices of the Riemann curvature tensor (putting $d = c$) in equation (1.12) and summing, we get

$$R_{ab} \equiv R^{c}{}_{abc} = g^{cf}\ (\Gamma_{acf,b} - \Gamma_{abf,c}) - g^{ce}\ g^{fh}\ (\Gamma_{acf}\ \Gamma_{beh} - \Gamma_{abf}\ \Gamma_{ceh}) \qquad (1.13)$$

Differentiating R_{ab} in equation (1.13) with respect to g_{mp}, $g_{mp,s}$ and $g_{mp,sr}$, respectively, we get

$$\frac{\partial R_{ab}}{\partial g_{mp}} = -\ g^{cm}\ R^{p}{}_{abc} + g^{ce}\ (\Gamma_{ac}{}^{m}\ \Gamma_{be}{}^{p} - \Gamma_{ab}{}^{m}\ \Gamma_{ce}{}^{p})$$

$$\frac{\partial R_{ab}}{\partial g_{mp,s}} = \{-\ \Gamma_{bc}{}^{p}\ \delta^{a}_{m}\ g^{cs} - \Gamma_{bc}{}^{p}\ \delta^{a}_{s}\ g^{cm} + \Gamma_{bc}{}^{s}\ \delta^{a}_{m}\ g^{pc} + \Gamma_{ab}{}^{p}\ g^{ms} + \Gamma_{ce}{}^{p}\ \delta^{a}_{m}\ \delta^{b}_{s}\ g^{ce}$$
$$-\ (1/2)\ \Gamma_{ab}{}^{s}\ g^{mp} - (1/2)\ \Gamma_{ce}{}^{s}\ \delta^{a}_{m}\ \delta^{b}_{p}\ g^{ce}\}$$

$$\frac{\partial R_{ab}}{\partial g_{mp,sr}} = (1/2)\ \{g^{mp}\ \delta^{a}{}_{s}\delta^{b}{}_{r} - g^{ps}\ \delta^{a}{}_{m}\delta^{b}{}_{r} - g^{pr}\ \delta^{b}{}_{m}\delta^{a}{}_{s} + g^{sr}\ \delta^{a}{}_{m}\delta^{b}{}_{p}\}$$

(1.14)

Performing the partial differentiations on the second and the third part of equation (1.14), we get

$$\frac{\partial R_{ab}}{\partial g_{mp}} = -\ g^{cm}\ R^{p}{}_{abc} + g^{ce}\ (\Gamma_{ac}{}^{m}\ \Gamma_{be}{}^{p} - \Gamma_{ab}{}^{m}\ \Gamma_{ce}{}^{p})$$

$$\frac{\partial}{\partial x^{s}}\ \frac{\partial R_{ab}}{\partial g_{mp,s}} = \{(-\Gamma_{bc}{}^{p}{}_{,s}\ \delta^{a}_{m}\ g^{cs} - \Gamma_{bc}{}^{p}{}_{,s}\ \delta^{a}_{s}\ g^{cm} + \Gamma_{bc}{}^{s}{}_{,s}\ \delta^{a}_{m}\ g^{pc} + \Gamma_{ab}{}^{p}{}_{,s}\ g^{ms} + \Gamma_{ce}{}^{p}{}_{,s}\ \delta^{a}_{m}\ \delta^{b}_{s}\ g^{ce}$$
$$-(1/2)\ \Gamma_{ab}{}^{s}{}_{,s}\ g^{mp} -\ (1/2)\ \Gamma_{ce}{}^{s}{}_{,s}\ \delta^{a}_{m}\ \delta^{b}_{p}\ g^{ce}) -\ \Gamma_{bc}{}^{p}\ \delta^{a}_{m}\ g^{cs}{}_{,s} - \Gamma_{bc}{}^{p}\ \delta^{a}_{s}\ g^{cm}{}_{,s}$$
$$+\Gamma_{bc}{}^{s}\ \delta^{a}_{m}\ g^{pc}{}_{,s} + \Gamma_{ab}{}^{p}\ g^{ms}{}_{,s} + \Gamma_{ce}{}^{p}\ \delta^{a}_{m}\ \delta^{b}_{s}\ g^{ce}{}_{,s} - (1/2)\ \Gamma_{ab}{}^{s}\ g^{mp}{}_{,s} - (1/2)\ \Gamma_{ce}{}^{s}\ \delta^{a}_{m}\ \delta^{b}_{p}\ g^{ce}{}_{,s})\}$$

$$\frac{\partial^{2}}{\partial x^{s}\ \partial x^{r}}\ \frac{\partial R_{ab}}{\partial g_{mp,sr}} = (1/2)\ \{g^{mp}{}_{,sr}\ \delta^{a}{}_{s}\delta^{b}{}_{r} - g^{ps}{}_{,sr}\ \delta^{a}{}_{m}\delta^{b}{}_{r} - g^{pr}{}_{,sr}\ \delta^{b}{}_{m}\delta^{a}{}_{s} + g^{sr}{}_{,sr}\ \delta^{a}{}_{m}\delta^{b}{}_{p}\}$$

(1.15)

Performing the calculation of the second and the third term in equation (1.15) and multiplying each term in the same equation by $(\frac{1}{2\kappa})\ (\sqrt{-g}\ g^{ab})$, then equation (1.10), yields

$$
\begin{aligned}
&(\frac{1}{2\kappa})\ (\sqrt{-g}\ \{-\ R^{amp}{}_{a} +\ 3\ R^{amp}{}_{a} - g^{mp}\ R^{ae}{}_{ea} -\ R^{rmp}{}_{r} + (1/2)\ R^{sq}{}_{qs}\ g^{mp} \\
&+g^{ab}\ g^{ce}\ (\Gamma_{ac}{}^{m}\ \Gamma_{be}{}^{p} - \Gamma_{ab}{}^{m}\ \Gamma_{ce}{}^{p}) - 2\ g^{ab}\ g^{ms}\ (-\Gamma_{sq}{}^{p}\ \Gamma_{ab}{}^{q} + \Gamma_{bq}{}^{p}\ \Gamma_{as}{}^{q}) \\
&-(g^{ab}\ g^{mp} - g^{mb}\ g^{pa})\ (-\Gamma_{as}{}^{s}{}_{,b}) - (g^{ab}\ g^{mp} - g^{mb}\ g^{pa})\ (-\Gamma_{bq}{}^{s}\ \Gamma_{as}{}^{q} + \Gamma_{sq}{}^{s}\ \Gamma_{ab}{}^{q}) \\
&+\Gamma_{ab}{}^{p}\ (g^{mb}\ g^{as})_{,s} - g^{mb}\ \Gamma_{ab}{}^{s}\ g^{pa}{}_{,s} - \Gamma_{ab}{}^{p}\ (g^{ab}\ g^{ms})_{,s} + (1/2)\ \Gamma_{ab}{}^{s}\ (g^{ab}\ g^{mp})_{,s} \\
&+(1/2)\ g^{sr}\ g^{pq}\ (-\Gamma_{tq}{}^{m}\ \Gamma_{sr}{}^{t} + \Gamma_{tq}{}^{m}\ \Gamma_{sq}{}^{t}) + (1/2)\ \Gamma_{sq}{}^{s}{}_{,r}\ (g^{mr}\ g^{pq} - g^{mp}\ g^{rq}) \\
&+(1/2)\ \Gamma_{rs}{}^{s}{}_{,q}\ (g^{mr}\ g^{pq} - g^{mp}\ g^{rq}) + (1/2)\ (\Gamma_{tq}{}^{s}\ \Gamma_{rs}{}^{t} - \Gamma_{st}{}^{s}\ \Gamma_{rq}{}^{t})\ (g^{mr}\ g^{pq} - g^{mp}\ g^{rq}) \\
&-(1/2)\ \Gamma_{sq}{}^{m}\ g^{sr}\ g^{pq}{}_{,r} - (1/2)\ \Gamma_{sq}{}^{p}\ g^{sr}\ g^{mq}{}_{,r} + (1/2)\ \Gamma_{sq}{}^{p}\ g^{mr}\ g^{sq}{}_{,r} + (1/2)\ \Gamma_{sq}{}^{s}\ g^{mr}\ g^{pq}{}_{,r} \\
&+(1/2)\ \Gamma_{sq}{}^{p}\ g^{ms}\ g^{rq}{}_{,r} + (1/2)\ \Gamma_{sq}{}^{r}\ g^{ms}\ g^{pq}{}_{,r} - (1/2)\ \Gamma_{sq}{}^{s}\ g^{mp}\ g^{rq}{}_{,r} - (1/2)\ \Gamma_{sq}{}^{r}\ g^{mp}\ g^{sq}{}_{,r} = 0
\end{aligned}
\qquad (1.16)
$$

In a local inertial frame of reference (LIF) at a point, the following expression is true:

$$g_{ij} = \eta_{ij} \text{ and } g_{ij,k} = 0$$

Where η_{ij} is the metric tensor of the flat spacetime.

The above condition of a LIF implies that all Christoffel's symbols are zero, and $\sqrt{-g} = +1$, (we are using spacetime with signature ($(+1,-1,-1,-1)$), then equation (1.16) becomes

$$(\frac{1}{2\kappa})\ (\sqrt{-g}\ \{-\ R^{amp}{}_{a} +\ 3\ R^{amp}{}_{a} - g^{mp}\ R^{ae}{}_{ea} -\ R^{rmp}{}_{r} + (1/2)\ R^{sq}{}_{qs}\ g^{mp}\} = 0 \qquad (1.17)$$

Dividing both sides by $(\frac{1}{2\kappa})\ (\sqrt{-g})$, and collecting same terms together, we get

$$(\frac{1}{2\kappa})\ (\sqrt{-g}\ \{R^{amp}{}_{a} - (1/2)\ g^{mp}\ R^{ae}{}_{ea}\} = 0 \qquad (1.18)$$

Which are again the Einstein's field equations in absence of energy-momentum fields.

Conclusion

Euler-Lagrange equations for the gravitational field could only be constructed from tensor quantities and not by direct differentiation of the scalar quantities on which the Lagrangian is based as in case of scalar field theories.

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