Meta mass function

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Abstract

In this paper, a meta mass function (MMF) is presented. A new evidence theory with complex numbers is developed. Different with existing evidence theory, the new mass function in complex evidence theory is modelled as complex numbers and named as meta mass function. The classical evidence theory is the special case under the condition that the mass function is degenerated from complex number as real number.

Keywords: Meta mass function, Complex evidence theory, Dempster-Shafer evidence theory, Quantum probability

1. Introduction

Dempster-Shafer theory evidence theory[1, 2] is widely used in many applications due to its efficiency to deal with uncertainty [3]. Some previous works are presented to deal with the problems in classical evidence

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theory. For example, generalized evidence theory is presented to address conflict management issue without the assumption that the mass function of EMPTY SET is set as ZERO [4]. In order to modelling the situation that the hypothesis is not exclusive with each other, D numbers theroy is presented [5].

The quantum mechanics is explained with complex probability [6]. In addition, quantum probability is used to model human decision and cognition[7]. Then, a question is that can we represented with mass function is a same way? The main contribution of this paper is to construct a so called meta mass function for the possbile extension of classical evidence theory with complex numbers. The paper is organized as follows. The preliminaries of complex numbers and Dempster-Shafer evidence theory are briefly introduced in Section 2. Section 3 presents the concept of meta mass function and the first result of complex evidence theory. Some numerical examples are illustrated in Section 4. Finally, this paper is concluded in Section 5.

2. Preliminaries

In this section, some preliminaries are briefly introduced.

2.1. Complex numbers [8]

A complex number is a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit, satisfying $i^2 = 1$.

Give two complex numbers $a + bi$ and $c + di$, the addition is defined as follows,

$$(a + bi) + (c + di) = (a + c) + (c + d)i$$

the subtraction is defined as follows,

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

the multiplication is defined as follows,

$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i$$

the division is defined as follows,

$$\frac{a + bi}{c + di} = \left(\frac{ac + bd}{c^2 + d^2}\right) + \left(\frac{bc - ad}{c^2 + d^2}\right) i$$

An important parameter is the absolute value (or modulus or magnitude) of a complex number $z = a + bi$ is

$$r = |z| = \sqrt{a^2 + b^2}$$

If z is a real number (i.e., $b = 0$), then $r = |a|$. The square of the absolute value is

$$|z|^2 = z\bar{z} = a^2 + b^2$$

where $\bar{z}$ is the complex conjugate of z.

2.2. Dempster-Shafer evidence theory [1, 2]

Dempster-Shafer evidence theory has many advantages to handle uncertain information. Some basic concepts in D-S theory are introduced as follows[1, 2].

Let X be a set of mutually exclusive and collectively exhaustive events, indicated by

$$X = \{\theta_1, \theta_2, \cdots, \theta_i, \cdots, \theta_{|X|}\} \tag{1}$$

where set X is called a frame of discernment. The power set of X is indicated by 2^X, namely

$$2^X = \{\emptyset, \{\theta_1\}, \cdots, \{\theta_{|X|}\}, \{\theta_1, \theta_2\}, \cdots, \{\theta_1, \theta_2, \cdots, \theta_i\}, \cdots, X\} \tag{2}$$

For a frame of discernment $X = \{\theta_1, \theta_2, \cdots, \theta_{|X|}\}$, a mass function is a mapping m from 2^X to $[0, 1]$, formally defined by[1, 2]:

$$m : \quad 2^X \to [0, 1] \tag{3}$$

which satisfies the following condition:

$$m(\emptyset) = 0 \quad and \quad \sum_{A \in 2^X} m(A) = 1 \tag{4}$$

A mass function is also called a basic probability assignment (BPA). Assume there are two BPAs indicated by m_1 and m_2, the Dempster's rule of combination is used to combine them as follows [1, 2]:

$$m(A) = \begin{cases} \frac{1}{1-K} \sum\limits_{B \cap C = A} m_1(B) m_2(C) \ , & A \neq \emptyset; \\ 0 \ , & A = \emptyset. \end{cases} \tag{5}$$

with

$$K = \sum_{B \cap C = \emptyset} m_1(B) m_2(C) \tag{6}$$

Note that the Dempster's rule of combination is only applicable to such two BPAs which satisfy the condition $K < 1$.

3. Meta mass function

The essence of complex numbers is still heavily studied. Some researches showed that quantum mechanics can be explained by complex probability, which means the probability function is not a real number but with imaginary part [6]. A straight question is that can we extend the mass function by complex numbers? To answer this question is the aim of this short paper. In this section, a so called complex evidence theory is presented.

Let X be a set of mutually exclusive and collectively exhaustive events, indicated by

$$X = \{\theta_1, \theta_2, \cdots, \theta_i, \cdots, \theta_{|X|}\} \tag{7}$$

where set X is called a frame of discernment. The power set of X is indicated by 2^X, namely

$$2^X = \{\emptyset, \{\theta_1\}, \cdots, \{\theta_{|X|}\}, \{\theta_1, \theta_2\}, \cdots, \{\theta_1, \theta_2, \cdots, \theta_i\}, \cdots, X\} \tag{8}$$

For a frame of discernment $X = \{\theta_1, \theta_2, \cdots, \theta_{|X|}\}$, a meta mass function is a mapping m from 2^X to $\mathbb{C}$ ($\mathbb{C}$ stands for the field of complex numbers), formally defined by:

$$m: \quad 2^X \to \mathbb{C} \tag{9}$$

which satisfies the following condition:

$$m(\emptyset) = 0 \quad and \sum_{A \in 2^X} |m(A)|^2 = 1 \tag{10}$$

where $|m(A)|$ is the absolute operation of complex numbers.

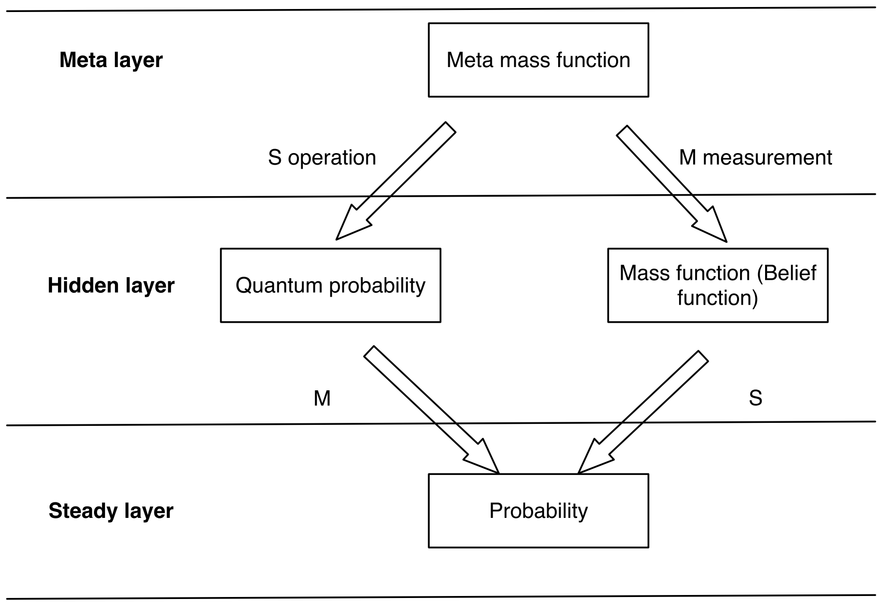

Figure 1: Relationships with meta mass function and other uncertain function

The reason to name this function as meta mass function is illustrated as Fig.1. There are three layers, namely meta layer, hidden layer and steady layer. With two different operations, namely measurement and specialization, the meta mass function in meta layer can transformed into the quantum probability (quantum mechanics) and belief function (Dempster Shafer evidence theory) in the hidden layer and finally to the probability in the steady layer. The measurement operation means the absolute value of the complex numbers while the specialization means the assignment of 1 to single basic event, or single hypothesis.

Assume there are two MMF indicated by m_1 and m_2, a rule of combination is used to combine them as follows:

$$m'(A) = \begin{cases} \sum\limits_{B \cap C = A} m_1(B)\, m_2(C), & A \neq \emptyset \\ 0, & A = \emptyset \end{cases} \tag{11}$$

The combined MMF is normalized as follow:

$$m(A) = \frac{m'(A)}{\sqrt{\sum\limits_{A \subset 2^X} |m'(A)|^2}} \tag{12}$$

where $F = \sqrt{\sum\limits_{A \subset 2^X} |m'(A)|^2}$ is the normalizing factor. It is to verify that $\sum\limits_{A \subset 2^X} |m(A)|^2 = 1$.

4. Numerical examples

Assume there are two meta belief function m_1 and m_2 with the frame of discernment $X = \{x, y\}$, of which the MMFs are denoted as follow

$$m_1(x) = \tfrac{1}{4} + \tfrac{1}{4}i;\ m_1(y) = \tfrac{1}{4} - \tfrac{1}{4}i;\ m_1(x, y) = \tfrac{\sqrt{3}}{2}$$

$$m_2(x) = \tfrac{1}{2} + \tfrac{1}{4}i;\ m_2(y) = \tfrac{1}{4}i;\ m_2(x, y) = \tfrac{\sqrt{10}}{4}i$$

It is easy to verify that $|m_1(x)|^2 + |m_1(y)|^2 + |m_1(x, y)|^2 = 1$ and $|m_2(x)|^2 + |m_2(y)|^2 + |m_2(x, y)|^2 = 1$.

According to the rule of combination using Eq.(11), we can obtain the combined MMF m'_{12} as follows:

$$\begin{aligned}
&m'_{12}(x)\\
&= m_1(x)\,m_2(x) + m_1(x)\,m_2(x,y) + m_2(x)\,m_1(x,y)\\
&= \left(\tfrac{1}{4}+\tfrac{1}{4}i\right)\times\left(\tfrac{1}{2}+\tfrac{1}{4}i\right)+\left(\tfrac{1}{4}+\tfrac{1}{4}i\right)\times\tfrac{\sqrt{10}}{4}i+\left(\tfrac{1}{2}+\tfrac{1}{4}i\right)\times\tfrac{\sqrt{3}}{2}\\
&= \left(\tfrac{1}{8}+\tfrac{1}{16}i+\tfrac{1}{8}i-\tfrac{1}{16}\right)+\left(\tfrac{\sqrt{10}}{16}i-\tfrac{\sqrt{10}}{16}\right)+\tfrac{\sqrt{3}}{4}+\tfrac{\sqrt{3}}{8}i\\
&= \left(\tfrac{1}{8}-\tfrac{1}{16}-\tfrac{\sqrt{10}}{16}+\tfrac{\sqrt{3}}{4}\right)+\left(\tfrac{1}{16}+\tfrac{1}{8}+\tfrac{\sqrt{10}}{16}+\tfrac{\sqrt{3}}{8}\right)i\\
&= \tfrac{1-\sqrt{10}+4\sqrt{3}}{16}+\tfrac{3+\sqrt{10}+2\sqrt{3}}{16}i
\end{aligned}$$

$$\begin{aligned}
&m'_{12}(y)\\
&= m_1(y)\,m_2(y) + m_1(y)\,m_2(x,y) + m_2(y)\,m_1(x,y)\\
&= \left(\tfrac{1}{4}-\tfrac{1}{4}i\right)\times\tfrac{1}{4}i+\left(\tfrac{1}{4}-\tfrac{1}{4}i\right)\times\tfrac{\sqrt{10}}{4}i+\tfrac{1}{4}i\times\tfrac{\sqrt{3}}{2}\\
&= \tfrac{1}{16}i+\tfrac{1}{16}+\tfrac{\sqrt{10}}{16}i+\tfrac{\sqrt{10}}{16}+\tfrac{2\sqrt{3}}{16}i\\
&= \tfrac{1+\sqrt{10}}{16}+\tfrac{1+\sqrt{10}+2\sqrt{3}}{16}i
\end{aligned}$$

$$\begin{aligned}
&m'_{12}(x,y)\\
&= m_1(x,y)\,m_2(x,y)\\
&= \tfrac{\sqrt{3}}{2}\times\tfrac{\sqrt{10}}{4}i\\
&= \tfrac{\sqrt{30}}{8}i
\end{aligned}$$

Then the normalized MMF m_{12} using Eq. (12) is obtained as follows:

$$\begin{aligned}
&m_{12}(x)\\
&= \frac{m'_{12}(x)}{\sqrt{\left|m'_{12}(x)\right|^2+\left|m'_{12}(y)\right|^2+\left|m'_{12}(x,y)\right|^2}}\\
&= \frac{\left(1-\sqrt{10}+4\sqrt{3}\right)+\left(3+\sqrt{10}+2\sqrt{3}\right)i}{16\times 1.1020}\\
&= 0.2703+0.5460\mathrm{i}
\end{aligned}$$

$$
\begin{aligned}
& m_{12}\left(y\right) \\
& = \frac{m'_{12}(y)}{\sqrt{\left|m'_{12}(x)\right|^2+\left|m'_{12}(y)\right|^2+\left|m'_{12}(x,y)\right|^2}} \\
& = \frac{\left(1+\sqrt{10}\right)+\left(1+\sqrt{10}+2\sqrt{3}\right)i}{16\times 1.1020} \\
& = 0.2361 + 0.4325\mathrm{i}
\end{aligned}
$$

$$
\begin{aligned}
& m_{12}\left(x,y\right) \\
& = \frac{m'_{12}(x,y)}{\sqrt{\left|m'_{12}(x)\right|^2+\left|m'_{12}(y)\right|^2+\left|m'_{12}(x,y)\right|^2}} \\
& = \frac{\frac{\sqrt{30}}{8}i}{1.1020} \\
& = 0.6213\mathrm{i}
\end{aligned}
$$

It is easy to verify that $\left|m_{12}\left(x\right)\right|^2 + \left|m_{12}\left(y\right)\right|^2 + \left|m_{12}\left(x,y\right)\right|^2 = 1$.

5. Conclusion

In this paper, a complex evidence theory is presented. Compared with the classical evidence theory, the mass function is extended by complex numbers. In the situation when the image part of the complex mass function is zero, the complex evidence theory is degenerated as Dempster Shafer evidence theory. The complex probability is applied in quantum mechanics [6]. Recently, evidence theory also is paid attention in this filed [9]. But in which field will the so called meta mass function can be applied? Some straight open issues are listed below for further research.

What's is the physical significance of the meta mass function in real world?

Is there any other rule to combine or deal with meta mass function?

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